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NON-CONGLOMERABILITY FOR FINITE-VALUED, FINITELY ADDITIVE PROBABILITY

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SUMMARY. We consider how an unconditional, finite-valued, finitely additive probability P on a countable set may localize its non-conglomerability (non-disintegrability). Non-conglomerability, a characteristic of merely finitely additive probability, occurs when the unconditional probability of an event $P(E)$ lies outside the closed interval of conditional probability values, $[\inf_{h\in\pi} P(E|h), \sup_{h\in\pi} P(E|h)]$, taken from a countable partition $\pi = \{h_j : j = 1, \ldots\}$. The problem we address is how to identify events and partitions where a finite-valued, finitely additive probability fails to satisfy conglomerability. We focus on the extreme case of 2-valued finitely additive probabilities that are not countably additive. These are, equivalently, non-principal ultrafilters. Evidently, the challenge we face is that given a countable partition, at most one of its elements has positive probability under P. Thus, we must find ways of regulating the coherent conditional probabilities, given null events, that cohere with the unconditional probability P. Our analysis of P proceeds by the use of combinatorial properties of the associated non-principal ultrafilter U_P. We show that when ultrafilter U_P is not minimal in the Rudin-Keisler partial order of $\beta(\omega)\backslash\omega$, we may locate a partition in which P fails to satisfy the conglomerability principle by examining (at most) countably many partitions. This result is then applied to finitely additive probabilities that assume only finitely many values. By contrast, if ultrafilter U_P is Rudin-Keisler minimal, then P is simultaneously conglomerable in each finite collection of partitions, though not simultaneously conglomerable in all partitions.

1. Introduction to Finitely Additive [f.a.] Probability

Let $\mathcal{F}$ be a σ-field of sets, of subsets of Ω. Kolmogorov's (1956) axiomatization of probability requires that $\forall(A, B) \in \mathcal{F}$:

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(1) $0 \le P(A) \le 1$.

(2) $P(\Omega) = 1$.

(3) If $A \cap B = \phi$, then $P(A) + P(B) = P(A \cup B)$.

A probability satisfying axioms (1)-(3) is said to be finitely additive [f.a.]. Last, consider a fourth axiom, σ-additivity (taken by Kolmogorov as an "expedient")

(4) If $(A_i \cap A_j) = \phi$ whenever $i \neq j$, then $P(\cup_i A_i) = \Sigma_i P(A_i)$.

What is distinctive about a f.a. probability that is not σ-additive, a *merely* finitely additive probability? The following illustrates a hallmark of merely finitely additive probability.

EXAMPLE 1 (deFinetti, 1930 and attributed to Levy by Cantelli, 1935). Consider a f.a. probability P on the set of all pairs $< s, t >$, for s and t positive integers, with the following two restrictions:

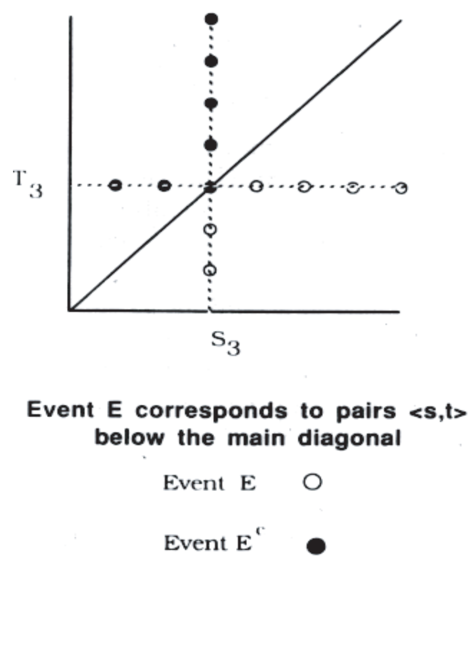
$$P(< s, t >) = 0, \text{ that is, } P \text{ is 0 on finite sets}$$

and

$$P(< s, t >| B) = 0 \text{ if } B \text{ is an infinite set.}$$

Define the events: $E = \{< s, t >: s > t]; S_m = \{< s, t >: s = m\}\ (m = 1, \ldots)$ and $T_n = \{< s, t >: t = n\}(n = 1, \ldots)$.

Diagram for deFinetti - Levy Example



Then

$$P(E \mid S_m) = 0 \text{ for } m = 1, \ldots$$

yet

$$P(E \mid T_n) = 1 \text{ for } n = 1, \ldots$$

Let π_s be the partition by vertical strips: $\pi_s = \{S_m : m = 1, \ldots\}$ and let π_t be the partition by horizontal strips: $\pi_t = \{T_n : n = 1, \ldots\}$. Thus, we see that the following principle (evidently valid for each countably additive probability) is invalid for each f.a. probability that satisfies the two constraints of Example 1, as such a probability must violate the principle in at least one of the two partitions π_s and π_t. Let $\pi = \{h_n : n = 1, \ldots\}$ be an exhaustive partition.

Principle of $(\omega-)$ *conglomerability for events* (deFinetti, 1972, p.99)
$\forall(A \in \mathcal{F})$ If $c_1 \leq P(A \mid h_n) \leq c_2$ $(n = 1, \ldots)$, then $c_1 \leq P(A) \leq c_2$.

Dubins (1975) strengthens the conglomerability principle to apply to all bounded random variables, rather than applying it solely to the indicator functions for events. He establishes that P is conglomerable (in the stronger sense) in a partition π iff P is disintegrable in π, which concept we review next.

For a f.a. probability P, partition $\pi = \{h_j : j = 1, \ldots\}$, and bounded random variable X, let $E_p[X]$ and $E_p[X \mid h]$ denote the P-expectation of X and conditional P-expectation of X given h, respectively. (We discuss coherence of conditional probability, below.)

DEFINITION. P is *disintegrable* in the partition π provided that, for each bounded random variable X,

$$E_p[X] = \int_{h \in \pi} E_P[X \mid h] \, dP(h).$$

In this paper, whenever we show that a f.a. probability P is conglomerable in a partition π, we do so for the strong (Dubins') sense of the principle and when we show that P fails to satisfy conglomerability in π, we give the failure with respect to an event, i.e., then we show that (deFinetti's) weak conglomerability principle fails.

Providing that preference is according to subjective expected utility, when conglomerability fails in a partition π, then such basic decision theoretic principles as simple dominance (or "admissibility") fail in π as well. Therefore, it is important to understand not only whether but where, i.e., in which partitions does a particular merely finitely additive probability fail to satisfy the principle of conglomerable.

In fact, (weak) conglomerability characterizes countable additivity. That is, non-conglomerability of merely finitely additive probability is a hallmark, as the following result reports.

THEOREM 1 (Schervish *et al.,* 1984). *Each merely f.a. probability fails conglomerability for some event in some denumerable partition.*

This result quantifies over all denumerable partitions and over all events. These are large sets, e.g., of cardinality of the continuum when the underlying set is countable. (Of course, if the underlying set is finite, there is no issue to discuss.) To locate where non-conglomerability occurs, according either to our proof of Theorem 1, or Zame's (1988) simplified proof, depends for some P upon how the *conditional* probabilities $P(\bullet \mid \bullet)$ are defined, separate from the *unconditional* probability, $P(\bullet)$.

For an illustration of this issue, reconsider Example 1. There are two constraints on P that lead to non-conglomerability in (at least) one of the two partitions, π_s and π_t, for the event E. The first is a constraint on the *unconditional* probability $P(\bullet)$: that P is 0 on finite sets. The second is a constraint on the *conditional* probabilities $P(\bullet \mid \bullet)$: that $P(< s, t >\mid B) = 0$ if B is an infinite set. The first constraint insures that P is a purely finitely additive probability on ω. However, if **Q**= {Q: Q is a finitely additive probability satisfying the first constraint} then only for a proper subset of **Q** does the unconditional probability Q specify even the two families of conditional probabilities, given π_s and given π_t. Only for some $Q \in \mathbf{Q}$ does the unconditional probability entail the two sets of conditional probabilities used in Example 1.

In this paper, we investigate the following issue:

Thematic question. Given a merely f.a. probability P, is it determined where conglomerability fails based solely on the *unconditional* probability values $P(\bullet)$? That is, given P as an *unconditional* finitely additive probability, can event A and partition π be found where P is not conglomerable in π for event A, i.e., where P is not π-conglomerable?

By contrast with the situation in Example 1, a positive answer to the question is available (Schervish *et al.*, 1984, p 210) whenever the range of P is an infinite set, i.e., whenever $P(\bullet)$ assumes infinitely many values. The following (see Dubins, 1975) illustrates what happens.

EXAMPLE 2. Let E_1 be the event of flipping a "fair" coin until heads shows. Let E_2 be the event of picking a positive integer "at random," according to some (purely) f.a. probability that assigns each integer 0 probability. (There are very many such purely f.a. probabilities, indeed!) Assume $P(E_1) = P(E_2) = .5$; for example, which of E_1 or E_2 occurs may be determined by an extraneous flip of a "fair" coin. Let x_1 be the random variable of the number of flips in case E_1 obtains, 0 otherwise, and let x_2 be the integer chosen at random in case E_2 obtains, 0 otherwise. Then $P(\{x_2 = n\}) = 0$, for $n = 1, \ldots$ and $P(\{x_1 = n\}) = 2^{-(n+1)}$. Let k denote the random variable of the positive integer that (with P-probability 1) results. So $P(k = n) = P(\{x_1 = n\}) = 2^{-(n+1)}$. Thus, $P(E_1 \mid k) = 1$ $(k = 1, 2, \ldots)$, and conglomerability fails in the partition $\pi = \{h_k : k = 1, \ldots\}$. Note that here each conditioning event, each partition element h_k, has positive (unconditional) probability: $P(h_k) = 2^{-(k+1)} > 0$. Here the conditional probabilities $P(\bullet \mid k)$ are fixed by the unconditional probabilities and non-conglomerability in π is fixed by the unconditional probability P. P cannot be made conglomerable in π.

When the (unconditional) probability P assumes infinitely many distinct values, the unconditional probabilities for events identify an event E and countable partition, $\pi = \{h_j : j = 1, \ldots\}$, where each h_j has positive P-probability, and where $P(E \mid h_j) > P(E) + \epsilon$, for $\epsilon > 0$ and $j = 1, \ldots$ That is, when the range of P is an infinite set, there exists a partition where P cannot be made conglomerable.

The remaining case, thus, is where a merely f.a. probability P assumes only finitely many values. A special sub-case that we consider in the next section is for a two-valued merely f.a. (unconditional) probability P: $\forall \in \mathbf{F}$ $P(A) = 0$ or $P(A) = 1$. This is the difficult case because then the conditional probabilities $P(\bullet \mid \bullet)$ are determined by the unconditional probability $P(\bullet)$ only up to conditioning sets of measure 1. With a two-valued probability, these form only a sparse collection. Given a partition $\pi = \{h_i : i = 1, \ldots\}$, either one or none of its elements has positive P-probability. To compensate, we use the following principle in dealing with conditional probabilities $P(\bullet \mid \bullet)$, especially for conditional probability given events of (unconditional) probability 0.

Principle of conditional coherence: For all pairs of events, A and B such that $A \cap B \neq \emptyset$, $Q(\bullet) = P(\bullet \mid B)$ is a finitely additive probability with $Q(B) = 1$, and $Q(\bullet \mid A) = P(\bullet \mid A \cap B)$.

When $P(A \cap B) > 0$, the principle applies, trivially. The principle of conditional coherence helps to formalize deFinetti's (1972) concern with conditional probability given *an event*, rather than given *a field*, in that $P(\bullet \mid B)$ does not depend upon how we partition the contraries to event B.

Dubins (1975, §3, Corollary 1) reproves an important result of P. Krauss (1968): for each finitely additive (unconditional) probability P, a full set of conditional probabilities may be define that satisfy the principle of conditional coherence. Moreover, Dubins (1975, Theorem 5) establishes that when P is disintegrable in a partition π, then there is a full set of coherent conditional probabilities extending the set of conditional probabilities $\{P(\bullet \mid h) : h \in \pi\}$. Throughout this paper we adopt the principle of conditional coherence, relying on Dubins' Theorem 5 to show that whenever P is conglomerable in π, then the set of conditional probabilities $\{P(\bullet \mid h) : h \in \pi\}$ that make it conglomerable are also coherent conditional probabilities.

REMARK. Though the merely f.a. probability P is two-valued, given B with $P(B) = 0$, a coherent conditional probability $P(\bullet \mid B)$ may have an infinite range and may be countably additive.

In section 2 we establish conditions when a two-valued merely f.a. probability P on ω cannot be made conglomerable in a specific partition, based on combinatoric properties of its associated non-principal ultrafilter U_P. We show that for each non-Ramsey ultrafilter U_P, either P cannot be made conglomerable in a specific partition (based on U_P's combinatorics), or else a countable sequence of partitions suffice to locate a partition where P does not satisfy the (weak) principle of conglomerability. We refer the reader to Comfort and Negrepontis (1974), especially chapters 9 and 16 for basic facts about ultrafilters and the Rudin-Keisler ordering of $\beta(\omega)\backslash\omega$.

2. Ultrafilters on ω

We investigate our thematic question first for two-valued, merely f.a. unconditional probabilities by considering combinatorial properties of their associated non-principal ultrafilters.

DEFINITIONS. An *ultrafilter* U (on ω) is a non-empty family of non-empty subsets (of ω), such that if $A, B \in U$, then $(A \cap B) \in U$; if $A \in U$ and $C \supset A$, then $C \in U$: and $\forall A(\subset \omega)A \in U$ or $A^c \in U$.

A *principal* ultrafilter is one generated by an element $i \in \omega$, i.e., the set of all subsets of ω that contain i.

A *non-principal* ultrafilter contains no finite subsets.

Evidently, each 2-valued, unconditional f.a. probability P on the powerset of ω, with $P(n) = 0$ $(n = 1, \ldots)$, corresponds uniquely to a non-principal ultrafilter U_P on ω, as determined solely by its unconditional probability values.

Fact (ZFC). There are $2^{2^{\omega}}$ non-principal ultrafilters on ω. (See Comfort-Negrepontis, 1974, p. 146.) Hereafter, we focus on non-principal ultrafilters in our discussion of ultrafilters on ω.

2.1 *Some elementary combinatorics.* Consider a function $f : \omega \to \omega$. The function f induces a denumerable (finite or countable) partition $\pi_f = \{h_n^f : n = 1, \ldots\}$ of ω by considering the inverse images, $f^{-1}(n)$.

DEFINITION. Call an ultrafilter U *selective* in a partition $\pi = \{h_1, h_2, \ldots\}$ if there is an $A \in U$ so that:

$$\exists h \in \pi \text{ with either } A \subseteq h \text{ or } \mid A \cap h_n \mid \leq 1 \quad (n = 1, \ldots).$$

DEFINITION. When an ultrafilter U_P is selective in a partition π, we call the following (one version of) its *natural conditional probability given* π:

In the first case, when $P(h) = 1$ then $P(\bullet \mid h) = P(\bullet)$ and for $h_n \neq h$, let $P(\bullet \mid h_n)$ be an arbitrary f.a. probability defined on h_n.

In the second case, let $P(\bullet \mid h_n)$ be concentrated, with probability 1, at the singleton $(A \cap h_n) = \{a_n\}$, if it exists. Otherwise, let $P(\{a_n\} \mid h_n) = 1$ for an arbitrary $a_n \in h_n$. Thus $P(\bullet \mid h_n)$ is a 0-1 principal ultrafilter probability.

If U_P is selective in partition π then there are many different versions of its natural conditional probability, depending upon which $A \in U_P$ is chosen. We note that each two versions differ only on a set of elements of π that lie outside the ultrafilter U_P, hence; each two versions differ for a set of conditioning events of P-measure 0.

LEMMA 1. *When U_P is selective in a partition π, it is conglomerable there using (any version of) its natural conditional probability for P given π.*

PROOF. In the first case, it is immediate that P is conglomerable in π from Dubins' equivalence with disintegrability and the obvious equality, $E_P[X \mid h] = E_P[X]$.

In the second case, let X be a bounded random variable with $E_P[X] = c$. Then, as P is an ultrafilter probability, for each $\epsilon > 0$, $P(\{a \in \omega : |X(a)| \leq c + \epsilon\}) = 1$. Each natural conditional probability satisfies $E_P[X \mid h_n] = E_p[X(a_n) \mid h_n] = X(a_n)$ (for $n = 1, \ldots$). However, $\cup\{a_n\} = A \in U_P$. Then $P(\cup h_n : |E_p[X|h_n]| \leq c + \epsilon) = 1$ and P is disintegrable in π; hence, P is conglomerable in π using (any version of) it natural conditional probability. □

DEFINITION. A non-principal ultrafilter U on ω is said to be a *Ramsey* ultrafilter if it satisfies Ramsey's partition theorem for a set in U. That is, for each integer n and binary partition $\{h_0, h_1\}$ of $[\omega]^n$, there is an $A \in U$ such that either $[A]^n \subset h_0$ or $[A]^n \subset h_1$.

THEOREM 2 (attributed to Kunen). *A non-principal ultrafilter on ω is Ramsey* **iff** *it is selective in each partition π.* (See Comfort-Negrepontis, 1974, p. 212.)

REMARK. In (ZFC + CH) there are 2^{2^ω} Ramsey ultrafilters on ω. (Comfort-Negrepontis, 1974, p. 220)

THEOREM 3. *If U_P is a Ramsey ultrafilter, P can be made conglomerable (simultaneously) in any finite number of partitions, $\pi_i (i = 1, \ldots, k)$, by using natural conditional probabilities.*

PROOF. Given $\pi_i = \{h^i_n : n = 1, \ldots\} (i = 1, \ldots, k)$, let $A_i \in U_P$ satisfy either $\exists h^i \in \pi_i$ with $A_i \subseteq h^i$ or $\mid A_i \cap h^i_n \mid \leq 1$ $(n = 1, \ldots)$. Let $\cap_i A_i = A \in U_P$. Then, corresponding to the respective case, given π_i for $i = 1, \ldots, k$: either $A \subseteq h^i$ or $\mid A \cap h^i_n \mid \leq 1 \,(n = 1, \ldots)$. Using A, consider a (version of the) natural conditional probability for P given $\pi_i (i = 1, \ldots, k)$. By Lemma 1, P is conglomerable (simultaneously) in each of the π_i. □

Hence, when Ramsey ultrafilters exist, they provide a negative answer to our thematic question. That is, given any specific partition π, each Ramsey ultrafilter probability has its associated (natural) conditional probability that makes it conglomerable in π. Next, we show that for non-Ramsey ultrafilters, their combinatorial properties pinpoint some of their non-conglomerability.

DEFINITION. An ultrafilter U is *weakly selective* in a partition π if there exists an $A \in U$ so that: either $\exists h_* \in \pi$ with $A \subseteq h_*$ or $\mid A \cap h_n \mid < \omega$ $(n = 1, \ldots)$.

REMARK. An ultrafilter U is a P-point **iff** it is weakly selective in each π. (Blass, 1973) Trivially, each Ramsey ultrafilter is a P-point.

COROLLARY 1. *If U_P is a P-point, but not Ramsey, then P fails to be conglomerable in each partition where it lacks the Ramsey (selective) property.*

PROOF. The claim is immediate from the following lemma.

LEMMA 2. *If U_P is weakly selective but not selective in partition π then U_P is not conglomerable in π, nor is it even approximately conglomerable in π (and the extent of non-conglomerability is 1).*

PROOF. (based on an argument by Dubins (1975, pp. 92-93). Let π be a partition where U_P is weakly selective but not selective. Thus, no element (h_n) of π belongs to U_P an and no graph of a function $g : \omega \to \omega$ belongs to U_P either. But, as U_P is weakly selective in π, there is a function $f : \omega \to \omega$ whose graph in π bounds U_P from above. That is, let $S_n = \{x : x \in h_n \text{ and } x \leq f(n)\}$ and define $S = \cup S_n$. Then $S \in U_P$.

Let Q be a finitely additive probability that is disintegrable in π so that, for event A, $Q(A) = \int_{h \in \pi} Q(A \mid h) dQ(h)$. Then we see that P and Q are singular, as follows. Fix $\epsilon > 0$ and let m satisfy $m\epsilon \geq 1$. Consider the finite partition of S according to the $1/m$ quantiles of the conditional distributions $Q(S \mid h)(h \in \pi)$. That is, given $h \in \pi$ and integer k $(1 \leq k \leq m)$ the $k/m - th$ $Q(\bullet \mid h)$-quantile point is the least element of h, x^k, such that $Q(\{x \leq x^k\} \mid h) \geq k/m$. By the reasoning above, for each k, the graph of the k/m-th quantile points does not belong to U_P. Hence, one of the (at most) m regions strictly between these (at most) m-graphs, call it R_k, belongs to U_P. But $Q(R_k) \leq \epsilon$ yet $P(R_k) = 1$. Hence, P is not at all approximable by f.a. probabilities Q that are conglomerable (in Dubins' sense) in π. That is, failure of conglomerability in π is maximal. For each coherent conditional probability $P(\bullet \mid \bullet)$, for each $\epsilon > 0$, there exists A with $P(A) = 1$, but $P(A \mid h) < \epsilon (\forall h \in \pi)$. □

Thus, for non-Ramsey P-points, we can identify a partition where conglomerability of events fails maximally.

3. Non-Ramsey Ultrafilters and Non-conglomerability

Next we explore consequences of Lemma 2 for the Rudin-Keisler partial order of ultrafilters. The Rudin-Keisler partial order of ultrafilters, $\preceq$, is defined as follows:

Let $f : \omega \to \omega$ and ultrafilter U be given. Define ultrafilter $V = f(U)$ by, $X \in V$ if $f^{-1}(X) \in U$. Then say that $V \preceq U$ if there is some $g : \omega \to \omega$ with $g(U) = V$.

The Rudin-Keisler partial order $\preceq$ is reflexive and transitive. Denote by $U \approx V$ the equivalence relation ($V \preceq U$ and $U \preceq V$) and denote by $V < U$ the strict partial order ($V \preceq U$ and $V \not\approx U$). It is well known that $U \approx V$ iff there is a function $g(U) = V$ where g is 1-1 on a set in U. (See Comfort-Negrepontis, 1974, p.209.) That is, $U \approx V$ obtains iff there is a mapping $g(U) = V$ where U is selective in π_g but no element of π_g belongs to U. Two other familiar results about the Rudin-Keisler partial order on $\beta(\omega)\backslash\omega$ are that Ramsey ultrafilters are minimal, and if U is a P-point and $V \leq U$, then V is a P-point too. Thus, Lemma 2 provides us with the following:

COROLLARY 2. *If $V \prec U_P$ and V is a non-Ramsey P-point then, by mapping U_P to V and locating where V is not-selective, we fix a partition π based on the combinatorial properties of U_P (and V) in which P is non-conglomerable.* □

Recall, too, that in Example 1 the non-conglomerability for event E is localized to one of two orthogonal partitions, i.e., partitions whose elements meet each other in singleton sets, at most. Since an ultrafilter is weakly selective in (at least) one of each pair of orthogonal partitions, we can generalize this feature of Example 1 using Lemma 2 as follows:

COROLLARY 3. *Let $V \prec U$ and $W \prec U$ with $f(U) = V$ and $g(U) = W$. If π_f and π_g are orthogonal partitions, then U is non-conglomerable in (at least) one of these two partitions.* □

Unfortunately, we do not know whether, for each non-Ramsey ultrafilter U there exist V and W satisfying the hypothesis of Corollary 3. Next, we show that when U_P is not a Ramsey ultrafilter and we use conditionally coherent versions of P's natural conditional probabilities whenever U_P is selective in a partition, then the combinatorial properties of U_P locate partitions and events where P does not satisfy conglomerability.

THEOREM 4. *If U_P is not Rudin-Keisler minimal then, either there exists a partition where U_P is weakly selective but not selective (where P is maximally not conglomerable), or else a coherent set of P's natural conditional probabilities, coming from a countable sequence of partitions where U_P is selective, lead to a failure of conglomerability also to the maximum possible extent.*

PROOF. By assumption, there exists $V \prec U_P$. Let π_0 be a partition

$$\pi_0 = \{h_j^0 : h_j^0 = \{f^{-1}(j)\}, j = 1, \ldots\}$$

induced by the mapping $f(U_P) = V$. Since $V \prec U_P$ and V is non-principal, U_P is not selective in π_0. Consider a (canonical) 1-1 map m between ω and $\omega \times \omega$ where $m(h_j^0) = \{(i,j) : i = 1, \ldots\}$ for $(j = 1, \ldots)$, and let Δ_0 be the diagonal of π_0 under m. Then, $m^{-1}(\Delta_0) \notin U_P$. If P is conglomerable in π_0 then (Lemma 2) U_P is not weakly selective in π_0 either. Hence, we may assume that (under m^{-1}) the set Y_0 of points above Δ_0 in π_0 belong to U_P. (Hereafter we suppress m in our discussion of partitions and, where notationally convenient, we identify unit sets with their members, as in the last sentence of this paragraph.) Let π_1

be the partition orthogonal to π_0, i.e., $\pi_1 = \{h_j^1 : h_j^1 = \{j^{th}$ element of $h_i^0 : i = 1, \ldots\}, j = 1, \ldots\}$ where some elements of π_0 and π_1 may be disjoint. Since U_P is not selective in π_0, for each $h^1 \in \pi_1, h^1 \notin U_P$. But U_P is weakly selective in π_1, since $Y_0 \in U_P$ and $| Y_0 \cap h_n^1 | < \omega$ $(n - 1, \ldots)$. Thus, if U_P is not selective in π_1 then P is not conglomerable in π_1. Hence, we assume there exists a set $A_1 \in U_P$ such that $| A_1 \cap h_n^1 | \leq 1\, (n - 1, \ldots)$. If we consider the mapping $f^1 : \omega \to \omega$ associated with π_1, then $f^1(U_P)$ is the ultrafilter U_1 and since f^1 is 1-1 on $A_1 \in U_P$, $U_P \approx U_1$. Based on A_1, choose a (version of the) natural conditional probability for P, given π_1, with set $B_1 = \{b_{1n} : b_{1n} \in h_n^1(n = 1, \ldots)\}$, $B_1 \supseteq A_1$ with $P(b_{1n} \mid h_n^1) = 1$.

Let $\mathbf{h}_{j1}^0$ be the least element of π_0 that meets B_1 at some element b_{1k^1} and define set $C_1 = B_1 - \{b_{1k^1}\}$. Obviously, $C_1 \in U_P$, though this depends upon the version of the natural conditional probability used. Call $h_{k^1}^1$ that element of π_1 which contains b_{1k^1}. Make a partially ordered tree T_1 from $h_{k^1}^1$ by rooting it in b_{1k^1} at level 0, i.e. setting level 0 equal to the unit set $\{b_{1k^1}\}$ and making level 1 the unit set $\{h_{k^1}^1 - \{b_{1k^1}\}\}$. This partial order coincides in an obvious way with the qualitative order from the natural conditional probability: lower levels have (much) higher probability. Specifically, $P(h_{k^1}^1 - \{b_{1k^1}\} \mid h_{k^1}^1) = 0$.

Iterate this procedure to create partitions $\pi_i (i = 2, \ldots)$ of the sets C_{i-1} (where $\lim_i C_i = \emptyset$) and where each π_i is orthogonal to π_0, i.e., where each element of the partition π_i meets each element of the partition π_0 in at most one element of ω. (See the figure below.) Thus, the set Y_i (the points above the diagonal Δ_i of $\pi_0 \times \pi_i$) belongs to U_P; hence, U_P is weakly selective in π_i. Then, P is not conglomerable in π_i unless U_P is selective in that partition. Then, if U_P is selective in π_i, the function f^i (which is associated with π_i) yields the ultrafilter $U_i = f^i(U_P)$ and $U_P \approx U_i$.

Assuming that P is selective in π_i, we arrive at the sets A_i and B_i, the element b_{ik^i}, and the set C_i, just as in the case $(i = 1)$, above. In this way we produce another R-K equivalent ultrafilter U_i, i.e., $U_P \approx U_i$. Also, we create the tree T_i, as described below. If this process continues, i.e., if U_P is selective in each $\pi_i (i = 1, \ldots)$ then the infinite set of trees, $\{T_i : i = 1, \ldots\}$, which partition ω, identify a partition $\pi^* = \{h_i^* : i = 1 \ldots\}$ in which the (chosen versions of the) natural conditional probabilities associated with the π_is fail deFinetti's conglomerability principle. Moreover, the extent of the failure is maximal, i.e. there is an event E with $P(E) = 1$ and $P(E \mid h_i^*) = 0$, $(i = 1, \ldots)$. Next, we give the details of the partial order for the tree T_i. We define $\pi_i, A_i, B_i, b_{ik^i}, T_i$, and C_i, inductively, as follows: For $C_{i-1} \in U_P$, let

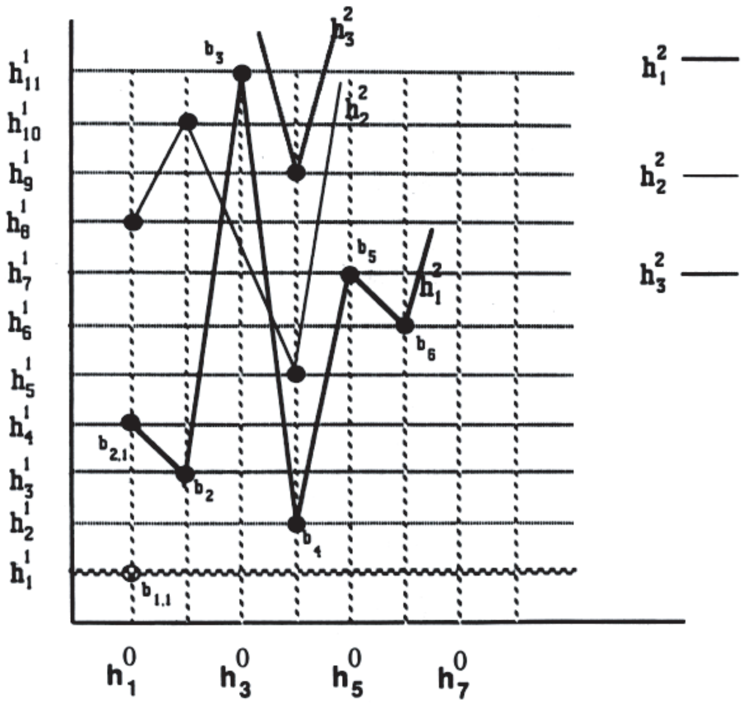
$$\pi_i = \{h_j^i : h_j^i = \{j^{th}\text{elements of } C_{i-1} \cap h_n^0 : n = 1, \ldots\}, j = 1, \ldots\}.$$

(*Note*: π_i is a partition of C_{i-1}, not of the full set ω.) Thus, each partition element $h^i \in \pi_i$ is orthogonal to π_0 and is the graph of a (partial) function. If $k > j$, then h_k^i lies above h_j^i in π_0. Also, the elements h^k (that is, functions) in partition π_k grow more rapidly than do those in π_j. Since U_P is not selective nor even weakly selective in π_0, U_P is weakly selective in each π_i. As argued in the base case $(i = 1), \Delta_i \notin U_P, Y_i \in U_P$, and for each $h^i \in \pi_i, h^i \notin U_P$. Thus, P is conglomerable in π_i if and only if U_P is selective there.

Segments of the first 3 partition elements of π_2 and elements of the set B_2 denoted by ●.

The circled integers belong to U. The tree T_1 is rooted in $b_{1,1}$ and its level 1 = $\{h^1_1 - b_{1,1}\}$.

The tree T_2 is rooted in $b_{2,1}$ and its level 1 = $\{h^1_4 - b_{2,1}, h^2_1 - b_{2,1}\}$
T_2's level 2 = $\{h^1_3 - b_2, h^1_{11} - b_3, \ldots, h^1_6 - b_6, \ldots\}$



Let A_i be a set in U_P meeting the condition that $\mid A_i \cap h^i_n \mid \leq 1\,(n = 1, \ldots)$. Since there exists a coherent version of the natural conditional probability for P, given π_i, there is a set $B_i = \{b_{in} : b_{in} \in h^i_n (n = 1, \ldots)\}$, $B_i \supseteq A_i$ with $P(b_{in} \mid h^i_n) = 1$. Let $\mathbf{h}^0_{ji}$ be the least element of π_0 that meets B_i at some element b_{ik^i} and let $C_i = B_i - \{b_{ik^i}\}$. Evidently, $C_i \in U_P$. Denote by $h^i_{k^i}$ that element of π_i which contains b_{ik^i}, and for $j < i$, denote by h^j_{ki} that element of π_j containing b_{ik^i}.

Make a tree T_i of height i by rooting it in b_{ik^i} at level 0 and making level 1 the i-element set $\{h^1_{k^i} - \{b_{ik^i}\}, h^2_{k^i} - \{b_{ik^i}\}, \ldots, h^i_{k^i} - \{b_{ik^i}\}\}$. Finite additivity assures that $P(\cup \text{ level } 1 \mid (\cup \text{ level } 1\ ) \cup \text{ level } 0\ ) = 0$. Level 2 of T_i is formed by adjoining to each $b \in h^j_{k^i} - \{b_{ik^i}\}(j = 2, \ldots, i)$ the $(j-1)$-many sets $\{h^{j-1} - \{b\}, h^{j-2} - \{b\}, \ldots, h^1 - \{b\}\}$, where $b \in h^n \in \pi^n (n = 1, \ldots, j-1)$. Again, finite additivity assures that: $P(h^{j-1} - \{b\} \cup h^{j-2} - \{b\} \cup \ldots \cup h^1 - \{b\} \mid h^{j-1} \cup h^{j-2} \cup \ldots \cup h^1) = 0$. Continue this way to extend the branches of T_i by adjoining to each $b \in$ level m $(m < i)$ the v-many sets $\{h^v - \{b\}, \ldots, h^1 - \{b\}\}$ for the v-many partition elements h that contain b $(1 \leq v \leq m-1)$. Finite additivity assures that $P(h^v - \{b\} \cup \ldots \cup h^1 - \{b\} \mid h^v \cup \ldots \cup h^1) = 0$. The tree T_i has branches ending at each level and the branches terminate in sets of the form $h^1 - b$, for $b \in h^1 \in \pi_1$.

Now, either this inductive procedure terminates after finitely many steps in a partition π_q where U_P is weakly selective but not selective, or else it leads to an infinite forest of trees $\{T_i : i = 1, \ldots\}$. In the latter case the trees partition the space, ω, because: (1) Elements of a tree are disjoint subsets of ω. (2) $(\cup_T i) \cap (\cup^j_T) = \emptyset$ whenever $i \neq j$. And (3) for each $b \in \omega$, b belongs only to a non-empty *finite* sequence of partition elements $\{h^j : b \in h^j \text{ and } h^j \in \pi_j : j = 1, \ldots, k\}$ where $P(b \mid h^i) = 1$, for $i < k$, and either b is the root of tree T_k or $P(b \mid h^k) = 0$.

Note that the union of sets in a tree does not belong to U_P since $\omega - C_{i+1} \supset \cup_{T^i}$ but $C_{i+1} \in U_P$. Moreover, the set of all tree-roots $R = \{b_{ik^i} : b_{ik^i} \text{ root of } T_i,\ i = 1, \ldots\}$ does not belong to U_P either. This is so because the b_{ik^i} are selected from decreasing sets C_i in order to have h^i_1 meet π_0 in its least element. Thus, either all but a finite number of the b_{ik^i} belong to one partition element of π_0, or else $\mid R \cap h^0_n \mid < \omega$ $(n = 1, \ldots)$. Since U_P is not weakly selective in $\pi_0, R \notin U_P$.

Last, consider the binary partition of $\omega \backslash R$ formed by taking the union of the sets in the odd levels of all trees, L_O, and the union of the sets in the even levels (excluding R, the set of roots) L_E. Exactly one of these two countable sets belongs to U_P, since R does not. Without loss of generality, assume that the union of sets from the odd levels is a set in U_P. Observe, next, that each b (an element of a set at level $2i$) has adjoined to it at level $2i+1$ the v-many sets $\{h^v - \{b\}, \ldots, h^1 - \{b\}\}$ for the v-many partition elements h that contain b $(1 \leq v \leq 2i)(i = 1, \ldots)$.

Consider the denumerable set, $\pi^\dagger = \{h^\dagger_j : j = 1, \ldots\}$ where each $h^\dagger_j$ contains finitely many subsets of ω, one of which is $\{b\}$ for some $b \in L_E$, that is, b is an element of a set from level $2i$ for some i (or from R), and the other sets in $h^\dagger_j$ are the finitely many disjoint sets (disjoint "events") that are adjoined to $\{b\}$ at level $2i+1$ (or at level 1). Evidently, $(\cup_{A \in h^\dagger i} A) \cap (\cup_{B \in h^\dagger j} B) = \emptyset$ whenever $i \neq j$. Let $g : \omega \to \omega$ be any function that is constant on each element of $h^\dagger_j$ for every j, such that $g^{-1}(\{i\}) \neq g^{-1}(\{j\})$ when $i \neq j$, and call V_p the ultrafilter defined by $g(U_P)$.

Since each $h^\dagger_j$ is a finite collection of disjoint subsets of ω, trivially, V_p is weakly selective in the partition $\pi^+ = \{h^+_j : h^+_j = \{i : g(m) = i, m \in B \in h^\dagger_j\}, j = 1, \ldots\}$. Either V_p is not selective in π^+ and P cannot be made conglomerable there, by the reasoning of Corollary 2, or else there is a set $A^+ \in V_p$ such that $\mid A^+ \cap h^+_j \mid \leq 1\, (j = 1, \ldots)$ and we may assume that $g^{-1}(A^+)$ is a subset of sets belonging to L_O. Thus for each set $h^j - \{b\}$ that meets $g^{-1}(A^+)$, from an odd level $2i + 1$ (or from level 1) there is a unique b, an element of an element of level $2i$ (or an element of R) where, $p(h^j - \{b\} \mid h^j) = 0$, $(i = 1, \ldots)$. Hence, using these natural conditional probabilities for P violates conglomerability in the partition, $\pi^* = \{h^j : h^j - \{b\} \in g^{-1}(A^+), j = 1, \ldots\}$. That is, there is a set $g^{-1}(A^+)$ belonging to U_P where, for each $h^j \in \pi^*, P(g^{-1}(A^+) \mid h^j) = 0$ but $P(g^{-1}(A^+)) = 1$. When $\cup L_E \in U_P$, just reverse the roles of even and odd levels in the trees. □

Next, we show that the combinatorial properties of U_P serve to localize the non-conglomerability in P even when the (coherent) conditional probabilities $P(\bullet \mid \bullet)$ are not so-called "natural."

THEOREM 5. *If U_P is not Rudin-Keisler minimal then, either (i) there exists a partition where U_P is weakly selective but not selective; i.e., a partition π^* in which P is singular with respect to each f.a. Q that is conglomerable in π^* (by Lemma 2), or else (ii) P's conditional probabilities (taken from no more than a sequence of partitions where U_P is selective) lead to a failure of conglomerability of extent at least 1/2, i.e.,*

$$\exists(E \in U_P)\exists\pi^*\forall(h \in \pi^*)P(E \mid h) \leq .5.$$

PROOF. We follow the reasoning of the previous theorem, identifying partitions $\pi_i (i = 1, \ldots)$ of the nested sets $C'_i \in U_P$ (where $C'_{i+1} \supset C'_i$) in which U_P is weakly selective. If U_P fails to be selective in π_i, clause (i) is established. However, unlike the situation in Theorem 4, when U_P is selective in π_i the conditional probability $P(\bullet \mid h^i_j)$ need not be the "natural" one. Nonetheless, one of two cases arises.

Case (a): $\exists(E \in U_P)$ such that for each $h^i \in \pi_i, P(E \mid h^i) \leq .5$. Then conditional probability given π_i, is assigned so that P fails to be conglomerable in π_i (by at least .5) - clause (ii).

Case (b): There exists $A_i \in U_P$ such that $\mid A_i \cap h^j_n \mid \leq 1\, (n = 1, \ldots)$ and the conditional probability, given π_i, satisfies $P(a_{in} \mid h^i_n) > .5$ whenever $A_i \cap h^i_n \neq \emptyset$.

We continue the argument with Case (b). Let $C_0 = \omega$ and denote by $B'_i = \{a_{in} : P(a_{in} \mid h^i_n) > .5\}$ so that $B'_i \in U_P$. As before, let $\mathbf{h^0_{j^i}}$ be the least element of π_0 that meets B'_i at some element b_{ik^i} and let $C'_i = B'_i - \{b_{ik^i}\}$. Evidently, $C'_i \in U_P$. Again, denote by $h^i_{k^i}$ that element of π_i which contains b_{ik^i}, and for $j < i$, denote by $h^j_{k^i}$ that element of π_j containing b_{ik^i}. We construct the trees T'_i as before; however, now the T'_i may fail to be a partition of ω. That is, let $D'_i = \{h^i : h^i \cap C'_i = \phi\}$ and $D'_i \neq \emptyset$ is possible. To accommodate the sets of these h^i, each a partition element where conditional probability fails to concentrate above .5 on any element, we form a second array of partially ordered sets, $S'_i(i = 1, \ldots)$, analogous to the T'_i. Each S'_i has a base (level 0) rooted in the set D'_i (rather than the singleton root b_{ik^i} of T'_i). The tree structure in S'_i above D'_i is analogous to that above b_{ik^i} in T'_i. There are two sub-cases to consider:

(b.1) $\mathcal{T} = \{T'_i\} \in U_P$. Then, by reasoning and notation of the previous theorem, i.e., dividing between odd and even levels of $\mathcal{T}$, $\exists(g^{-1}(A^+) \in U_P)\exists\pi^*\forall(h^* \in \pi^*)\ P(g^{-1}(A^+) \mid h^*) \leq .5$.

(b.2) $\mathcal{S} = \{S'_i\} \in U_P$. Whereas the roots R of T (or roots R' of T') do not belong to U_P, the set $D = \cup_i D_i$ may belong to U_P. However, as D is a collection of (disjoint) partition elements $\{h^i : h^i \in D_i; i = 1, \ldots\}$, with each h^i orthogonal to π_0, consider the partition π_D of D formed by these sets h^i. If $D \in U_P$, then U_P is weakly selective in π_D. So, either

(b.2.1) U_P is not selective in π_D and P is not even approximately π_D-conglomerable or

(b.2.2) $\exists(A \in U_P) \mid A \cap h^i \mid \leq 1$ for each $h^i \in \pi_D$. But, since $\pi_D \cap C'_i = \emptyset$ $(i = 1, \ldots)$, for each $h^i \in \pi_D$, $P(A \mid h^i) \leq .5$ and the conditional probability P, given π_D, fails the conglomerability principle by an extent .5 (at least).

Thus, without loss of generality in case (b.2), assume that $D \notin U_P$. That is, the union of the level 0 sets of $\mathcal{S}$ does not belong to U_P. Then, as the structure of $\mathcal{S}$ at higher levels if the same as in $\mathcal{T}$, conclude by reasoning (with notation as in Theorem 4) that upon dividing between the odd and even levels of $\mathcal{S}$, $\exists(g^{-1}(A^+) \in U_P)\exists\pi^*\forall(h^* \in \pi^*)\ P(g^{-1}(A^+) \mid h) \leq .5$. □

4. Non-conglomerability for Finitely Valued Merely f.a. Probabilities

Let P be a merely f.a. probability that assumes only finitely many values. Then (Schervish *et al.*, 1984 p. 213) P may be written as $P = \sum_{i=1}^{k} \gamma_i P_i$, where $\gamma_i > 0$, $\sum_i \gamma_i = 1$, and each P_i is an ultrafilter probability. Since P is merely finitely additive, there exists an integer k_1, $0 \leq k_1 < k$ where each $P_i(i \leq k_1)$ is a principal ultrafilter probability, and each $P_i(k_1 + 1 \leq i \leq k)$ is a non-principal ultrafilter probability. Denote each of these ultrafilters by U_i. Theorem 3.3. (Schervish *et al.*, 1984) establishes that $\beta = \gamma_{k^1+1} + \ldots + \gamma_k$ is the least upper bound on the extent of non-conglomerability possible with P, over all events and all countable partitions. It is an elementary fact that we may find k disjoint sets $\{A_i : A_i \in U_i(i = 1, \ldots, k)$, with $A_i \cap A_j = \emptyset$ for $i \neq j\}$. Thus, $\gamma_i P_i(\bullet) = P(\bullet \mid A_i)$, so that $P(\bullet) = \sum_i \gamma_1 P(\bullet \mid A_i)$.

Let M be the (possibly empty) set of integers that index the non-Ramsey (non-principal) ultrafilters, and let $\gamma_M = \sum_{i \in M} \gamma_i$. Then we may apply Theorem 4 to obtain the following result.

COROLLARY 4. *Let P be as above, together with its decomposition as a mixture of ultrafilter probabilities. Then, (i) either there is a determinate partition where P fails to be even approximately conglomerable (up to the extent γ_M), or (ii) based on the natural conditional probabilities for P_i, there is a determinate partition where P's extent of non-conglomerability is γ_M.*

PROOF. Use the fact that the A_is are disjoint to apply Theorem 4 to each $U_i (i \in M)$. Whenever (i): according to the proof of that theorem we encounter a partition of A_i where U_i is weakly selective but not selective, there P fails to be (even approximately) conglomerable to the extent γ_i. We may concatenate these (disjoint) partitions to form a single partition where P cannot be made conglomerable. If (ii): for a given $U_i (i \in M)$, it is selective in each of the (countably many) partitions used in the proof of Theorem 4, then as previously shown, when P_i's natural conditional probabilities are used, it fails conglomerability in the partition π^* of A_i, and the extent of nonconglomerability there is the maximum possible value, 1. □

Likewise, we may apply Theorem 5 to obtain the following:

COROLLARY 5. *Let P be as above, together with its decomposition as a mixture of ultrafilter probabilities. Then, either there is a determinate partition where P fails to be even approximately conglomerable (up to the extent γ_M), or, based on the conditional probabilities for P_i, there is a determinate partition where P's extent of non-conglomerability is $\gamma_M/2$, at least.*

PROOF. Use Theorem 5 with each of the disjoint sets $A_i (i \in M)$. □

5. Conclusion

We have shown how to locate partitions in which a finite-valued, merely finitely additive probability P displays non-conglomerability. Our approach is to use some combinatorial properties of the associated non-principal ultrafilters for P to regulate all the coherent conditional probabilities for P. These combinatorial properties of the associated ultrafilters are given by the unconditional probability P. This analysis improves upon our previous result in two ways. It demonstrates where P displays non-conglomerability according to its unconditional probability even when P is only two-valued, and it avoids quantifying over a continuum of partitions and conditional probabilities. Also, we hope we have indicated how some basic set-theoretic combinatorial properties of ultrafilters carry interesting consequences for finitely additive probabilities.

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