ON D-OPTIMAL DESIGNS FOR ESTIMATING SLOPE

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SUMMARY. The concepts of D-, E- and A-optimality are extended to consider designs for estimating the slopes of a response surface. Optimal designs under the D-optimality criterion are obtained for second-order models over spherical regions.

1. Introduction

In experimental designs the interest is usually in comparison of treatment effects. One exception is the field of response surface designs where usually the interest is in the treatment effects themselves. However, even in response surface designs often the difference between estimated responses at two points may be of greater interest rather than the response at individual locations. (c.f. Herzberg (1967), Box and Draper (1980), Huda (1985), Huda and Mukerjee (1984), Huda (1997)). If differences at points close together in the factor space are involved, the estimation of local slopes of the response function is of interest.

Atkinson (1970) initiated research in designs for estimating slope and subsequently there have been important contributions by Ott and Mendenhall (1972), Murthy and Studden (1972), Myres and Lahoda (1975), Hader and Park (1978), Mukerjee and Huda (1985), Park (1987), Huda and Shafiq (1992), Huda and Al-Shiha (1998) among others.

In this paper we extend the concepts of D-, E- and A-optimality criteria to designs for estimating the slopes of a response surface and consider the problem of deriving optimal designs under the D-optimality criterion.

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2. The Criteria

Suppose that the response \( y \) depends upon \( k \) quantitative factors \( x_1, \ldots, x_k \) through a smooth functional relationship \( y = \eta(x) \) where \( x = (x_1, \ldots, x_k)' \). Let \( y_i \) be the observation on response at the point \( x_i = (x_{i1}, \ldots, x_{ik})' \). It is assumed that \( y_i = \eta(x_i) + e_i \) where the \( e_i \)'s are uncorrelated random errors with zero mean and constant variance \( \sigma^2 \) taken to be unity without loss of generality. Suppose further that \( \eta(x) \) is a linear parametric function given by \( \eta(x) = f'(x)\theta \) where \( f'(x) \) contains \( p \) linearly independent functions of \( x \) and \( \theta \) is the corresponding column vector of unknown parameters.

A design \( \xi \) is a probability measure on the experimental region \( \mathcal{X} \). If \( N \) observations are taken according to \( \xi \) then \( Ncov(\hat{\theta}) = M^{-1}(\xi) \), where \( \hat{\theta} \) is least squares estimator of \( \theta \) and \( M(\xi) = \int_X f(x)f'(x)\eta(dx) \) is the information matrix of \( \xi \). Under the traditional D-, E- and A-optimality criteria the objective is to minimize \( |M^{-1}(\xi)| = |\prod_{i=1}^{p} \mu_i|, \max(\mu_1, \ldots, \mu_p) \) and \( \text{tr} \, M^{-1}(\xi) = \sum_{i=1}^{p} \mu_i \), respectively where \( \mu_i \) are the eigen-values of \( M^{-1}(\xi) \).

The estimated response at a point \( x \) in the factor space is \( \hat{y}(x) = f'(x)\hat{\theta} \). The column vector of estimated slopes along the factor axes at \( x \) is given by \( d\hat{y}/dx = (\partial\hat{y}/\partial x_1, \ldots, \partial\hat{y}/\partial x_k)' = H\hat{\theta} \) where \( H \) is a \( k \times p \) matrix with the \( i \)-th row given by \( \partial f'(x)/\partial x_i \), \( i = 1, \ldots, k \). Thus the variance-covariance matrix of \( d\hat{y}/dx \) is given by \( Ncov(d\hat{y}/dx) = HM^{-1}(\xi)H' \).

If the primary interest of the experimenter is in estimating slopes rather than the absolute responses, then, assuming the model to be correct, it is reasonable to use designs that would in some sense minimize \( HM^{-1}(\xi)H' \) rather than \( M^{-1}(\xi) \). Let \( \beta_i (i = 1, \ldots, k) \) be the \( e \)-values of \( HM^{-1}(\xi)H' \).

Consider
\[
|HM^{-1}(\xi)H'| = \Pi_{i=1}^{k} \beta_i, \\
\max(\beta_1, \ldots, \beta_k), \\
\text{tr}HM^{-1}(\xi)H' = \sum_{i=1}^{k} \beta_i.
\]

The above quantities arise quite naturally and in analogy with the traditional set-up designs minimizing them may be called D-, E- and A-optimal designs for estimating the slopes, respectively. Clearly, the above quantities depend upon the point \( x \) through the matrix \( H \) and on the design \( \xi \) through \( M(\xi) \). If the region of interest in the factor space is \( R \) then one obvious choice is to define the above loss functions for \( \int_R HM^{-1}(\xi)H'\eta(dx) \), the average value of \( HM^{-1}(\xi)H' \) with respect to some measure \( \eta \) on \( R \). Another interesting possibility is to consider the values of the loss functions maximized over the region \( R \), i.e. to consider
\[
\max_{x \in R} |HM^{-1}(\xi)H'|, \max_{x \in R} \max(\beta_1, \ldots, \beta_k) \quad \text{and} \quad \max_{x \in R} \text{tr} \, HM^{-1}(\xi)H'.
\]
In what follows we consider the second choice, assume \( R = \mathcal{X} \) and derive the optimal designs for second-order models over spherical regions. Without loss of generality we take \( \mathcal{X} \) to be the unit sphere centered at the origin, i.e. \( \mathcal{X} = \{x : x'x \leq 1\} \) and for convenience write \( f(x) = (1, x_1^2, \ldots, x_k^2; x_1, \ldots, x_k, x_1x_2, \ldots, x_{k-1}x_k) \) for the \((k+1)(k+2)/2\) terms of a full second degree polynomial in \((x_1, \ldots, x_k)\).

Without loss of generality we restrict to symmetric and permutation invariant designs. Then the only non-zero elements of \( M(\xi) = \int_{\mathcal{X}} x_i^2 \xi(dx) \), \( \alpha_2 = \int_{\mathcal{X}} x_i^2 x_j^2 \xi(dx) \) and \( \alpha_{22} = \int_{\mathcal{X}} x_i^2 x_j^2 \xi(dx)(i \neq j = 1, \ldots, k) \) and \( M(\xi) \) may be written

\[
M(\xi) = diag\{M, \alpha_2 I_k, \alpha_{22} I_{k'}\} \quad \text{where} \quad M = \begin{bmatrix}
1 & \alpha_2 1_k' \\
\alpha_2 1_k & (\alpha_4 - \alpha_{22}) I_k + \alpha_{22} E_k
\end{bmatrix},
\]

\( I_k \) is identity matrix of order \( k \), \( 1_k \) is the \( k \)-component column vector of 1’s, \( E_k = 1_k 1_k' \) and \( k' = k(k-1)/2 \).

3. **D-optimal designs**

Let \( V(\xi, x) = H M^{-1}(\xi) H' \). Then after some algebra it can be shown that for a second-order permutation invariant symmetric design \( \xi \) we may write

\[
V(\xi, x) = \left( \frac{1}{\lambda_2} + \frac{\rho^2}{\lambda_4} \right) I_k + \left( \frac{4}{\alpha_4 - \alpha_{22}} - \frac{2}{\alpha_{22}} \right) \text{diag}\{x_1^2, \ldots, x_k^2\} + \left( \frac{1}{\alpha_{22}} + \frac{4}{\rho^2} \left( \frac{\alpha_4}{(\alpha_4 + (k-1)\alpha_{22} - k\alpha_2^2)} - \frac{1}{(\alpha_4 - \alpha_{22})} \right) \right) xx' \quad \ldots (1)
\]

say, where \( \rho^2 = x'x \).

Box and Hunter (1957) introduced rotatable designs for which variance of the estimated response is constant at points equidistant from the origin. From Kiefer (1960) it is known that for full polynomial models over spherical regions the D-optimal designs, minimizing \( |M^{-1}(\xi)| \), belong to the class of rotatable designs and in particular for the second-order model the D-optimal design has \( \alpha_4 = 3\alpha_{22} = 3\alpha_2/(k+2) = 3(k+3)/\{(k+1)(k+2)^2\} \). In view of this we first restrict attention to the rotatable designs.

3.1 **Rotatable D-optimal design for slope.** If \( \xi \) is a second-order rotatable design then in the traditional notation \( \alpha_4 = 3\alpha_{22} = 3\lambda_4 \) and \( \alpha_2 = \lambda_2 \), say and the objective function (1) reduces to

\[
V(\lambda_2, \lambda_4, x) = \left( \frac{1}{\lambda_2} + \frac{\rho^2}{\lambda_4} \right) I_k + \frac{k\lambda_4 - (k-2)\lambda_2^2}{\lambda_4\{(k+2)\lambda_4 - k\lambda_2^2\}} xx'. \quad \ldots (2)
\]

For a rotatable design \( k\lambda_2^2 \leq (k+2)\lambda_4 \leq \lambda_2 \leq 1/k \) and for non-singularity of the design the first and the last inequality must be strict inequality. It follows that for a nonsingular design coefficients of \( I_k \) and \( xx' \) in (2) are both positive. It can be readily shown that the eigen-values of \( V(\lambda_2, \lambda_4, x) \) are \( \left( \frac{1}{\lambda_2} + \frac{\rho^2}{\lambda_4} \right) + \frac{k\lambda_4 - (k-2)\lambda_2^2}{\lambda_4\{(k+2)\lambda_4 - k\lambda_2^2\}} \).
when \( \lambda \) maximized when \( k - 1 \) corresponding to e-vectors orthogonal to \( x \). Therefore, from (2) we get

\[ |V(\lambda_2, \lambda_4, x)| = \left( \frac{1}{\lambda_2} + \frac{\rho^2}{\lambda_4} \right)^{k-1} \left[ \frac{1}{\lambda_2} + \frac{1}{\lambda_4} + \frac{k\lambda_4 - (k-2)\lambda_2^2}{\lambda_4((k+2)\lambda_4 - k\lambda_2^2)} \right] \] \( \ldots (3) \)

which we wish to maximize with respect to \( x \varepsilon X \). Since \( X \) is the unit ball (3) is maximized when \( x'x = \rho^2 = 1 \) and then objective function reduces to

\[ |V(\lambda_2, \lambda_4)| = \left( \frac{1}{\lambda_2} + \frac{1}{\lambda_4} \right)^{k-1} \left[ \frac{1}{\lambda_2} + \frac{4}{k\lambda_4} + \frac{2(k-1)}{k\lambda_4} \frac{4}{((k+2)\lambda_4 - k\lambda_2^2)} \right] \] \( \ldots (4) \)

which we wish to minimize with respect to \( \lambda_2 \) and \( \lambda_4 \) subject to \( 0 < k\lambda_2^2 < (k+2)\lambda_4 \leq \lambda_2 < 1/k \).

Now (4) may be rewritten as

\[ |V(\lambda_2, \lambda_4)| = \left( \frac{1}{\lambda_2} + \frac{1}{\lambda_4} \right)^{k-1} \left[ \frac{1}{\lambda_2} + \frac{2(k-1)}{k\lambda_4} + \frac{4}{k((k+2)\lambda_4 - k\lambda_2^2)} \right] \] \( \ldots (5) \)

which is clearly minimized when \( \lambda_4 \) is as large as possible, i.e. \( \lambda_4 = \lambda_2/(k+2) \). Thus the D-optimal rotatable design for estimating the slope, like the usual D-optimal design, must put all its mass on the surface of \( X \) and the origin. Substituting the optimal value of \( \lambda_4 \) reduces (5) to

\[ |V(\lambda_2)| = \{ (k+3)/\lambda_2 \}^{k-1} \{ (2k+3)/\lambda_2 + 4/(1-k\lambda_2) \}. \] \( \ldots (6) \)

Differentiating (6) with respect to \( \lambda_2 \), it can be seen that \( |V(\lambda_2)| \) is minimized when

\[ \lambda_2 = \frac{1}{k} \left[ \frac{(2k^3 + 3k^2 - 2k + 2) - 2(2k^3 + 4k^2 - 2k + 1)^{1/2}}{(2k^3 + 3k^2 - 4k)} \right]. \] \( \ldots (7) \)

Thus the D-optimal rotatable design for estimating the slope \( \xi_{RDS} \) puts a mass \( m = k\lambda_2 \) evenly distributed over the surface of \( X \) and a mass \( 1-m \) at the origin where \( \lambda_2 \) is given by (7).

Let \( \xi_D \) denote the D-optimal design for estimating the response. For an arbitrary design \( \xi \) the D-efficiency is defined as \( E_D(\xi) = \{|M(\xi)|/|M(\xi_D)|\}^{1/p} \). Analogously, we may define \( E_{RDS}(\xi) = \{|V(\lambda_2)|/\max_{x\varepsilon X} |V(\xi, x)|\}^{1/k} \) as efficiency of \( \xi \) in comparison with \( \xi_{RDS} \) where \( |V(\lambda_2)| \) is given by (6) and \( \lambda_2 \) is given by (7). In particular, it is of interest to compare \( \xi_D \) with \( \xi_{RDS} \). Table 1 which follows provides \( E_D(\xi_{RDS}) \) and \( E_{RDS}(\xi_D) \) for \( k = 2 \) to \( k = 10 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>2</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( E_{RDS}(\xi_D) )</td>
<td>97.11</td>
<td>97.81</td>
<td>98.03</td>
<td>98.13</td>
<td>98.20</td>
<td>98.25</td>
<td>98.30</td>
<td>98.35</td>
<td>98.40</td>
</tr>
<tr>
<td>( E_D(\xi_{RDS}) )</td>
<td>98.47</td>
<td>98.87</td>
<td>99.04</td>
<td>99.10</td>
<td>99.17</td>
<td>99.22</td>
<td>99.27</td>
<td>99.32</td>
<td>99.33</td>
</tr>
</tbody>
</table>
The table shows that both $\xi_D$ and $\xi_{RDS}$ perform well under the other optimality criterion but the performance of $\xi_{RDS}$ is consistently slightly better. 3.2

Unrestricted $D$-optimal design for slope. Equation (1) can be rewritten in the form

$$V(\xi, x) = D + \gamma xx'$$ \ldots (8)

where

$$D = \text{diag}\{d_1, \ldots, d_k\}, \quad d_i = \frac{1}{\alpha_2} + \frac{\rho^2}{\alpha_{22}} + 2\left(\frac{2}{\alpha_4 - \alpha_{22}} - \frac{1}{\alpha_{22}}\right)x_i^2$$

and

$$\gamma = \frac{1}{\alpha_{22}} + \frac{4}{k} \left[\frac{1}{(\alpha_4 + (k-1)\alpha_{22} - k\alpha_2^2)} - \frac{1}{(\alpha_4 - \alpha_{22})}\right].$$

For a given design $\xi$ consider maximizing $|V(\xi, x)|$ as given in (8) with respect to $x$ subject to $x'x = \rho^2$. Now subject to $x'x = \rho^2$,

$$\text{tr}V(\xi, x) = \text{tr}(D) + \gamma \sum_{i=1}^{k} x_i^2 = \sum_{i=1}^{k} d_i + \gamma \rho^2$$

is a constant. But by the arithmetic mean-geometric mean inequality, for any symmetric non-negative definite matrix $A$, $|A|$ is maximized subject to $\text{tr}(A) = k$ constant when all the eigen-values of $A$ are equal. Hence for a given design $\xi$ and given $x'x = \rho^2$, $|V(\xi, x)|$ would be maximized with respect to $x$ if all the eigen-values of $V(\xi, x)$ were equal. Unfortunately, such equality of all eigen-values can not be achieved for the matrix $V(\xi, x)$ in general. It can be seen that the equality is attainable at $x' = (\pm 1, \ldots, \pm 1)/\rho/\sqrt{k}$ if and only if the design is such that $\gamma = 0$. Hence $|V(\xi, x)|$ is maximized with respect to $x$ by choosing $x$ to make the eigen-values as nearly equal as possible.

Now for $\gamma \neq 0$, at most $(k - 1)$ of the eigen-values can be made equal. This happens if $x = (\pm 1, \ldots, \pm 1)/\rho/\sqrt{k}$ or if $x = \rho e_i$ where $e_i$ is the unit vector along the $i$-th axis. In the first case the eigen-values are

$$a = 1/\alpha_2 + [2(k-1)/\alpha_{22} + 4/(\alpha_4 + (k-1)\alpha_{22} - k\alpha_2^2)]\rho^2/k$$

and

$$b = 1/\alpha_2 + [(k-2)/\alpha_{22} + 4/(\alpha_4 - \alpha_{22})]\rho^2/k$$

with multiplicity $k - 1$.

In the second case the eigen-values are

$$c = 1/\alpha_2 + [(k-1)/(\alpha_4 - \alpha_{22}) + 1/(\alpha_4 + (k-1)\alpha_{22} - k\alpha_2^2)]4\rho^2/k$$

and

$$d = 1/\alpha_2 + \rho^2/\alpha_{22}$$

with multiplicity $k - 1$.

Note that the eigen-values are all maximized when $\rho^2 = 1$. Therefore, $|V(\xi, x)|$ maximized with respect to $x$ subject to $x \in \mathcal{X}$ is given by

$$|V(\xi)| = \max\{ab^{k-1}, cd^{k-1}\} = |V(\alpha_2, \alpha_{22}, \alpha_4)|,$$ \ldots (9)
say, where \( a, b, c \) and \( d \) are as given above with \( \rho^2 = 1 \).

Now, for regression on the unit sphere, the restrictions on \( \alpha_2, \alpha_2 \) and \( \alpha_4 \) for non-singularity of the design are \( 0 < \alpha_2 < 1/k, 0 < \alpha_{22} < \alpha_4 \) and \( k\alpha_2^2 < \alpha_4 + (k-1)\alpha_{22} \leq \alpha_2 \). Our objective is to minimize (9) which for given \( \alpha_2, \alpha_{22} \) is strictly decreasing in \( \alpha_4 \). Substituting the minimizing value \( \alpha_4 = \alpha_2 - (k-1)\alpha_{22} \) reduces \( |V(\alpha_2, \alpha_{22}, \alpha_4)| \) to

\[
|V(\alpha_2, \alpha_{22})| = \max\{a_m b_m^{k-1}, c_m d_m^{k-1}\}, \quad \ldots (10)
\]

Where

\[
\begin{align*}
a_m &= 1/\alpha_2 + 1/\alpha_{22} + (k-2)/k\alpha_{22} + 4/k\alpha_2 + 4/(1-k\alpha_2), \\
b_m &= 1/\alpha_2 + 1/\alpha_{22} + 2/(\alpha_2 - k\alpha_{22}) - 1/\alpha_{22} 2/k, \\
c_m &= 1/\alpha_2 + 4(k-1)/k(\alpha_2 - k\alpha_{22}) + 4/k\alpha_2 + 4/(1-k\alpha_2), \\
d_m &= 1/\alpha_2 + 1/\alpha_{22}.
\end{align*}
\]

Our problem is to minimize (10) or equivalently to minimize \( U(\alpha_2, \alpha_{22}) = |V(\alpha_2, \alpha_{22})|^{1/k} = \max\{U_1(\alpha_2, \alpha_{22}), U_2(\alpha_2, \alpha_{22})\} \) subject to \( 0 < \alpha_{22} < \alpha_2/k < 1/k^2 \) where \( U_1(\alpha_2, \alpha_{22}) = (a_m b_m^{k-1})^{1/k} \) and \( U_2(\alpha_2, \alpha_{22}) = (c_m d_m^{k-1})^{1/k} \).

This minimization algebraically is rather a formidable task. However, the minimization can be readily done numerically. The values of \( \alpha_2 \) and \( \alpha_{22} \) minimizing \( U(\alpha_2, \alpha_{22}) \) along with the minimum values are presented in Table 2 for \( k = 2 \) to \( k = 10 \).

<table>
<thead>
<tr>
<th>( k )</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_2 )</td>
<td>0.5897</td>
<td>0.2829</td>
<td>0.2231</td>
<td>0.1837</td>
<td>0.1558</td>
<td>0.1352</td>
<td>0.1194</td>
<td>0.1068</td>
<td>0.0967</td>
</tr>
<tr>
<td>( \alpha_{22} )</td>
<td>0.0652</td>
<td>0.0566</td>
<td>0.0377</td>
<td>0.0262</td>
<td>0.0195</td>
<td>0.0150</td>
<td>0.0119</td>
<td>0.0097</td>
<td>0.0086</td>
</tr>
<tr>
<td>( U )</td>
<td>21.4865</td>
<td>29.7016</td>
<td>40.4276</td>
<td>53.3225</td>
<td>68.2864</td>
<td>85.2810</td>
<td>104.2886</td>
<td>125.3012</td>
<td>148.3143</td>
</tr>
</tbody>
</table>

A quick glance at Table 2 reveals a surprising fact. The optimal designs are really the rotatable D-optimal designs derived in Section 3.1, for which \( U_1(\alpha_2, \alpha_{22}) = U_2(\alpha_2, \alpha_{22}) \). It is not clear as to why the optimal designs belong to the class of rotatable designs. To get a better insight into what might be happening we derived designs minimizing \( U_1(\alpha_2, \alpha_{22}) \) alone and \( U_2(\alpha_2, \alpha_{22}) \) alone and present these in Tables 3.1 and 3.2, respectively.

For the designs of Table 3.1, \( U_2 \) is always greater than \( U_1 \) and \( \alpha_{22} < \alpha_2/(k+2) \) while for designs of Table 3.2, \( U_1 \) is always greater than \( U_2 \) and \( \alpha_{22} > \alpha_2/(k+2) \). For rotatable designs \( \alpha_{22} = \alpha_2/(k+2) \). The class of designs with \( \alpha_4 = \alpha_2 - (k-1)\alpha_{22} \) is partitioned into three sets. The designs minimizing \( U_1 \) alone belong to the set with \( \alpha_{22} < \alpha_2/(k+2) \), designs minimizing \( U_2 \) alone belongs to the set with \( \alpha_{22} > \alpha_2/(k+2) \) while designs minimizing \( \max(U_1, U_2) \) belongs to the rotatable set (with \( \alpha_{22} = \alpha_2/(k+2) \)) in the middle. As \( k \) increases, the
efficiencies (in terms of minimizing $\max(U_1, U_2)$ of designs of Tables 3.1 and 3.2 increase and approach unity. However, designs of Table 3.1 seem to perform consistently better than those of Table 3.2.

Table 3.1 Designs minimizing $U_1$

<table>
<thead>
<tr>
<th>$k$</th>
<th>2</th>
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<tbody>
<tr>
<td>$\alpha_2$</td>
<td>0.3900</td>
<td>0.2845</td>
<td>0.2237</td>
<td>0.1839</td>
<td>0.1560</td>
<td>0.1354</td>
<td>0.1194</td>
<td>0.1069</td>
<td>0.0967</td>
</tr>
<tr>
<td>$\alpha_{22}$</td>
<td>0.0648</td>
<td>0.0499</td>
<td>0.0347</td>
<td>0.0251</td>
<td>0.0189</td>
<td>0.0148</td>
<td>0.0117</td>
<td>0.0096</td>
<td>0.0080</td>
</tr>
<tr>
<td>$U_1$</td>
<td>20.5710</td>
<td>29.2811</td>
<td>40.1513</td>
<td>53.1180</td>
<td>68.1251</td>
<td>85.1601</td>
<td>104.1760</td>
<td>125.2855</td>
<td>148.239</td>
</tr>
<tr>
<td>$U_2$</td>
<td>24.5735</td>
<td>31.3039</td>
<td>41.7418</td>
<td>54.4880</td>
<td>69.3488</td>
<td>85.9892</td>
<td>105.2660</td>
<td>126.1096</td>
<td>148.8846</td>
</tr>
</tbody>
</table>

Table 3.2 Designs minimizing $U_2$

<table>
<thead>
<tr>
<th>$k$</th>
<th>2</th>
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<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_2$</td>
<td>0.3900</td>
<td>0.2895</td>
<td>0.2275</td>
<td>0.1868</td>
<td>0.1582</td>
<td>0.1370</td>
<td>0.1207</td>
<td>0.1078</td>
<td>0.0975</td>
</tr>
<tr>
<td>$\alpha_{22}$</td>
<td>0.1300</td>
<td>0.0735</td>
<td>0.0463</td>
<td>0.0317</td>
<td>0.0230</td>
<td>0.0174</td>
<td>0.0136</td>
<td>0.0109</td>
<td>0.0090</td>
</tr>
<tr>
<td>$U_1$</td>
<td>24.5454</td>
<td>34.8257</td>
<td>47.3240</td>
<td>62.0592</td>
<td>78.8628</td>
<td>97.4806</td>
<td>117.8920</td>
<td>139.7355</td>
<td>164.5775</td>
</tr>
<tr>
<td>$U_2$</td>
<td>29.5710</td>
<td>28.2085</td>
<td>38.4279</td>
<td>50.8455</td>
<td>65.3495</td>
<td>81.8945</td>
<td>100.4607</td>
<td>121.0399</td>
<td>143.6231</td>
</tr>
</tbody>
</table>

4. Discussion

For second-order polynomial regression model over spherical region the optimal designs for estimating slope, like the usual optimal designs, put all the mass at the centre and surface of the experimental region. For estimating slope the best designs within the rotatable class are also the over all best ones under D-optimality criterion.

Park and Kwon (1998) have recently introduced the concept of slope-rotatability with equal maximum directional variance for second-order response surface models. They also mention the criteria considered in our paper. As the referee pointed out, the E-criterion in the present context does not have a lot of physical significance. For optimal designs under that criterion the interested readers are requested to see Huda and Al-Shiha (1997).

Under A-optimality criterion our objective is to minimize $\max_{x \in \mathbb{R}} \text{tr} \left( H M^{-1}(\xi) H' \right)$ with respect to the design $\xi$. But with $R = \mathcal{X}$ this is really the minimax criterion introduced in Mukerjee and Huda (1985) for estimating slope. Thus the A-optimal second- and third-order designs for spherical regions are the designs derived in Mukerjee and Huda (1985) while those for hypercubic regions are the ones presented in Huda and Shafiq (1992) and Huda and Al-Shiha (1998), respectively.

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References


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