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CENTRAL LIMIT THEOREM FOR A DOUBLE ARRAY OF HARRIS CHAINS

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SUMMARY. Let $X_n = \{X_{nj}\,;\, j \geqslant 0\}$ be a sequence of Harris recurrent Markov chains with X_n having a general state space $(S_n, \mathcal{S}_n)$. A central limit theorem for $\sum_{j=0}^{N_n} f_n(X_{nj})$ is established in this paper where f_n is a sequence of suitable measurable functions from S_n to.

1. Introduction

In a recent paper Kulperger and Prakasa Rao (1989) studied the method of bootstrap for finite state Markov chains. While extending their results to the countable state space case, Athreya and Fuh (1989) established a central limit theorem for a double array of Markov chains with countable state space. The present paper extends that result to more general state spaces under the assumption of existence of recurrent singletons. This is not as restrictive as it may seem at first. It is known that if a Markov chain $\{X_j\}$ on a state space S with a countably generated σ-algebra is Harris recurrent (that is, there exists a σ-finite measure φ on S such that $\varphi(A) > 0$ implies $P_x(T_A < \infty) \equiv 1$ for all x in S, where P_x refers to starting at x and $T_A \equiv \inf\{j : j \geqslant 1, X_j \in A\}$), then there exists an integer $n_0 \geqslant 1$ such that the sequence $Y_n = X_{nn_0}$, $n = 0, 1, 2, \ldots$ is a regenerative sequence of random variables. In fact, there exist random terms $\alpha_1, \alpha_2, \alpha_3, \ldots$ such that $\{Y_{\alpha_i+j} : 0 \leqslant j \leqslant \alpha_{i+1}-\alpha_i-1,\ \alpha_{i+1}-\alpha_i\}$, $i = 1, 2, \ldots$ are independent and identically distributed cycles. The reader is refered to Athreya and Ney (1978) and Nummelin (1984) for details.

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The motivation for proving a central limit theorem of the kind presented in this paper comes from an attempt to prove the consistency of bootstrap methods for a Harris recurrent Markov chain. In that context, for each n one has an estimate P_n of the transition probability and then bootstraps using this P_n. It is of interest to show that the boogstrap is consistent if P_n is. This leads to a double array situation.

The central limit theorem and a weak law of large numbers are stated in the next section, their proofs are given in Section 3. Alternative assumptions for the validity of the weak law is discussd in Section 4. Comprehensive background discussions on Harris recurrent Markov chain are available in Nummelin (1984).

2. Statement of the results

Let $\boldsymbol{X}_n \equiv \{X_{nj}\,;\, j \geqslant 0\}$ be a sequence of Markov chains with state spaces $(S_n, \mathcal{S}_n)$ such that for each n there exists a singleton $\Delta_n \in \mathcal{S}_n$ such that $P_{n\Delta_n}(T_{n\Delta_n} < \infty) = 1$, where P_{nx} refers to the probability distribution of $\boldsymbol{X}_n$ under $X_{n0} = x$, and $T_{n\Delta_n}$ be the first hitting time of the Harris chain $\boldsymbol{X}_n$ to its recurrent point Δ_n on the state space S_n, namely,

$$T_{n\Delta_n} = \begin{cases} \inf\{\, j \geqslant 1, X_{nj} = \Delta_n\}\,; \\ \infty, \quad \text{if no such } j \text{ exist.} \end{cases}$$

Let

$$\nu_n(A) \equiv E_{n\Delta_n}\left(\sum_{j=0}^{T_{n\Delta_n}-1} I_A(X_{nj}) \right) \quad \text{for } A \in \mathcal{S}_n.$$

It is known (Nummelin, 1984) that $\nu_n(.)$ is σ-finite and invariant for X_n, that is,

$$\nu_n(.) = \int \nu_n(dy)\, P_n(y, .)$$

where $P_n(y, .) = P(X_{n1} \in . \,|\, X_{n0} = y)$ is the transition kernel function of the chain X_n.

Assume that $\nu_n(S_n) \equiv E_{n\Delta_n} T_{n\Delta_n} < \infty$ and set $\pi_n(.) \equiv \nu_n(.)/\nu_n(S_n)$. Let $f_n : S_n \to \boldsymbol{R}$ be $\mathcal{S}_n$ measurable. We are interested in proving a central limit theorem for $\sum_{j=0}^{N_n} f_n(X_{nj})$, where $N_n \to \infty$ as $n \to \infty$. We assume in what follows that $N_n \to \infty$ as $n \to \infty$.

An important step in our proof of the central limit theorem is the weak law below that is also of some independent interest.

Theorem 1. *Let for $\rho > 0$ and $x \in \boldsymbol{X}_n$,*

$$\delta_n(\rho, x) \equiv \sup_{m \geqslant \rho N_n} \left| \frac{1}{m} \sum_{s=1}^{m} P_n^{(s)}(x, \Delta_n) - \pi_n(\Delta_n) \right| .$$

Let $m_n = \sum_{j=0}^{N_n} I_{\Delta_n}(X_{nj})$ *be the number of visits to* Δ_n *by* $\{X_{nj} : 0 \leqslant j \leqslant N_n\}$ *and* $\tilde{\pi}_n(\Delta_n) = N_n^{-1} m_n$. *Assume that the sequence of initial distrributions of* X_{n0} *satisfy*

$$\lim_{\rho \to 0} \limsup_{n \to \infty} E\delta_n(\rho, X_{n0}) = 0. \quad \ldots \quad (1)$$

Then

$$\tilde{\pi}_n(\Delta_n) - \pi_n(\Delta_n) \to 0 \text{ in probability,}$$

where $\pi_n(\Delta_n) = (E_{n\Delta_n} T_{n\Delta_n})^{-1}$.

In Section 4, a variety of sufficient conditions to ensure (1) are discussed.

The next result is the central limit theorem.

Theorem 2. *Assuming the following conditions hold,*

(i) $f_n \in L_2(\pi_n)$, $\int f_n \, d\pi_n = 0$ *and* $\dfrac{\int |f_n| \, d\pi_n}{\sigma_n \sqrt{N_n \pi_n(\Delta_n)}} \to 0$,

(ii) $\liminf_n \pi_n(\Delta_n) > 0$,

(iii) *for each* $\epsilon > 0$, *as* $n \to \infty$, *we have*

$$\frac{1}{\sigma_n^2} E_{n\Delta_n} \{(\eta_{n1}(f_n))^2 ; |\eta_{n1}(f_n)| > \epsilon \sigma_n \sqrt{[N_n \pi_n(\Delta_n)]}\} \to 0,$$

where

$$\eta_{n1}(f_n) = \sum_{j=0}^{T_{n\Delta_n}-1} f_n(X_{nj}),$$

$$\begin{aligned} \sigma_n^2 &= E_{n\Delta_n} (\eta_{n11}(f_n))^2 \\ &= 2\int f_n(x) (T_n f_n)(x) \, \pi_n(dx) - \int f_n^2(x) \, \pi_n(dx), \end{aligned}$$

and

$$(T_n f_n)(x) = E_{nx} \left(\sum_{j=0}^{T_{n\Delta_n}-1} f_n(X_{nj}) \right)$$

Then, under (1) (*of Theorem* 1),

$$\frac{1}{\sigma_n \sqrt{N_n \pi_n(\Delta_n)}} \sum_{j=0}^{N_n} f_n(X_{nj}) \to N(0, 1) \text{ in distribution.} \quad \ldots \quad (2)$$

3. Proofs of the results

Proof of Theorem 1.

$$\pi^*_{n\Delta_n} = E\tilde{\pi}_n(\Delta_n) = E\left[\frac{1}{N_n} \sum_{s=1}^{N_n} P_n^{(s)}(X_{n0}, \Delta_n). \right]$$

Clearly,

$$|\pi^*_{n\Delta_n} - \pi_n(\Delta_n)| \leqslant E\delta_n(\rho, X_{n0}) \text{ for all } \rho < 1.$$

and so, it is enough to prove

$$\tilde{\pi}_n(\Delta_n)-\pi^*_{n\Delta_n} \to 0 \text{ in probability.}$$

Now, for any given $\epsilon > 0$

$$P(|\tilde{\pi}_n(\Delta_n)-\pi^*_{n\Delta_n}| > \epsilon) \leqslant \frac{1}{\epsilon^2} \operatorname{var}(\tilde{\pi}_n(\Delta_n))$$

$$= \frac{1}{\epsilon^2}\Big[\frac{1}{N_n^2}\sum_{j=1}^{N_n} \operatorname{var}(I_{\Delta_n}(X_{nj}))+\frac{2}{N_n^2}\sum_{k=1}^{N_n}\sum_{s=1}^{N_n-k} \operatorname{cov}(I_{\Delta_n}(X_{nk}, I_{\Delta_n}(X_{n(k+s)}))\Big]$$

$$= \frac{1}{\epsilon^2}\Big[\frac{1}{N_n}+\frac{2}{N_n^2}\sum_{k=1}^{N_n}\sum_{s=1}^{N_n-k} E(P_n^{(k)}(X_{n0}, \Delta_n)(P_n^{(s)}(\Delta_n, \Delta_n)-P_n^{(s+k)}(X_{n0}, \Delta_n))\Big]$$

$$= \frac{1}{\epsilon^2 N_n}+\frac{2\Gamma_n}{\epsilon^2}, \text{ say.}$$

It suffices to show that $\Gamma_n \to 0$.

Let $$S_{nx}(m) \equiv \sum_{r=1}^{m} P_{nx}(X_{nr} = \Delta_n).$$

Let $$\Gamma_n(x) \equiv \frac{1}{N_n^2}\sum_{k=1}^{N_n} P_n^{(k)}(x, \Delta_n)\,(S_{n\Delta_n}(N_n-k)-S_{nx}(N_n)+S_{nx}(k))$$

$$= \frac{1}{N_n^2}\sum_{k=1}^{N_n} P_n^{(k)}(x, \Delta_n)\Big[\Big(\frac{S_{n\Delta_n}(N_n-k)}{N_1-k}-\pi_n(\Delta_n)\Big)(N_n-k)$$

$$-\Big(\frac{S_{nx}(N_n)}{N_n}-\pi_n(\Delta_n)\Big)N_n+\Big(\frac{S_{nx}(k)}{k}-\pi_n(\Delta_n)\Big)k\Big]$$

Thus, for $0<\rho<1$,

$$|\Gamma_n(x)| \leqslant \frac{1}{N_n^2}\sum_{k\leqslant(1-\rho)N_n} P_n^{(k)}(x, \Delta_n)(N_n-k)\delta_n(\rho, x)$$

$$+\frac{1}{N_n^2}\sum_{k>(1-\rho)N_n}^{N_n}(N_n-k)+\frac{\delta_n(\rho, x)N_n}{N_n^2}\sum_{k=1}^{N_n} P_n^{(k)}(x, \Delta_n)$$

$$+\frac{1}{N_n^2}\sum_{k<\rho N_n} k+\frac{\delta_n(\rho, x)}{N_n^2}\sum_{k\geqslant\rho N_n} P_n^{(k)}(x, \Delta_n)k$$

$$\leqslant \delta_n(\rho, x)\Big(2+\frac{1}{N_n^2}\sum_{j=\rho N_n}^{N_n} j\Big)+\frac{2}{N_n^2}\sum_{1}^{\rho N_n} j$$

$$\leqslant C_1\,\delta_n(\rho, x)+C_2\rho^2,$$

where C_1 and C_2 are constants and hence from (1)

$$\limsup_n |\Gamma_n| = 0. \quad \square$$

For the proof of Theorem 2, we need the following three lemmata.

Lemma 1. *Assume the hypotheses (i), (ii) of Theorem 2 and $0 < \sigma_n^2 < \infty$. Let $T^k_{n\Delta_n}$ be the k-th recurrent time to state Δ_n. Then, for any initial distribution of X_{n0},*

$$\frac{1}{\sigma_n \sqrt{N_n \pi_n(\Delta_n)}} \sum_{j=T^{m_n}_{n\Delta_n}}^{N_n} f_n(X_{nj}) \to 0 \text{ in probability as } n \to \infty,$$

where $m_n = \sum_{j=0}^{N_n} I_{\Delta_n}(X_{nj})$ as in Theorem 1.

Proof. Let P denote the probability with distribution of $\boldsymbol{X}_n$ for a given initial distribution of X_{n0}. Then, for any $\epsilon > 0$,

$$P\left\{\left|\sum_{j=T^{m_n}_{n\Delta_n}}^{N_n} f_n(X_{nj})\right| > \epsilon\sigma_n \sqrt{N_n\pi_n(\Delta_n)}\right\}$$

$$= \sum_{r=0}^{N_n} P\left\{\left|\sum_{j=N_n-r}^{N_n} f_n(X_{nj})\right| > \epsilon\,\sigma_n \sqrt{N_n\pi_n(\Delta_n)}\,;\ T^{m_n}_{n\Delta_n} = N_n - r\right\}$$

$$\leqslant \sum_{r=0}^{N_n} P\left\{\sum_{j=N_n-r}^{T^{m_n+1}_{n\Delta_n}} |f_n(X_{nj})| > \epsilon\,\sigma_n \sqrt{N_n\pi_n(\Delta_n)}\,;\quad T^{m_n}_{n\Delta_n} = N_n - r\right\}$$

$$= P_{n\Delta_n}\{\eta_{n1}(|f_n|) > \epsilon\sigma_n \sqrt{N_n\pi_n(\Delta_n)}\} \sum_{r=0}^{N_n} P\{T^{m_n}_{n\Delta_n} = N_n - r\}$$

$$\leqslant P_{n\Delta_n}\{\eta_{n1}(|f_n|) > \epsilon\sigma_n \sqrt{N_n\pi_n(\Delta_n)}\}$$

$$\leqslant \frac{E(\eta_{n1}(|f_n|))}{\epsilon\sigma_n\sqrt{N_n\pi_n(\Delta_n)}}$$

$$= \frac{\int |f_n|\, d\nu_n}{\epsilon\sigma_n\sqrt{N_n\pi_n(\Delta_n)}}$$

$$\to 0. \quad \square$$

Lemma 2. *Let P be as in the proof of Lemma 1 above and let (1) hold. Then, under hypothesis (ii) of Theorem 2 and $0 < \sigma_n^2 < \infty$, we have for any $\epsilon > 0$,*

$$P\left(\left| \frac{1}{\sigma_n \sqrt{N_n \pi_n(\Delta_n)}} \right| \sum_{j=0}^{m_n} \eta_{nj}(f_n) - \sum_{j=0}^{N_n \pi_n(\Delta_n)} \eta_{nj}(f_n) \Big| > \epsilon \right) \to 0.$$

Proof. By Theorem 1, we have for any $\epsilon > 0$, there exists an integer n_0 such that

$$P\{|m_n - [N_n \pi_n(\Delta_n)]| > \epsilon^3 N_n\} < \epsilon \text{ for all } n > n_0.$$

Clearly, for such n,

$$P\left(\left| \sum_{j=0}^{m_n} \eta_{nj}(f_n) - \sum_{j=0}^{N_n \pi_n(\Delta_n)} \eta_{nj}(f_n) \right| > \epsilon \, \sigma_n \sqrt{N_n \pi_n(\Delta_n)} \right)$$

$$\leqslant P\{|m_n = [N_n \pi_n(\Delta_n)]| > \epsilon^3 N_n\}$$

$$+P\left\{ \max_{|r-[N_n\pi_n(\Delta_n)]| \leqslant \epsilon^3 N_n} \left| \sum_{j=[N_n \pi_n(\Delta_n)]+1}^{r} \eta_{nj}(f_n) \right| > \epsilon \sigma_n \sqrt{N_n \pi_n(\Delta_n)} \right\}$$

$$< \epsilon + 2P\left\{ \max_{1 \leqslant r \leqslant \epsilon^3 N_n} \left| \sum_{j=1}^{r} \eta_{nj}(f_n) \right| > \epsilon \sigma_n \sqrt{N_n \pi_n(\Delta_n)} \right\} \text{ (by Theorem 1)}$$

$$< \epsilon + \frac{2E\left(\sum_{j=1}^{[\epsilon^3 N_n]+1} \eta_{nj}(f_n) \right)^2}{\epsilon^2 \sigma_n^2 N_n \pi_n(\Delta_n)} \text{ (by Kolmogorov's inequality)}$$

$$= \epsilon + 2\epsilon \frac{1}{\pi_n(\Delta_n)}$$

$$\leqslant \text{(Constant) } \epsilon. \quad (\text{since } \liminf_n \pi_n(\Delta_n) > 0). \quad \square$$

Lemma 3. *The variance σ_n^2 of $\eta_{n1}(f_n)$ is given by*

$$\sigma_n^2 \equiv E_{n\Delta n}(\eta_{n1}(f_n))^2$$

$$= 2 \int f_n(x)\, (T_n f_n)(x) v_n(dx) - \int f_n^2(x) \nu_n(dx),$$

where $(t_n f_n)(x) \equiv E_{nx} \left(\sum_{j=0}^{T_{n\Delta n}-1} f_n(X_{nj}) \right)$.

Proof. $E_{n\Delta_n}(\eta_{n1}(f_n))^2$

$$= E_{n\Delta_n}\Big(\sum_{j=0}^{T_{n\Delta_n}-1} f_n(X_{nj})\Big)^2$$

$$= E_{n\Delta_n}\Big(\sum_{i,j=0}^{T_{n\Delta_n}-1} f_n(X_{ni}) f_n(X_{nj})\Big)$$

$$= E_{n\Delta_n}\Big(\sum_{i=0}^{T_{n\Delta_n}-1} f_n^2(X_{ni})\Big) + 2E_{n\Delta_n} \sum_{i=0}^{T_{n\Delta_n}-1} \sum_{j=i+1}^{T_{n\Delta_n}-1} f_n(X_{ni}) f_n(X_{nj})\Big)$$

$$= E_{n\Delta_n}\Big(\sum_{i=0}^{T_{n\Delta_n}-1} f_n^2(X_{ni})\Big)$$

$$+2E_{n\Delta_n}\Big(\sum_{i=0}^{\infty} I(T_{n\Delta_n} > i) \sum_{j=i+1}^{\infty} f_n(X_{ni}) f_n(X_{nj})\, I(T_{n\Delta_n} > j)\Big)$$

$$= E_{n\Delta_n}\Big(\sum_{i=0}^{T_{n\Delta_n}-1} f_n^2(X_{ni})\Big)$$

$$+2\sum_{i=0}^{\infty} E_{n\Delta_n}\Big[(I(T_{n\Delta_n} > i)\,(f_n X_{ni})$$

$$. E\Big(\sum_{j=i+1}^{\infty} f_n(X_{nj}) I(T_{n\Delta_n} > j) \,|\, \mathcal{F}_{ni}\Big)\Big].$$

where $\mathcal{F}_{ni}$ is the σ-algebra generated by $\{X_{nj}\,;\, j \leqslant i\}$. But

$$E_{n\Delta_n}\Big[\sum_{j=i+1}^{\infty} f_n(X_{nj}) I(T_{n\Delta_n} > j \,|\, \mathcal{F}_{ni}) I(T_{n\Delta_n} > i)\Big]$$

$$= E_{nX_{ni}} \sum_{r=1}^{\infty} f_n(X_{nr}) I(T_{n\Delta_n} > r)\, I(T_{n\Delta_n} > i)$$

$$= ((T_n f_n)(X_{ni}) - f_n(X_{ni})) I(T_{n\Delta_n} > i).$$

Hence,

$E_{n\Delta_n}(\eta_{n1}(f_n))^2$

$$= E_{n\Delta_n}\Big(\sum_{i=0}^{T_{n\Delta_n}-1} f_n^2(X_{ni})\Big)$$

$$+2\sum_{i=0}^{\infty} E_{n\Delta_n}[I(T_{n\Delta_n} > i) f_n(X_{ni})][((T_n f_n)(X_{ni}(X_{ni})) I(T_{n\Delta_n} > i)]$$

$$= 2\sum_{i=0}^{\infty} E_{n\Delta_n}\Big[\Big(I(T_{n\Delta_n} > i) f_n(X_{ni}(T_n f_n)(X_{ni}) - E_{n\Delta_n}\Big(\sum_{i=0}^{T_{n\Delta_n}-1} f_n^2(X_{ni})\Big)$$

$$= 2\int f_n(x)\,(T_n f_n)(x)\,\nu_n(dx) - \int f_n^2(x)\,\nu_n(dx). \quad \square$$

Proof of Theorem 2. We decompose $(1/\sigma_n\sqrt{N_n\pi_n(\Delta_n)}\sum_{j=0}^{N_n} f_n(X_n)$ as follows :

$$\frac{1}{\sigma_n\sqrt{N_n\pi_n(\Delta_n)}}\sum_{j=0}^{N_n} f_n(X_{nj})$$

$$=\frac{1}{\sigma_n\sqrt{N_n\pi_n(\Delta_n)}}\Big(\sum_{j=0}^{m_n}\eta_{nj}(f_n)+\sum_{n=T_{n\Delta_n}^{m_n}}^{N_n} f_n(X_{nj})\Big)$$

$$=\frac{1}{\sigma_n\sqrt{N_n\ \pi_n(\Delta_n)}}\sum_{j=0}^{[N_n\pi_n(\Delta_n)]}\eta_{nj}(f_n) \qquad \cdots \ (3)$$

$$+\frac{1}{\sigma_n\sqrt{N_n\pi_n(\Delta_n)}}\Big(\sum_{j=0}^{m_n}\eta_{nj}(f_n)-\sum_{j=0}^{N_n\pi_n(\Delta_n)}\eta_{nj}(f_n)\Big) \qquad \cdots \ (4)$$

$$+\frac{1}{\sigma_n\sqrt{N_n\pi_n(\Delta_n)}}\sum_{j=T_{n\Delta}^{m_n}}^{N_n} f_n(X_{nj}). \qquad \cdots \ (5)$$

By Lemma 1 and assumption (i), we have that (5)$\to 0$ in probability, by Lemma 2, we have that (4)$\to 0$ in probability. Now, we need only to verify that

$$\frac{1}{\sigma_n\sqrt{[N_n\pi_n(\Delta_n)]}}\sum_{j=0}^{[N_n\pi_n(\Delta_n)]}\eta_{nj}(f_n)\to N(0,1) \text{ in distribution.}$$

Let

$$Y_{nj}\equiv\frac{\eta_{nj}(f_n)}{\sigma_n\sqrt{[N_n\pi_n(\Delta_n)]}}.$$

Then, we have for each n

(1) $\{Y_{nj};j=1,\ldots,[N_n\pi_n(\Delta_n)]\}$ are i.i.d. random variables,

(2) $EY_{nj}=0$,

(3) $\displaystyle\sum_{j=1}^{[N_n\pi_n(\Delta_n)]}EY_{nj}^2=\frac{E(\eta_{n1}(f_n))^2}{\sigma_n^2}=1$ (by assumption (iii)).

Therefore, it is enough to show that $\{Y_{nj}\}$ satisfies the Lindeberg condition. That is, for any fixed $\epsilon>0$, we have

$$\sum_{j=1}^{[N_n\pi_n(\Delta_n)]}E[Y_{nj}^2I(|Y_{nj}|>\epsilon)]\to 0 \text{ as } n\to\infty.$$

But, the left side of the above equation is equal to

$$\frac{1}{\sigma_n^2} E[(\eta_{n1}(f_n))^2 \, ; \, |\eta_{n1}(f_n)| > \epsilon \sigma_n \sqrt{[N_n \pi_n(\Delta_n)]}]$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty \text{ (by assumption (iii)).} \quad \square$$

4. Some sufficient conditions for (1)

The assumption (1) of Theorem 1 relates the length of the observation N_n to the rate at which the n^{th} chain approaches (in Césaro mean) its stationary distribution. If P_n converges to a P that is nice, then it is reasonable to expect that this assumption holds.

Let $\boldsymbol{X} \equiv \{X_j\}$ be a Harris chain with state space $(S, \mathcal{S})$ and transition kernel $P(.,.)$. Let $\boldsymbol{X}_n \equiv \{X_{nj}\, ; j = 1, 2, \ldots, N_n, n = 1, 2, \ldots\}$ be a sequence of Harris chains all with state space $(S, \mathcal{S})$, and transition kernel $P_n(., .)$. As mentioned earlier, we assume without loss of generality that $\boldsymbol{X}_n, \boldsymbol{X}$ have recurrent singletons.

Lemma 4. *For* $\rho N_n > L$,

$$\delta_n(\rho, x) \leqslant 2 \frac{L}{\rho N_n} + \Delta(n, x, L)$$

where

$$\Delta(n, x, L) = \|P_n^{(L)}(x, .) - \pi_n(.)\|$$

and $\|.\|$ *is the total variation norm.*

Proof. By stationarity for π_n, we have for $s > L$

$$P_n^{(s)}(x,.) - \pi_n(.) = \int P_n^{(s-L)}(y, .)\, P_n^{(L)}(x, dy) - \int P_n^{(s-L)}(y, .)\, \pi_n(dy)$$

$$= \int (P_n^{(L)}(x, dy) - \pi_n(dy)) P_n^{(s-L)}(y, .).$$

which implis for $x > L$,

$$\|P_n^{(s)}(x, \Delta_n) - \pi_n(\Delta_n)\| \leqslant \|P_n^{(L)}(x, .) - \pi_n(.)\|.$$

Thus,

$$\delta_n(\rho, x) = \sup_{m > \rho N_n} \left\| \frac{1}{m} \sum_1^m P_n^{(s)}(x, .) - \pi_n(.) \right\|$$

$$\leqslant 2 \frac{L}{\rho N_n} + \Delta(n, x, L) \quad \square$$

This leads to the following sufficient condition for (1).

Condition I. $\lim_{L\to\infty}\limsup_{n\to\infty} E\Delta(n, X_{n0}, L) = 0.$

Now

$$\Delta(n, x, L)$$
$$\leqslant \|P_n^{(L)}(x, .)-P^{(L)}(x, .)\|+\|P^{(L)}(x, .)-\pi(.)\|+\|\pi(.)-\pi_n(.)\|$$

Thus, anothr set of sufficient conditions for (1) is

Condition II. (i) $\lim_{L\to\infty}\limsup_{n\to\infty} E\|P_n^{(L)}(X_{n0}, .)-P^{(L)}(X_{n0}, .)\| = 0,$

(ii) $\limsup_{L\to\infty} E\|P^{(L)}(X_{n0}, .)-\pi(.)\| = 0,$

(iii) $\lim_{n\to\infty}\|\pi(.)-\pi_n(.)\| = 0.$

Sufficient conditions for the above are given by

Condition III. (i_1) for all $x \in S, \|P_n(x, .)-P(x, .)\| \to 0$ as $n \to \infty$,

(ii_1) for all $x \in S, \|P^{(L)}(x, .)-\pi(.)\| \to 0$ as $L \to \infty$,

(iii_1) $\|\pi(.)-\pi_n(.)\| \to 0$ as $n \to \infty$,

(iv_1) there exist a σ finite measure μ on $(S, \mathcal{S})$ such that $P(X_{n0} \in dy) = f_n(y)\,\mu(dy)$, and there exist $F \in L_1(\mu)$ such that for all $n, |f_n(y)| \leqslant |f(y)|$ $a.e.\mu$.

To see this, we first note that (ii) is implied by (ii_1), (iv_1) and the Lebesgues dominated convergence theorem. Next we see that (i_1) implies

$$P_n^{(2)}(x, .)-P^{(2)}(x, .)$$
$$= \int P_n(y, .)\, P_n(x, dy)-\int P(y, .)\, P(x, dy)$$
$$= \int P_n(y, .)\,(P_n(x, dy))-P(x, dy))+\int (P_n(y, .))\, P(x, dy).$$

and so,

$$\|P_n^{(2)}(x, .)-P^{(2)}(x, .)\|$$
$$\leqslant \|P_n(x, .)-P(x, .)\|+\int\|P_n(y, .)-P(y), .)\|P(x, dy).$$

By bounded convergence theorem and (i_1), the rightside goes to 0 for each $x \in S$. Similary,

$$\|P_n^{(k+1)}(x, .)-P^{(k+1)}(x, .)\|$$
$$\leqslant \|P_n^{(k)}(x, .)-P^{(K)}(x, .)\|+\int P_n(y, .)-P(y, .)P^{(k)}(x, dy).$$

and by induction, we get that i_1 implies

$$\|P_n^{k)}(x, .)-P^{(k)}(x, .)\| \to 0 \text{ for all } x \in S \text{ and for all } k.$$

Now, (iv_1) and dominated convergence theorem yields (1).

Finally, a fourth set of sufficient conditions for (1) is

Condition IV. (i_2) All $\boldsymbol{X}_n$ and $\boldsymbol{X}$ have a common recurrent point Δ,

$$(ii_2)\quad \lim_{\rho\to 0}\ \limsup_{n\to\infty}\ \sup_{m\geqslant \rho N_n} E\left|\frac{1}{m}\sum_{1}^{m} P_n^{(s)}(X_{n0},\Delta)-\frac{1}{m}\sum_{1}^{m} P^{(s)}(X_{n0},\Delta)\right| = 0,$$

$$(iii_2)\quad \lim_{\rho\to 0}\ \limsup_{n\to\infty}\ \sup_{m\geqslant \rho N_n} E\left|\frac{1}{m}\sum_{1}^{m} P^{(s)}(X_{n0},\Delta)-\pi(\Delta)\right| = 0,$$

$$(iv_2)\quad |\pi_2(\Delta)-\pi(\Delta)|\to 0 \text{ as } n\to\infty.$$

Clearly, we can replace (ii_2) and (iii_2) by pointwise convergence (for each $X_{n_0} = x$) and then impose (iv) fo Condition III.

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