

# A UNIFORM $L_1$ LAW OF LARGE NUMBERS FOR FUNCTIONS ON A TOTALLY BOUNDED METRIC SPACE\*

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**SUMMARY.** Let  $S$  be a totally bounded metric space and  $B(S)$  the space of bounded real function on  $S$ . We obtain a  $L_1$  law of large numbers in the uniform (sup) norm for averages associated with arrays in  $B(S)$ .

## 1. Introduction

Khinchin's weak law of large numbers (W-LLN) and its many applications in the field of statistics and probability are well known. The  $L_1$  version of LLN for i.i.d. random variables proved in Dharmadhikari (1976) also serves as a somewhat simpler alternative to the classical proof of W-LLN (see, e.g., Feller, 1968). An extension to the non-identically distributed case was given in Fabian and Hannan (1985, Theorem 2.3.9) under the uniform integrability condition for the centered summands. Later, Chandra (1989) obtained a similar proof under the assumption of an average uniform integrability condition on the centered summands. The pointwise (i.e., non-uniform) case of the result obtained in this paper incorporates averages with respect to more general probability measures. The pointwise  $L_1$ -LLN result is translated to a uniform  $L_1$ -LLN result with respect to an index parameter lying in a totally bounded space with the additional equi-continuity assumption that exploits the pointwise martingale structure of the summands. Consequently it is weaker than ones available in the literature of generic uniform LLN. See Gine' and Zinn (1984) and Pollard (1990) for other versions of uniform laws of large numbers using Banach space and empirical process methodologies.

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For each integer  $n$ , we consider sequences of random variables  $Y_k(s)$ ,  $k \geq 1$ , indexed by a parameter  $s$  in a totally bounded metric space  $S$ . We are interested in  $L_1$  convergence of averages of martingale differences associated with  $Y_1(s)$ ,  $Y_2(s)$ ,  $\dots$ , where the deviations are measured uniformly in  $s$  (see the next section for a formal description). We introduce another level of uniformity in the result through the underlying probability distribution  $P$  governing the  $Y_k(s)$ . we let  $P$  belong to a collection  $\mathcal{P}_n$  and obtain  $L_1$  convergence which holds uniformly in  $P \in \mathcal{P}_n$ . In particular, the result applies to an array.

Hoadley (1971) introduced a WLLN for independent summands where the convergence is uniform in an index parameter. Andrews (1987) describes it as frequently used in proving consistency in non-linear econometric models and devotes extensive effort to a lengthy analysis of the implications of assumption (b) in Hoadley (1971). The main result of the present paper is a late stage in the evolution of a strengthened replacement (Hannan 1972, unpublished manuscript) for the uniform LLN in Hoadley (1971, Theorem A.5). That replacement eliminated the need for his Theorem A.3 and A.4 and grossly excessive almost sure equicontinuity assumption (b). The penultimate stage in this evolution is Theorem A.3 of Datta's 1988 thesis, where the summands were independent random elements in the space of all continuous functions on a compact metric space. In Andrews (1992), generic equicontinuity conditions on the summands are provided which could translate a pointwise LLN result into a uniform LLN. While this could be of use in some cases, the generic equicontinuity condition may still be unduly restrictive when additional structure is present. As for example, in the case of independent summands and compact  $S$ , our equicontinuity condition reduces to (see Section 3),

$$\lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n E \left( \sup_{d(s,s_0) < \rho} |Y_k(s) - Y_k(s_0)| \right) = 0, \text{ for each } s_0 \in S. \dots (1.1)$$

However, the corresponding equicontinuity condition following the treatment in Andrews (1992) would require

$$\lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n E \left( \sup_{s_0 \in S} \sup_{d(s,s_0) < \rho} |Y_k(s) - Y_k(s_0)| \right) = 0. \dots (1.2)$$

Clearly (1.2) implies (1.1). The following example illustrates that the converse is not true. Let  $\{U_k\}$  be a sequence of i.i.d. uniform on  $[0, 1]$  random variables. Define  $Y_k(s) = 1_{\{s\}}(U_k)$ ,  $s \in S = [0, 1]$ ,  $k \geq 1$ . It is not hard to check that (1.1) holds but not (1.2) for this example. We note that Andrews (1992) does not actually have condition (1.2) because he considered convergence in probability rather than  $L_1$  convergence; but it is precisely the  $L_1$  version of his termwise stochastic equicontinuity condition (TSE). One can observe an

“extra” supremum inside the expectation in (1.2). Even in the non-compact case, we only require (1.1) to hold uniformly in  $s_0 \in S$ , i.e., the supremum over  $s_0 \in S$  only needs to be outside the expected value making it considerably weaker than (1.2). Thus our results are not subsumed by earlier generic uniform laws of large numbers results employing excessive equicontinuity conditions on the summands.

To illustrate the usefulness of our result, we briefly describe in Section 4 two statistical applications in recent literature (Datta, 1991 and 1995) where special cases of it have been used.

## 2. The Formal Context, Definitions and the Main Result

Let  $(S, d)$  be a totally bounded metric space and  $\| \cdot \|$  be the sup norm on  $\mathfrak{R}^S$ , the space of all real functions on  $S$ . Let  $\mathbb{N}$  be the set of all positive integers and for each  $n \in \mathbb{N}$ , let  $A_n$  be a probability measure on  $\mathbb{N}$  whose support is denoted by  $K_n$  and  $a_{k,n} = A_n(\{k\})$ ,  $k \in K_n$ , such that

$$\sum_{k \in K_n} a_{k,n}^2 \rightarrow 0, \text{ as } n \rightarrow \infty. \quad \dots(2.1)$$

We will use  $E^{A_n}$  to denote the expectation determined by  $A_n$ .

We consider for each  $n \in \mathbb{N}$ , a family of probability measures  $\mathcal{P}_n$  and for  $P \in \mathcal{P}_n$ ,  $k \in K_n$  and  $s \in S$ , a random variable  $Y_k(s)$  under  $P$ . (In particular,  $Y$  may also be indexed by  $n$  and  $P$ , which will henceforth be undisplayed.) Note that for each  $k \in K = K_n$ ,  $Y_k = \{Y_k(s) : s \in S\}$  is a  $\mathfrak{R}^S$  valued random element under  $P$ . Given  $s, n$ , and  $P$ , let  $\{\mathcal{F}_{k,s} : k \in K\}$  be a collection of sub  $\sigma$ -algebras of the  $\sigma$ -algebra  $\mathcal{F}$  on which  $P$  is defined such that  $\mathcal{F}_{k,s}$ 's are increasing in  $k$  and  $\mathcal{F}_{k,s}$  contains the  $\sigma$ -algebra generated by  $\{Y_j(s) : j \in K, j < k\}$ . Let  $E_{k,s}$  denote the conditional expectation given the sub  $\sigma$ -algebra  $\mathcal{F}_{k,s}$  and  $X_k^c(s) = X_k(s) - E_{k,s} X_k(s)$ , for a  $\sigma(\{Y_j : j \in K, j \leq k\})$  measurable and  $P$  integrable random variable  $X_k(s)$ . Note that, in particular,  $\{Y_k^c(s) : k \in K\}$  is a martingale difference array, for each  $s \in S$ . Given  $s \in S$ ,  $k \in K$  and  $\rho > 0$ , define  $V_k(s; \rho) = \{|Y_k^c(t) - Y_k^c(s)| : d(t, s) < \rho\}$ . All suprema encountered in this paper are assumed to be measurable. Let  $x_+ = \max(0, x)$ , for  $x \in \mathfrak{R}$ .

**THEOREM 2.1.** *Suppose*

$$\overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} E^{A_n} E(\|Y_k^c(s)\| - M)_+ \rightarrow 0, \text{ as } M \rightarrow \infty, \quad \dots(2.2)$$

and

$$\lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} E\|E^{A_n} E_{k,s} V_k(s; \rho)\| = 0. \quad \dots(2.3)$$

Then

$$\lim_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} E \| E^{A_n} Y_k^c(s) \| = 0. \quad \dots (2.4)$$

PROOF. First consider the degenerate (i.e., real) case when  $\mathcal{S}$  is a singleton set  $\{s_0\}$ . Denote, for simplicity,  $Y_k(s_0)$  by  $Y_k$  and  $E_{k s_0}$  by  $E_k$ . For  $0 < M < \infty$ , represent (as in the proof of Theorem 2.3.9 of Fabian and Hannan (1985)),  $Y_k^c = U_k + W_k = U_k^c + W_k^c$ , with  $U_k = (-M \vee Y_k^c) \wedge M$ . Thus, by subadditivity,

$$\begin{aligned} E |E^{A_n} Y_k^c| &\leq E |E^{A_n} U_k^c| + E |E^{A_n} W_k| + E |E^{A_n} E_k W_k|, \\ &\leq E |E^{A_n} U_k^c| + 2E^{A_n} E |W_k|. \end{aligned} \quad \dots (2.5)$$

Since  $EU_k^c U_k^c = 0$ , if  $k \neq k'$ , and  $E(U_k^c)^2 \leq EU_k^2 \leq M^2$ , we have

$$(E |E^{A_n} U_k^c|)^2 \leq E (E^{A_n} U_k^c)^2 \leq M^2 \sum_{k \in K_n} a_{k,n}^2. \quad \dots (2.6)$$

Combining (2.5) and (2.6), and using  $|W_k| = (|Y_k^c| - M)_+$ , we get that

$$E |E^{A_n} Y_k^c| \leq M \left( \sum_{k \in K_n} a_{k,n}^2 \right)^{1/2} + 2E^{A_n} E (|Y_k^c| - M)_+.$$

The conclusion of the theorem for the present case now follows from (2.1) and (2.2).

Next for the general case, given  $\rho > 0$ , find by the total boundedness, a finite cover of  $\mathcal{S}$  by  $\rho$ -spheres centered at  $s_1, s_2, \dots, s_g$ , say. Then

$$\| E^{A_n} Y_k^c(s) \| \leq \max_{1 \leq i \leq g} \{ |E^{A_n} Y_k^c(s_i)| + E^{A_n} V_k(s_i; \rho) \} \quad \dots (2.7)$$

$$\leq \sum_{i=1}^g \{ |E^{A_n} Y_k^c(s_i)| + |E^{A_n} V_k^c(s_i; \rho)| \} + \| E^{A_n} E_{k s} V_k(s; \rho) \|. \quad \dots (2.8)$$

Since  $|Y_k^c(s_i)| \leq \|Y_k^c(s)\|$ ,  $(E_{k s_i} \|Y_k^c(s)\| - M)_+ \leq E_{k s_i} (\|Y_k^c(s)\| - M)_+$ , and  $|V_k^c(s_i; \rho)| \leq 2(\|Y_k^c(s)\| + E_{k s_i} \|Y_k^c(s)\|)$ , it follows that each of  $Y_k^c(s_i)$  and  $V_k^c(s_i; \rho)$  inherit the uniform integrability property (2.2) of  $\|Y_k^c(s)\|$ . Hence, by the degenerate case of the theorem, it follows that

$$\overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} E \sum_{i=1}^g \{ |E^{A_n} Y_k^c(s_i)| + |E^{A_n} V_k^c(s_i; \rho)| \} = 0. \quad \dots (2.9)$$

The proof now ends using (2.8) and (2.9) by first letting  $n \rightarrow \infty$  and then letting  $\rho \rightarrow 0$  and using (2.3). □

REMARK 2.1. For pairwise independent  $Y_k$ , we can obtain an analogous theorem by changing  $E_{k_s}$  to  $E$  throughout (including in the definition of the centering). It is easy to see that the same proof works with  $E_{k_s}$  changed to  $E$ .

### 3. A Discussion of the Sufficient Conditions

In this section we would like to investigate the conditions of Theorem 2.1. First, we consider (2.3) which is the equicontinuity condition on the centered summands. We will note a number of sets of sufficient conditions for this which may be easier to verify in practice. Moreover, in certain important special cases, it will be weakened considerably.

We first list down a number of "natural" equicontinuity conditions involving various levels of smoothness of the summands.

$$C1. \lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} E \|E^{A_n} E_{k_s} V_k(s; \rho)\| = 0.$$

$$C2. \lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} \|E^{A_n} E V_k(s; \rho)\| = 0.$$

$$C3. \lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} E \|E^{A_n} (E - E_{k_s}) V_k(s; \rho)\| = 0.$$

$$C4. \lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} E^{A_n} E V_k(s_0; \rho) = 0, \text{ for each } s_0 \in \mathcal{S}.$$

$$C5. \lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} P [\|E^{A_n} E_{k_s} V_k(s; \rho)\| > \epsilon] = 0, \text{ for each } \epsilon > 0.$$

Note that C1 is the equicontinuity condition (2.3) used in Theorem 2.1. Since for a non-negative process  $Z = Z_k(s; \rho)$  on  $\mathcal{S}$ ,

$$E^{A_n} E Z_k(s_0; \rho) \leq \|E^{A_n} E Z_k(s; \rho)\| \leq E \|E^{A_n} E_{k_s} Z_k(s; \rho)\|,$$

for each  $s_0 \in \mathcal{S}$  and  $\rho > 0$ , it follows that C1 implies C2, and C2 implies C4. The first implication, together with the triangle inequality shows that C1 is equivalent to C2 and C3 together.

Consider the important special case when  $\mathcal{S}$  is compact. In this case C2 and C4 are equivalent. To see this note that we can find  $\rho_m \downarrow 0$ ,  $n_m \uparrow \infty$ , and  $s_m \in \mathcal{S}$ , such that

$$\sup_{P \in \mathcal{P}_{n_m}} E^{A_{n_m}} E V_k(s_m; \rho_m) \rightarrow \lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} \|E^{A_n} E V_k(s; \rho)\|.$$

Because  $\mathcal{S}$  is compact, we can assume without loss of generality that  $s_m \rightarrow s_\infty \in \mathcal{S}$ . Then for any  $\rho > 0$ ,  $V_k(s_m; \rho_m) \leq 2 V_k(s_\infty; \rho)$ , eventually (uniformly in  $k$ ) and hence

$$\lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} \|E^{A_n} E V_k(s; \rho)\| \leq 2 \lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} E^{A_n} E V_k(s_\infty; \rho) = 0$$

if C4 holds.

Next we note that, in the case where for each  $k$ ,  $Y_k$  is independent of  $\mathcal{F}_{k_s}$  for every  $s$ , the condition C3 is trivially satisfied. This is the case, in particular, if

$Y_k$  is independent of the past  $Y$ 's and  $\mathcal{F}_{k_s} = \sigma(\{Y_j(s) : j < k, j \in K, s \in S\})$  is the  $\sigma$ -algebra generated by the  $Y$ 's prior to  $k$ . Thus, in the independent case, C1 can be weakened to C2, and if  $S$  is compact, it can be further weakened to C4.

When  $E_{k_s}$  is constant in  $s$ , which is the case for example, if  $\mathcal{F}_{k_s} = \sigma(\{Y_j(s) : j < k, j \in K, s \in S\})$  is the  $\sigma$ -algebra generated by the  $Y$ 's prior to  $k$ ,

$$\left( \|E^{A_n} E_k V_k(s; \rho)\| - M \right)_+ \leq 2E^{A_n} E_k (\|Y_k^c(s)\| - M/2)_+.$$

Therefore,  $\|E^{A_n} E_k V_k(s; \rho)\|$  inherits the uniform integrability property (2.2) of  $\|Y_k^c(s)\|$ . Hence, using the elementary inequality

$$\|E^{A_n} E_k V_k(s; \rho)\| \leq \epsilon + M \left[ \|E^{A_n} E_k V_k(s; \rho)\| > \epsilon \right] + \left( \|E^{A_n} E_k V_k(s; \rho)\| - M \right)_+,$$

for  $\epsilon > 0, 0 < M < \infty$ , and non-negative processes  $Z$  on  $S$ , we see that in this case C1 can be weakened to C5 provided (2.2) holds.

In the rest of this section, consider the above situation when  $E_{k_s} \equiv E_k$ , i.e.,  $E_{k_s}$  is constant in  $s$ . In some applications, it may be more convenient to verify conditions without the centering.

Suppose, condition (2.2) holds with  $Y_k^c$  replaced by  $Y_k$ . Since  $\|E_k Y_k(s)\| \leq E_k \|Y_k(s)\|$ , we have  $(\|E_k Y_k(s)\| - M)_+ \leq E_k (\|Y_k(s)\| - M)_+$ , and hence

$$\overline{\lim}_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_n} E^{A_n} E \left( \|E_k Y_k(s)\| - M \right)_+ \rightarrow 0, \text{ as } M \rightarrow \infty.$$

Therefore, by subadditivity, (2.2) holds.

Next let  $C1^+ - C4^+$  denote C1-C4, with  $V$  replaced by  $W$ , where  $W_k(s; \rho) = \sup\{|Y_k(t) - Y_k(s)| : d(t, s) < \rho\}$ . If  $C1^+$  holds then  $V_k(s) \leq W_k(s) + E_k W_k(s)$  and hence  $E_k V_k(s) \leq 2E_k W_k(s)$ . Therefore C1 holds. Similarly,  $C2^+, C4^+$  and  $C5^+$  imply C2, C4 and C5, respectively.

#### 4. Statistical Applications

Uniform laws of large numbers are useful in consistency proofs for estimators obtained from estimating equations and also minimum dispersion estimators, among other things. In this section we briefly mention two statistical applications where special cases of the general theorem obtained in this paper have been used.

4.1. *Consistency of posterior mixture and asymptotic optimality of Bayes compound estimators.* In Datta (1991a), a general consistency result for a posterior mixture in the framework of a compound decision problem was established which was in turn used in Datta (1991b) to prove the asymptotic

optimality of certain Bayes compound estimators of parameters in an exponential family model. In a compound decision problem one observes a number, say  $n$ , independent random elements  $X_1, \dots, X_n$  taking values in a common measurable space  $\mathcal{X}$ , where  $X_i$  has distribution  $P_{\theta_i}$ ,  $1 \leq i \leq n$ . It will be assumed that the index parameters  $\theta_1, \dots, \theta_n$  belong to a compact metric space  $\Theta$  and that the family of probability measures  $\{P_\theta : \theta \in \Theta\}$  is dominated by a  $\sigma$ -finite measure  $\mu$ . The joint distribution of  $\mathbf{X} = (X_1, \dots, X_n)$  is denoted by  $P_\theta = \prod_{i=1}^n P_{\theta_i}$ , the empirical distribution of  $\theta_1, \dots, \theta_n$  by  $G_n$  and the set of all probability distributions (priors)  $G$  on  $\Theta$  by  $\mathcal{G}$ . The  $G$ -mix of the  $P_\theta$  is denoted by  $P_G = \int P_\theta dG$  and its  $\mu$ -density by  $p_G$ .

The independent case of Theorem 2.1 was used to prove the uniform, with respect to  $G \in \mathcal{G}$ ,  $L_1$ -LLN, of the average  $n^{-1} \sum \log(p_G(X_i)/p_{G_n}(X_i))$ , i.e.,

$$\sup_{\theta} E_{\theta} \left( \sup_{G \in \mathcal{G}} \left| n^{-1} \sum \log(p_G(X_i)/p_{G_n}(X_i)) - \int \log(p_G/p_{G_n}) dP_{G_n} \right| \right) \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{4.1}$$

The details can be found in Datta (1991a). This convergence result was a key step in proving the above mentioned posterior consistency.

4.2. *Consistency of the mle in a sequential design problem.* In a sequential design problem, the likelihood takes the product form

$$L_n(\theta) = \prod_{i=1}^n f(X_i, E_i, \theta), \tag{4.2}$$

where  $X_1, \dots, X_n$  are sequential observations defined on a common measurable space  $(\Omega, \mathcal{F})$  taking values in a sample space  $\mathcal{X}$ , and  $E_1, \dots, E_n$  are experiments taking values in an experiment space  $\mathcal{E}$  such that  $E_i$  is  $\mathcal{F}_{i-1} = \sigma(\mathcal{F}_0 \cup \sigma(X_1, \dots, X_{i-1}))$  measurable,  $i \geq 1$ , where  $\mathcal{F}_0$  is a sub  $\sigma$  algebra of  $\mathcal{F}$ . Also, under  $\theta \in \Theta$ , given  $\mathcal{F}_{i-1}$ , the  $i$  th observation  $X_i$  has  $\mu$ -density  $f(X_i, E_i, \theta)$ ,  $i \geq 1$ . In Datta (1995), consistency of the mle  $\hat{\theta}$  (a maximizer of (4.2)) was established under regularity conditions by first obtaining a uniform LLN for the average  $n^{-1} \sum f(X_i, E_i, \theta)$  by an application of Theorem 2.1 for the case where  $P_{k_s}$  is constant in  $s$  and the conditions are in terms of the summands without the centering.

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