CONNECTEDNESS AND PATTERNS OF CONFOUNDING IN A
CLASS OF CYCLIC PBIB DESIGNS

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SUMMARY. This paper presents a necessary and sufficient condition for connectedness of a class of PBIB designs with higher associate cyclic association scheme which have inter-effect orthogonality. The problem of identifying the confounded contrasts in these designs, when they are disconnected, is also considered. A method based on association matrices is also given for constructing such symmetrical block designs with prespecified confounding patterns.

1. Introduction

Use of connected and disconnected incomplete block designs having inter effect orthogonality for arranging factorial experiment has been discussed by many authors (Cotter, John and Smith, 1973; Mukerjee, 1979, 1980; Chauhan and Dean, 1986). If a disconnected incomplete block design is used all the treatment contrasts are not estimable; because some of the contrasts are confounded with blocks. So, before arranging factorial experiment in a block design one needs to know whether the design is connected or not and what are the contrasts confounded.

Bose and Kishen (1940), Bose (1947a), White and Haltquist (1965) gave a single-replicate block design where all contrasts belonging to some of the factorial effects are confounded. John and Dean (1975), Dean and John (1975) studied confounding patterns in single-replicate generalized cyclic (GC/n) designs. Bailey (1977) made use of the annihilator concept (Halmos, 1958) for studying confounding patterns of single-replicate designs generated by DSIGN algorithm of Patterson (1976). Voss and Dean (1988) also utilized annihilators for construction of single-replicate GC/n designs with prespecified confounding patterns. Note that single-replicate incomplete block designs are disconnected,
which means these authors started with disconnected block designs and studied confounding patterns in them.

In case of multireplicate non-resolvable type incomplete block designs having inter-effect orthogonality, before studying confounding patterns, one needs to verify the connectedness of the designs. So, it would be better, if, for a class of block designs known to have inter-effect orthogonality, connectedness is studied first and then, if the designs are disconnected, contrasts confounded in them are identified.

It is known (Adhikary and Saha, 1990, 1991a) that the class of partially balanced incomplete block designs (PBIBD) having higher associate cyclic (HAC) association scheme (Adhikary, 1967) has inter-effect orthogonality and hence can be used widely for both symmetric and asymmetric factorial experimentation. In section 3 a necessary and sufficient condition for connectedness of the block designs of this class is considered. If some block designs are found to be disconnected, then utilizing the concept of annihilator the condition for identifying the confounded contrasts of these designs is dealt with in section 4. In section 5 the construction of disconnected designs of this class with prespecified confounding patterns is discussed.

Because of the suitability of PBIB designs having HAC association scheme for factorial experimentation and for brevity the class of PBIB designs having HAC association scheme hereafter will be referred to as class of cyclic factorial designs.

2. Notations

Let \((F_i)^{x_i}\) stands for \(x_i\)-th contrast of the main effect of the \(i\)-th factor \(F_i\), \(x_i = 1, 2, ..., s_i - 1\).

For \(x = (x_1, x_2, ..., x_n)\) let \((F)^x = (F_1)^{x_1} (F_2)^{x_2} ... (F_n)^{x_n}\) denotes a \(k\)-factor interaction contrast, which is interaction of \(x_{i1}\)-th contrast of main effect of \(F_{i1}\), \(x_{i2}\)-th contrast of main effect of \(F_{i2}\), ..., \(x_{ik}\)-th contrast of main effect of \(F_{ik}\), say, if \(x_i = x_{i1}, x_{i2}, ..., x_{ik}\) for \(i = i_1, i_2, ..., i_k\) and \(x_i = 0\) for \(i(\neq i_1, i_2, ..., i_k) = 1, 2, ..., n, \ x_{it} = 1, 2, ..., s_{it} - 1, \ for \ t = 1, 2, ..., k, \ G \equiv (a_1^{s_1} a_2^{s_2} ... a_n^{s_n}, j_i = 0, 1, 2, ..., s_i - 1, 1 \leq i \leq n; a_1^{s_1} = a_2^{s_2} = ... = a_n^{s_n} = 1)\) denotes a multiplicative abelian group of order \(v = \prod_{i=1}^{n} s_i\). The element \(a_1^{s_1} a_2^{s_2} ... a_n^{s_n} = a_1^{0} a_2^{0} ... a_n^{0} = 1\) denotes the unit element of group \(G\).

The symbol \((\times)\) denotes the direct product of the groups or sets of elements of \(G\).

3. Condition for Connectedness

A block design is said to be connected if, for any two treatments \(\alpha\) and \(\theta\), there exists a chain of treatments \(\alpha, \alpha_1, \alpha_2, ..., \alpha_t, \theta\) for some \(t \leq v - 2\), such that every consecutive pair of treatments in the chain appears together in a block.
Let $A_1, A_2, \ldots, A_m$ be any decomposition of non-unit elements of $G$ for an $m$-associate HAC association scheme, where $G$ is of order $v = \Pi_{i=1}^n s_i$. Associate one treatment with each element of $G$. Let $v, b, r, k, \lambda_1, \lambda_2, \ldots, \lambda_m$ be the parameters of a cyclic factorial design having $m$-associate HAC association scheme involving the elements of $G$. If elements of $G$ are arranged lexicographically then it may be noted that the first row of $NN'$ matrix of the design gives concurrences of the non-unit elements with the unit element ‘1’. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are all non-zero then all non-unit elements of $G$ occur with the element 1 in some blocks; and hence the design is connected. But if some of the $\lambda_i$’s are zero even then the design may be connected.

Let $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_p}$ be non-zero and $\lambda_{i_{p+1}}, \ldots, \lambda_{i_{p+q}}$ be zero, where $p + q = m$. That means, we can divide the non-unit elements of $G$ into sets :

$$A_1^* = \bigcup_{j=1}^p A_{i_j} \text{ and } A_2^* = G - 1 - A_1^* = \bigcup_{j'=1}^q A_{i_{p+j'}},$$

where each element of $A_1^*$ occurs at least once with the element 1 in some blocks, but elements of $A_2^*$ do not occur with the element 1 in any block. We shall prove the following theorem :

Theorem 3.1. A cyclic factorial design is connected if and only if each element of $A_2^*$ can be expressed as product or product of powers of the elements of $A_1^*$.

Proof. It is clear from the definition of HAC association scheme (see Adhikary and Saha, 1991a, b) that if two elements $\alpha, \beta \in G$ are $i$-th associate in HAC association scheme then the elements 1 and $\alpha \beta^{-1}$ are also mutually $i$-th associate; which means, if the elements $\alpha$ and $\beta$ occur together in $\lambda_i$ blocks in a cyclic factorial design then the elements 1 and $\alpha \beta^{-1}$ also occur together in $\lambda_i$ blocks.

If part. Without any loss of generality let $A_1^* = (\alpha_1, \alpha_2, \ldots, \alpha_y)$ for some $y \leq v - 1$ where $\alpha_i \in G$ for all $i = 1, 2, \ldots, y$. For some integers $t_1, t_2, \ldots, t_y$ let any element $\theta \in A_2^*$ can be expressed as $\theta = \alpha_1^{t_1} \alpha_2^{t_2} \ldots \alpha_y^{t_y}$ or $\alpha_1 \beta_1 = \theta$, where

$$\alpha_1^{t_1-1} \alpha_2^{t_2} \ldots \alpha_y^{t_y} = \beta_1 (\text{say}) \in G \text{ or, } \theta/\beta_1 = \alpha_1. \quad \ldots (3.1)$$

From the definition of HAC association scheme elements $h$ and $g \in G$ are $i$-th associate if $h/g \in A_i$. Let $\alpha_1 \in A_{i_j}$, $1 \leq j \leq p$. From (3.1) it is then clear that $\theta$ and $\beta_1$ are mutually $i_j$-th associate. Since $A_{i_j}$ is a subset of $A_1^*$, $\lambda_{i_j}$ is non-zero, we can say that the elements $\theta$ and $\beta_1$ occur jointly in $\lambda_{i_j} (> 0)$ blocks, i.e., these two elements occur together in some blocks. So $\theta$ and $\beta_1$ are connected.

Again $\beta_1/(\alpha_1^{t_1-1} \alpha_2^{t_2} \ldots \alpha_y^{t_y}) = \alpha_1 \in A_{i_j}$; which means, the elements $\alpha_1^{t_1-1}$ $\alpha_2^{t_2} \ldots \alpha_y^{t_y}$ and $\alpha_1^{t_1-2} \alpha_2^{t_2} \ldots \alpha_y^{t_y}$ are connected. Proceeding likewise it can be shown that the elements $\theta$ and $\alpha_y$ are connected. But since $\alpha_y \in A_1^*$, elements 1 and $\alpha_y$ are connected. So we can say that a treatment-block-treatment chain, for the treatments 1 and $\theta$, can be formed, i.e., elements 1 and $\theta$ are connected. This is true for all $\theta \in A_2^*$. Hence all elements of $A_2^*$ are connected with the
element 1. Hence the design is connected.

Only if part. Suppose the design is connected. If all $\lambda_i$'s are non-zero, i.e., if $A^*_2$ is an empty set then the condition is trivially true.

Suppose some of the $\lambda_i$'s are zero and division of the non-unit elements of $G$ into sets $A^*_1$ and $A^*_2$ can always be made. Since the design is connected, all the non-unit elements of $G$ are connected with the element 1, where elements of $A^*_1$ are directly connected (i.e., each of the elements of $A^*_1$ occurs with the element 1 in some blocks) and elements of $A^*_2$ are indirectly connected (i.e., a chain of directly connected elements can be formed for any element of $A^*_2$ and element 1, though they do not occur together in a block).

Let $\theta \in A^*_2$ and since the design is connected we can always form a treatment-block-treatment chain as follows:

$$1 = \alpha_0, \alpha_{i_1} \alpha_{i_2}, \ldots, \alpha_{i_d}, \beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_f}, \theta$$

where any two consecutive elements of the chain are connected, for elements $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_d} \in A^*_1$ and elements $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_f}, \theta \in A^*_2$. 

Note that any two consecutive elements of the chain are connected means, any two consecutive elements occur together at least in one block. Let

$$\alpha_{i_2}/\alpha_{i_1} = w_1.$$  

(3.3)

Now since $\alpha_{i_1}$ and $\alpha_{i_2}$ are connected, they jointly occur in some blocks, say in $\lambda(>0)$ blocks. Then from (3.1) element $w_1$ also jointly occurs with the element 1 in $\lambda$ blocks i.e., $w_1$ is also connected with 1. That means

$$w_1 \in A^*_1.$$  

(3.4)

Let since $\alpha_{i_3}/\alpha_{i_2} = w_2$, and it can be shown that $w_2 \in A^*_1$.

From (3.3) $\alpha_{i_3} = w_2 \alpha_{i_2} = w_1 w_2 \alpha_{i_1}$. Proceeding likewise it can be shown that

$$\alpha_{i_d} = w_1 w_2 \ldots \alpha_{i_1},$$  

(3.5)

where $w_i$'s belong to $A^*_1$.

Let $\beta_{j_1}/\alpha_{i_d} = y'_1$ and obviously $y'_1 \in A^*_1$, so $\beta_{j_1} = y'_1 \alpha_{i_d}$. 

(3.6)

Similarly $\beta_{j_2}/\beta_{j_1} = y'_2$ or, $\beta_{j_2} = y'_2 \beta_{j_1} = y'_1 y'_2 \alpha_{i_d}$ from (3.6). Proceeding likewise we shall get $\theta/\beta_{j_f} = y'_{f+1}$ or,

$$\theta = y'_{f+1} \beta_{j_f} = y'_1 y'_2 \ldots y'_f y'_{f+1} \alpha_{i_d}.$$  

(3.7)

where $y'_i$'s belong to $A^*_1$. From (3.5) and (3.7) $\theta$ can be expressed as

$$\theta = w_1 w_2 \ldots w_{d-1} y'_1 y'_2 \ldots y'_f y'_{f+1} \alpha_{i_1}.$$
Note that some of the $u_i$'s and $y_j$'s may be equal. So we can say that $\theta$ can be expressed as a product or product of powers of the elements of $A_2^*$. This is true for all $\theta \in A_2^*$.

**Example 3.1.** For a $3 \times 2 \times 4$ factorial experiment take $G \equiv (a^i b^j c^k, 0 \leq i \leq 2, 0 \leq j \leq 1, 0 \leq k \leq 3; a^0 = b^1 = c^1 = 1)$. Let $A_1 = (a, a^2) A_2 = (b, ab, a^2 b) A_3 = (c, ac, a^2 c, bc, abc, a^2 bc, c^3, a^2 c^3, bc^3, abc^3, a^2 bc^3) A_4 = (c^2, ac^2, a^2 c^2, bc^2, abc^2, a^2 bc^2)$ be any decomposition of the non-unit elements of $G$.

Developing the initial block $(1, a, a^2, b, ab, a^2 b, c^2)$ (i.e., multiplying by the elements of $G$) we get a four associate cyclic factorial design with the following parameters : $v = 3 \times 2 \times 4, b = 24, r = 7 = k, \lambda_1 = 6, \lambda_2 = 6, \lambda_3 = 0, \lambda_4 = 2$. For this design $A_2^* = A_3$. None of the elements of $A_2^*$ can be expressed as a product of powers of the elements of $A_2^* = G - 1 - A_2^*$. Hence the design is disconnected.

But the cyclic factorial design obtained from the initial block $(1, a, a^2, b, ab, a^2 b, c^3)$ having the parameters $v = 3 \times 2 \times 4, b = 24, r = 7 = k, \lambda_1 = \lambda_2 = 6, \lambda_3 = 1, \lambda_4 = 0$ is a connected design, since in this case elements of $A_2^* = A_4$ can be expressed as product of powers of the elements of $A_2^* = G - 1 - A_2^*$.

**Remark 3.1.** It is known from Adhikary and Saha (1991b) that the association schemes involved in $GC/n$ designs are HAC schemes. In fact $GC/n$ designs give a particular method of construction of PBIB designs having HAC association scheme. Dean and Lewis (1986) considered the problem of connectedness of $GC/n$ designs obtained from the initial blocks of the type $R(\times)S$ (note that following the multiplicative approach their initial block can be written as $R(\times)S$), where $S$ is a subgroup of $G$ and $R$ is a subset of $G$. Let $R^*$ be a group generated by the elements of $R$ and $R^{**} = R^*(\times)S$. Dean and Lewis (1986) have proved that $GC/n$ designs obtained from initial blocks of the type $R(\times)S$ are connected if and only if $R^{**} = G$. It can be shown that the condition $R^{**} = G$ can be justified from Theorem 3.1.

4. **Patterns of confounding**

Consider an $n$-factor factorial experiment arranged in a cyclic factorial design involving the elements of the abelian group $G \equiv ((a_1^{j_1} a_2^{j_2} ... a_n^{j_n}), 0 \leq j_i \leq s_i - 1; a_i^{s_i} = 1, 1 \leq i \leq n)$, where levels of the $i$-th factor $F_i$ are given by the elements of the group $G_i \equiv (a_i^{j_i}, 0 \leq j_i \leq s_i - 1; a_i^{s_i} = 1)$. $(a_1^{j_1} a_2^{j_2} ... a_n^{j_n})$ denotes any particular combination of levels of $n$ factors and these $v = \prod_{i=1}^{n} s_i$ level combinations are considered as treatments of the block designs considered here.

Let

\[ G_x \equiv (a^x \mid a^x = (a_1^{x_1} a_2^{x_2} ... a_n^{x_n}), x_i = 0, 1, 2, ..., s_i - 1; a_i^{s_i} = 1, i = 1, 2, ..., n, x \neq (0, 0, ..., 0)) \]
blocks is constructed, where Gilchrist and Patterson (1977) we know that if a single-replicate block design with

\[ G \]

are confounded with blocks. In this case one of the confounded contrasts is given

\[ y \]

particular subset, since \( M \) contains \( p \) distinct values, \( s \) appears \( v/p \) times.

\[ W \]

For any \( x, a^y \in G_x \) it can be verified that \( [y : x] \) takes \( p \) distinct values, \( p = 1, 2, ..., t \) and frequency of each of these values is \( v/p \) i.e., each of these distinct values appear equal number of times.

\[ D_n = \text{diag}(t/s_1, t/s_2, ..., t/s_n) \]

For any \( x, a^y \in G_y \) it can be verified that \( [y : x] \) takes \( p \) distinct values, \( p = 1, 2, ..., t \) and frequency of each of these values is \( v/p \) i.e., each of these distinct values appear equal number of times.

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For any \( x, a^y \in G_x \) it can be verified that \( [y : x] \) takes \( p \) distinct values, \( p = 1, 2, ..., t \) and frequency of each of these values is \( v/p \) i.e., each of these distinct values appear equal number of times.
With this given partition of elements of $G_y$ we get another $p - 2$ values of $x$, say $x_{(2)}; x_{(3)}; \ldots, x_{(p-1)}$ for which

$$M^0_{x(j)} = (a^y \in G_y : [y : x_{(j)}] = 0 \mod t).$$

for all $j, 1 \leq j \leq p - 1, \text{and for } i \neq 0 \text{ i.e., for } m_i \neq 0 \text{ for } i = 1, 2, \ldots, p - 1$

$$M^0_{x(j)} = (a^y \in G_y : [y : x_{(j)}] = m_j \mod t), 1 \leq j \leq p - 1.$$  

(That is, in the subset $M^0_{m_i}, m_i \neq 0$, if we put $x_{(j)}$ instead of $x_{(1)}$ we get $[y : x_{(j)}] = m_j \mod t$ instead of $[y : x_{(1)}] = m_i \mod t$.)

Now we are in a position to say that the $p - 1$ contrasts confounded with the blocks are given by $(F)^{x(1)}(F)^{x(2)}, \ldots, (F)^{x(p-1)}$. For the above example another confounded contrast is $(F_1)^2(F_2)$ for which $x = (2, 1, 0)$.

Consider a disconnected cyclic factorial design involving the elements of $G_y$ in $b$ blocks of size $k$ and replication $r$. Let an $n$-factor factorial experiment be arranged in this block design and $p - 1$ contrasts are confounded with blocks. Since in this design $p - 1$ contrasts are confounded, the $b$ blocks of this design can be divided into $p$ sets, say $B(0), B(1), \ldots, B(p-1)$, each containing $b/p$ blocks, such that the blocks in $B(i), i = 0, 1, 2, \ldots, p-1$ give a connected segment of the given disconnected cyclic factorial design. For each $i, i = 0, 1, \ldots, p - 1$, there will be $v/p$ distinct elements of $G_y$ each replicated $r$ times in the $b/p$ blocks of $B(i)$; and no two segments $B(i)$ and $B(j)$ will have any common element for $i \neq j, 0 \leq i, j \leq p - 1$, since the design is disconnected.

Let $T(i)$ be the set of distinct elements in the $b/p$ blocks of $B(i), i = 0, 1, \ldots, p - 1$. Without any loss of generality let $T(0)$ contains element $a^0 = 1$, i.e., $y = (0, 0, 0 \ldots, 0)$. It can be verified that there are $p - 1$ values of $x$, say $x_{(1)}, x_{(2)}, \ldots, x_{(p-1)}$, $a^{x(j)} \in G_x, j = 1, 2, \ldots, p - 1$, such that $T(0, x_{(j)}) = (a^y \in G_y : [y : x_{(j)}] = 0 \mod t, 1 \leq j \leq p - 1$ and $T(i, x_{(j)}) = (a^y \in G_y : [y : x_{(j)}] = m_i \mod t, 1 \leq i, j \leq p - 1$.

From the earlier discussion in this section it is now clear that $p - 1$ contrasts which are confounded in the disconnected cyclic factorial design are identified as $(F)^{x(1)}, (F)^{x(2)}, \ldots, (F)^{x(p-1)}$. Note that for all the $p-1$ values of $x : x_{(1)}, x_{(2)}, \ldots, x_{(p-1)} | y : x_{(j)}$ = 0 mod $t$ for all $y, a^y \in T(0)$, i.e., $[y : x_{(j)}] = 0 \mod t$ for all distinct elements of $B(0)$ and for all $x_{(j)}, 1 \leq j \leq p - 1$. But for the elements $a^y \in T(i), i = 1, 2, \ldots, p - 1$, $[y : x_{(j)}]$ takes different values for different $x_{(j)}, j = 1, 2, \ldots, p - 1$. From this we can say that the confounding patterns of any disconnected cyclic factorial design can be identified from the elements of $T(0)$.

It may be noted that blocks in $B(0)$ give a connected segment of the original disconnected design where unit element $a^0 = 1$ occurs. This means, all the non-unit elements of $T(0)$ are connected with element $a^0 = 1$. This can be checked.
from Theorem 3.1 that none of the elements of \( G_x - T(0) \) can be expressed as a product of powers of the elements of \( T(0) \). In other words, \( T(0) \) is a subgroup of \( G_x \). Let out of the \( v/p - 1 \) non-unit elements of \( T(0) \), \( u \) are directly connected with the element 1 and the remaining \( v/p - u - 1 \) non-unit elements of \( T(0) \) are indirectly connected with the element 1.

Let

\[
A_1^* = (\text{Set of all non-unit elements of } T(0) \text{ which are directly connected with element } 1),
\]

\[
A_1^{**} = (\text{Set of } v/p - u - 1 \text{ non-unit elements of } T(0) \text{ which are indirectly connected with element } 1).
\]

\( A_1^* = T(0) - A_1^* - 1 \). It is obvious that each of the elements of \( A_1^{**} \) can be expressed as product of powers of elements of \( A_1^* \). Let \( A = A_1^* \cup A_1^{**} = T(0) - 1 \). We shall prove the following lemma.

Lemma 4.1. For any \( x, a^x \in G_x \) if \([y : x] = 0 \mod t\) for all \( y, a^y \in A_1^* \) then \([y : x] = 0 \mod t\) for all \( y, a^y \in A \).

Proof. To prove the lemma we are to show that for all \( y, a^y \in A_1^{**} \) \([y : x] = 0 \mod t\). Let \( a^{y(1)}, a^{y(2)}, \ldots, a^{y(\alpha)} \) be the elements of \( A_1^* \). For some integers \( t_1, t_2, \ldots, t_\alpha \), \( a^{y} \in A_1^{**} \), where \( a^{y} = a^{t_1 y(1)} a^{t_2 y(2)} \ldots a^{t_\alpha y(\alpha)} \).

Since for any \( x, a^x \in G_x \), \([y(1) : x] = 0 \mod t\), \([y(2) : x] = 0 \mod t\), \ldots, \([y(\alpha) : x] = 0 \mod t\), immediately we get \([t_1 y(1) + t_2 y(2) + \ldots + t_\alpha y(\alpha) : x] = 0 \mod t\). This is true for all \( a^y \in A_1^{**} \). Hence the lemma.

Thus we find confounding patterns of a disconnected cyclic factorial design can be identified from the elements of \( A_1^* \); which means, the confounding patterns of a disconnected cyclic factorial design can be identified from the non-unit elements of the concerned abelian group which occur with the element 1 in some blocks, i.e., elements which are directly connected with element 1.

Thus the contrasts confounded in a disconnected cyclic factorial design are obtained from the bilinear form \([y : x] = 0 \mod t\) for all \( y, a^y \in A_1^* \), for \( x, a^x \in G_x \). In other words, set of contrasts confounded in a disconnected cyclic factorial design is given by:

\[
((F)^x \text{ confounded } | a^x \in G_x : [y : x] = 0 \mod t \text{ for all } y, a^y \in A_1^*)
\]

Let

\[
H = (a^x \in G_x : [y : x] = 0 \mod t \text{ for all } y, a^y \in A_1^*)
\]

\[
= (a^x \in G_x : [y : x] = 0 \mod t \text{ for all } y, a^y \in 1 \cup A_1^*)
\]

\[
= (a^x \in G_x : [y : x] = 0 \mod t \text{ for all } y, a^y \in 1 \cup A_1^* \cup A_1^{**}) \quad \ldots (4.1)
\]

\[
= (a^x \in G_x : [y : x] = 0 \mod t \text{ for all } y, a^y \in T(0)).
\]

If \( S \) is any subset of \( G_x \), the annihilator of \( S \), say \( S^A \), is the set of all \( x \) in \( 1 \cup G_x \) such that \([y : x] = 0 \mod t\) for all \( x \) in \( S^A \) (Halmos, 1958; Bailey, 1977). Clearly \( S^A \) is a subgroup of \( 1 \cup G_x \) and \((S^A)^A\) contains \( S \). Thus from (4.1) we find that \( H \) is annihilator of both \( A_1^* \) and \( T(0) \), and we have the following theorem.
Theorem 4.1. Confounding patterns of a disconnected cyclic factorial design are identified by the annihilator of $A_1^*$.

Remark 4.1. For any cyclic factorial design $A_1^*$ gives vital information, whether the design is connected or not, and if disconnected, what are the contrasts confounded.

Example 4.1. $G \equiv (a^j b^i, j_1 = 0, 1, 2, 3; j_2 = 0, 1, 2, 3, 4, 5; a^4 = 1 = b^6)$ $A_1 = (a, a^3), A_2 = (a^2), A_3 = (b, b^2, b^4, b^5), A_4 = (b^3), A_5 = (ab, ab^2), A_6 = (a^2 b, a^2 b^2, a^2 b^4, a^3 b^3, A_7 = (a^2 b^2, a^2 b^4, a^3 b^3), A_8 = (a^2 b^2)$

Following blocks give a non-initial block solution (A non-initial block solution means a solution which has been obtained not by developing initial blocks) of an 8-associate cyclic factorial design with parameters $v = 4 \times 6, b = 8, r = 2, k = 6, \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 1, \lambda_4 = \lambda_5 = 0, \lambda_6 = 0$, which is given by $(1, 2, 3, 4, 5, 6) \equiv (a, a^3, a^2 b, a^2 b^2, a^3 b, a^3 b^2, a^3 b^3, a^3 b^4, a^4 b, a^4 b^2, a^4 b^3, a^4 b^4, a^5 b, a^5 b^2, a^5 b^3, a^5 b^4)$, namely $G = 4\times 8$, $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 1, \lambda_4 = \lambda_5 = 0, \lambda_6 = 0$.

Clearly here $G = 4\times 8$, $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 1, \lambda_4 = \lambda_5 = 0, \lambda_6 = 0$.

For this design $A_1^* = (b, b^2, b^4, b^5, a^2 b, a^2 b^2, a^2 b^4, a^3 b^3, a^4 b^3)$ and $A_2^* = (b^3, a, ab, ab^2, ab^3, ab^5, a^2 b^3, a^2 b^4, a^3 b^3, a^3 b^4, a^3 b^5)$. Only two elements of $A_2^*$, namely $b^3$ and $a^2 b^3$, can be expressed as a product of two elements of $A_1^*$, and other elements of $A_2^*$ cannot be expressed as a product or product of powers of the element of $A_1^*$. Hence the design is disconnected.

Example 4.2. $G \equiv (a^i b^j c^k, 0 \leq i, j, k \leq 2, 0 \leq k \leq 3; a^3 = b^3 = c^4 = 1)$. $A_1 = (a, a^2, b, b^2), A_2 = (c, c^2, c^3, ab, a^2 b^2, abc^3, a^2 b^2 c), A_3 = (abc, a^2 b^2 c^3), A_4 = (ab c^2, a^2 b^2 c^2). A_5 = G - 1 - \cup_{i=1}^4 A_i$.

Consider the cyclic factorial design obtained by cyclical development of the initial block $(1, abc, a^2 b^2 c^2)$ with parameters $v = 3 \times 3 \times 4, b = 36, r = 3 = k, \lambda_1 = \lambda_2 = \lambda_5 = 0, \lambda_3 = 2, \lambda_4 = 1$. For this design $A_1^* = (a^i b^j c^k, a^2 b^2 c^2), A_2^* = G - 1 - A_1^*$.

Only seven elements of $A_2^*$, namely $c, c^2, a^2 b^2, c^3, abc^3, a^2 b^2 c, ab$ can be expressed as product of powers of the elements of $A_1^*$. Hence from Theorem 3.1 the design is disconnected.

Clearly $A_1^* = (c, c^2, c^3, ab, a^2 b^2, abc^3, a^2 b^2 c)$ and so $T(0) = 1 \cup A_1^* \cup A_1^*$. Confounding patterns of this disconnected design are characterized by the annihilator of $A_1^*$, which is given by $(1, a^2 b, ab^2)$ i.e., $x = (000, 210, 120)$. So, two contrastts, namely $(F_1)^2(F_2)$ and $(F_1)(F_2)^2$ are confounded.
5. Cyclic factorial designs with prespecified patterns of confounding

The following theorem can be proved by utilizing similar analogy as used in the proof of Theorem 4.1 of Bailey (1977).

Theorem 5.1. For a disconnected cyclic factorial design \( H^A = T(0) \), where \( H^A \) is annihilator of \( H \), which is a sub-group of \( G \).

Theorem 5.1 will be useful in constructing cyclic factorial designs with prespecified patterns of confounding. Let \( H^* \) be the set of contrasts to be confounded so that \( H = 1 \cup H^* \) is a subgroup of \( G \). First we are to find annihilator of \( H \) i.e., \( H^A = T(0) \), say, which is a subgroup of \( G \). In the literature different methods are available for construction of annihilators (Collings, 1984; Mossadeq, et al., 1985; Voss and Dean, 1988). Now the question is to construct cyclic factorial design for given \( T(0) \), where non-unit elements of \( T(0) \) occurs with the element 1 in some blocks and the elements of \( G - T(0) \) do not occur with the element 1 in any block.

Let \( T(0) = 1 \cup A_1^* \cup A_2^* \), where elements of \( A_1^* \) can be expressed as a product of powers of the elements of \( A_1^* \).

Developing the initial block \( T(0) \) cyclically and discarding the duplicate blocks a single-replicate cyclic factorial design can be obtained with prespecified confounding patterns \( H^* \). Voss and Dean (1988) have utilized this method for construction of single-replicate \( GC/n \) designs with prespecified confounding patterns.

In a HAC with \( m \)-associate classes the \( v - 1 \) non-unit elements of \( G \) are divided into \( m \) subsets satisfying some conditions. Let \( A_1, A_2, ..., A_m \) be any decomposition of non-unit elements of \( G \) satisfying these conditions.

Let \( A_1^* = A_{i_1} \cup A_{i_2} \cup .... \cup A_{i_p} \) and \( A_1^{*^*} = A_{j_1} \cup A_{j_2} \cup .... \cup A_{j_q} \) for some \( (i_1, i_2, ..., i_p) \subset (1, 2, ..., m) \) and \( (j_1, j_2, ..., j_q) \subset (1, 2, ..., m) \), \( (i_1, i_2, ..., i_p) \) and \( (j_1, j_2, ..., j_q) \) do not have any common element; and \( n_1, n_2, ..., n_m \), \( p_{i_k} \) be the parameters of the HAC scheme.

Let \( B_0 = I_v, B_1, B_2, ..., B_m \) be the association matrices of the \( m \)-associate HAC system. Following theorem gives cyclic factorial designs with prespecified patterns of confounding.

Theorem 5.2. \( N = \delta B_0 + B_{i_1} + B_{i_2} + .... + B_{i_p} \) gives an incidence matrix of an \( m \)-associate cyclic factorial design with parameters \( v = s_1 \times s_2 \times .... \times s_m, b = v, r = \sum_{i_d \in (i_1, i_2, ..., i_p)} n_{i_d} + \delta = k, \lambda_0 = r, \)
with confounding patterns $H^*$ where $\delta = 0$ or $1$.

Example 5.1. $G = (a^ib^j, 0 \leq i \leq 3, 0 \leq j \leq 4; a_4^4 = 1 = b^5)$

$$A_1 = (a^2), A_2 = (b, b^4, a_2^2 b^4), A_3 = (b^2, b^3, a_2^2 b^2, a_2^3 b^3),$$
$$A_4 = (ab, ab^4, a^3 b^4, ab^2, a_3^3 b^2, a_3^3 b^3, a^3).$$

We want to construct a 4-associate cyclic factorial design involving the elements of $G$ where one contrast of main effect of $F_1$, namely $(F_1)^2$, will be confounded. That means, in this case $H^* = (a_2 b^0)$ i.e., $H = (a_0 b^0, a_2 b^0) = (1, a^2)$.

Note that the annihilator subgroup of $(1, a^2)$ is given by

$$(1, a^2, b, b^4, a_2^2 b^4, b^2, b^3, a_2^2 b^2, a_2^3 b^3) = T(0).$$

Take $A_1^* = (a_2^2, b, b^4, a_2^2 b^4) = A_1 U A_2, A_1^{**} = (b^2, b^3, a_2^3 b^2, a_2^3 b^3) = A_3$ so that $T(0) = 1 \cup A_1^* \cup A_1^{**}$.

It can be verified that the elements of $A_1^{**}$ can be expressed as product of powers of the elements of $A_1^*$. The above decomposition of non-unit elements of $G$ give a 4-associate HAC scheme with parameters $v = 4 \times 5$, $n_1 = 1$, $n_2 = 4 = n_3$, $n_4 = 10$,

$$
\begin{pmatrix}
(p_{jk}^1) = \\
0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 10
\end{pmatrix}
\begin{pmatrix}
(p_{jk}^2) = \\
0 & 1 & 0 & 0 \\
1 & 0 & 2 & 0 \\
0 & 2 & 0 & 2 \\
0 & 0 & 0 & 10
\end{pmatrix}
=$$

$$
\begin{pmatrix}
(p_{jk}^3) = \\
0 & 0 & 1 & 0 \\
0 & 2 & 2 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 0 & 10
\end{pmatrix}
\begin{pmatrix}
(p_{jk}^4) = \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 4 \\
0 & 0 & 0 & 4 \\
1 & 4 & 4 & 0
\end{pmatrix}
$$

From Theorem 5.2, $N = B_1 + B_2$ (when $\delta = 0$) gives an incidence matrix of cyclic factorial design with parameters $v = 4 \times 5$, $b = 20$, $r = 5$, $k = 5$, $\lambda_1 = p_{11}^1 + p_{22}^2 + 2p_{12}^1 = 4$, $\lambda_2 = p_{11}^2 + p_{22}^2 + 2p_{12}^2 = 2$, $\lambda_3 = p_{11}^3 + p_{22}^2 + 2p_{12}^3 = 2$ and $\lambda_4 = 0$, with the confounding patterns $H^*$ i.e., where contrast $(F_1)^2$ is confounded.

Again from Theorem 5.2 (when $\delta = 1$) $N = B_0 + B_1 + B_2$ gives an incidence matrix of cyclic factorial design with parameters $v = 4 \times 5$, $b = 20$, $r = 6$, $k = 6$, $\lambda_1 = p_{11}^1 + p_{22}^2 + 2(1 + p_{12}^1) = 6$, $\lambda_2 = p_{11}^2 + p_{22}^2 + 2(1 + p_{22}^2) = 4$, $\lambda_3 = p_{11}^3 + p_{22}^3 + 2p_{12}^3 = 2$ and $\lambda_4 = 0$, with the confounding patterns $H^*$. 
Example 5.2. Suppose for $2 \times 3 \times 4$ factorial experiment we want to construct a cyclic factorial design where three contrasts, namely: the only contrast of main effect of $F_1$ (namely, $(F_1)$), one contrast of main effect of $F_3$ (namely, ($(F_3)^2$)), one contrast of interaction effect $F_1F_3$ (namely, $(F_1)(F_3)^2$) are confounded. That means, $H^* = (ab^0c^2, ab^0c^2, ab^0c^2)$ i.e., $H = 1 UH^* = (1, a, c^2, ac^2)$, where $G = (a^2 = b^3 = c^4 = 1)$, two subsets are $A_1 = (b, b^2, c^2, bc^2, b^2c^2)$ and $A_2 = G - 1 - A_1$.

These two subsets give a two associate HAC scheme with parameters $v = 2 \times 3 \times 4$, $n_1 = 5$, $n_2 = 18$, $T(0) = \left(\begin{array}{c}4 & 0 \\ 0 & 18\end{array}\right)$.

It can be verified that the annihilator of $H$ is given by

$$T(0) = (1, b, b^2, c^2, bc^2, b^2c^2).$$

Take $A_1^* = (b, b^2, c^2, bc^2, b^2c^2) = A_1$ and $A_1^{**} = (\phi)$, empty subset; so that $T(0) = 1 \cup A_1^* \cup A_1^{**}$. From Theorem 5.2. $N = B_1$ gives an incidence matrix of a two associate cyclic factorial design with parameters $v = 2 \times 3 \times 4$, $b = 24$, $r = 5 = k$, $\lambda_1 = p_{11}^1 = b = 24$, $r = 6 = k$, $\lambda_1 = p_{11}^1 = 2 = 6, \lambda_2 = p_{11}^2 = 0$, where the confounding patterns are as prespecified by $H^*$.

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References


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