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A CLASS OF BIVARIATE HEAVY-TAILED DISTRIBUTIONS

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SUMMARY. One-dimensional heavy-tailed distributions have been used for modelling problems in which there may be outlying observations or parameters (c.f. O'Hagan (1990)). In this paper we tackle the corresponding problems in two dimensions, by studying a class of functions analogous to those used in the one-dimensional case. Unlike those in the latter case the functions we require are only partially ordered and this leads to more intricate results.

1. Introduction

Heavy-tailed distributions are of interest in Bayesian analysis because they offer a great variety of methods, and in particular can produce inferences that are inherently robust to outliers or other anomalies. In the simple case of an observation x distributed given θ as $N(\theta, 1)$, and a $N(0,1)$ prior distribution for θ, the posterior distribution for θ given x is $N(x/2, 1/2)$. If $|x|$ is large there is conflict between the prior and likelihood, the former saying that θ is almost certainly in $(-3, 3)$ and the latter suggesting strongly that θ is in $(x - 3, x + 3)$. In this situation the posterior says that θ is now almost certainly in $((x-3)/2, (x+3)/2)$, which disagrees entirely with both of the sources of information that it claims to synthesise. Dawid (1973) and Hill (1974) showed that if the prior distribution were heavy-tailed, for instance a t distribution, then as $|x| \to \infty$ the posterior distribution tends to $N(x, 1)$. (More formally, the posterior distribution of $\theta - x$ given x tends to $N(0, 1)$.) In effect, the prior information is ultimately disregarded and the posterior tends to the normalized likelihood. Conversely, if the likelihood were made a t distribution instead of the prior, then as $|x| \to \infty$ the posterior distribution tends to the prior, $N(0, 1)$,

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and the observation is rejected. Building on this basic idea, O'Hagan (1979, 1988), West (1984, 1985), Meinhold and Singpurwalla (1989), Angers and Berger (1991), Carlin and Polson (1991), Fan and Berger (1992) and Geweke (1992) demonstrate that modelling with heavy tails in Bayesian analysis can produce a wide variety of novel and robust inferences.

Apart from some generalized results in Hill (1974), theoretical developments to date have been restricted to one-dimensional distributions. The present paper initiates a study of multivariate heavy-tailed distributions, beginning with an interesting and flexible class of bivariate heavy-tailed distributions. The practical potential of this class is illustrated by O'Hagan and Le (1994). Our development builds on the one-dimensional theory of O'Hagan (1990), and we begin here by summarizing some results from that paper.

Consider the functions

$$\psi_c(x) = \left(1 + x^2\right)^{-\frac{c}{2}}, \qquad c > 1. \qquad \ldots(1)$$

For any two functions on $\mathbb{R}$, define $f \succeq g$ if there exists a constant $K > 0$ such that $f(x) \geq Kg(x)$ for all $x \in \mathbb{R}$. We write $f \preceq g$ if $g \succeq f$ and $f \approx g$ if both $f \succeq g$ and $f \preceq g$. Then if t_d is the Student t density with d degrees of freedom it is easily shown that $t_d \approx \psi_{d+1}$. If f is a density function on $\mathbb{R}$ and $f \approx \psi_c$ then O'Hagan (1990) defines f (and the corresponding distribution) to have *credence* c. A key result is that if $f \approx \psi_c$ and $g \approx \psi_{c'}$ then $f \star g \approx \psi_{c^*}$, where $f \star g$ denotes the convolution of f and g, and where $c^* = \min\{c, c'\}$. Note that Berman in (1992) also studies the tails of convolutions of members of a class of one-dimensional distributions. O'Hagan (1990) goes on to prove a variety of results concerning posterior distributions when the various sources of information have distributions with specified credences. In particular, if an observation x has density (or likelihood) given θ represented by $p(x|\theta) = f(x-\theta)$, with f having credence c, and if the prior density of θ has credence c', then as $|x| \to \infty$ the prior information is effectively rejected if $c' < c$, and the observation is effectively rejected if $c' > c$.

2. Definitions and Basic Properties

In one dimension the class of functions $\Psi = \{\psi_c : c > 1\}$ is completely ordered by the $\succeq$ relation, since $\psi_c \succeq \psi_{c'}$ if and only if $c \leq c'$. As c decreases, the tails of ψ_c become steadily heavier. In two or more dimensions the situation is very much more complex, since tail thickness can be different in different directions. First consider the natural generalisation of the 'dominance' relation $\succeq$.

DEFINITION. *For any two functions f and g on a given space Ω, we say that f dominates g, denoted by $f \succeq g$ or $g \preceq f$, if there exists a constant $K > 0$ such* *that* $f(\omega) \geq K g(\omega)$ $\forall \omega \in \Omega$. *If* f *and* g *dominate each other, we say that* f *is equivalent to* g *and write* $f \approx g$.

As a simple but non-trivial generalisation of the one-dimensional 'Student t-like' ψ_c, consider density functions on $\mathbb{R}^2$ which are equivalent to

$$\left(1+x^2+y^2\right)^{-\frac{c}{2}}\left(1+x^2\right)^{-\frac{c_1}{2}}\left(1+y^2\right)^{-\frac{c_2}{2}} \qquad \ldots(2)$$

for some c, c_1 and c_2. Practical circumstances in which such distributions may arise have been presented in O'Hagan and Le (1994). The components $\left(1+x^2\right)^{-c_1/2}\left(1+y^2\right)^{-c_2/2}$ correspond to independent one-dimensional t distributions, but $\left(1+x^2+y^2\right)^{-c/2}$ is equivalent to a bivariate t distribution. Their product (2) will be shown to have properties much more complex than either of those cases.

At this point, it is appropriate to contrast (2) with two other families of multivariate heavy-tailed distributions. The class of v-spherical distributions of Fernandez, Osiewalski and Steel (1995) is a very flexible family of distributions with densities of the form

$$f(\mathbf{x}) = g\left\{\mathrm{v}(\mathbf{x}-\mu)\right\}$$

where the function v operates like a metric with the property $\mathrm{v}(a\mathbf{y}) = a\mathrm{v}(\mathbf{y})$, but is otherwise arbitrary, and where g is a univariate function which is also arbitrary (subject to the result being a proper density function) and can be chosen so that $f(\mathbf{x})$ is heavy-tailed. However, the key property of v implies that tail thickness is the same in every direction. The multivariate t function has the same property (and is a member of the v-spherical class), but our intention in exploring the distributions (2) is to present a family with non-uniform tail thicknesses.

The stable distributions have also been suggested as alternative heavy-tailed distributions, and multivariate stable laws are considered by Samorodnitsky and Taqqu (1994). However, in addition to a similar uniformity of tail behaviour, stable distributions (apart from the normal) all have infinite variance. In one dimension, they all have tails at least as heavy as the Cauchy (Feller, 1971). They cannot therefore exhibit the same range of tail behaviour as the family (2).

It will be more convenient to write the function represented by (2) as

$$\chi_{\mathbf{s}}(x,y) = \left\{\frac{1+y^2}{1+x^2+y^2}\right\}^{\frac{s_1}{2}}\left\{\frac{1+x^2}{1+x^2+y^2}\right\}^{\frac{s_2}{2}}\left\{\frac{1+x^2+y^2}{(1+x^2)(1+y^2)}\right\}^{\frac{s_3}{2}}, \qquad \ldots(3)$$

where $\mathbf{s} = (s_1, s_2, s_3)$ and $s_1 = c + c_1$, $s_2 = c + c_2$, $s_3 = c + c_1 + c_2$.

We denote by fg the simple algebraic product of the two functions f and g, so that $(fg)(x,y) = f(x,y)\,g(x,y)$. Similarly, f^{-1} is not the inverse function but simply defined by $(f^{-1})(x,y) = \{f(x,y)\}^{-1}$. The convolution $f \star g$ is defined by

$$(f \star g)(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u,v)\, g(x-u, y-v)\, du\, dv$$

when the integrals exist. The following lemma summarizes some very simple properties for which the proof is trivial.

LEMMA 1. (*a*) *If* $f_1 \succeq g_1$ *and* $f_2 \succeq g_2$, *then* $f_1 f_2 \succeq g_1 g_2$;
(*b*) *If* $f_1 \approx g_1$ *and* $f_2 \approx g_2$, *then* $f_1 f_2 \approx g_1 g_2$;
(*c*) *If* $f_1 \succeq g_1$, $f_2 \succeq g_2$ *and* $f_1 \star f_2$ *exists, then* $g_1 \star g_2$ *exists and* $f_1 \star f_2 \succeq g_1 \star g_2$;
(*d*) *If* $f_1 \approx g_1$, $f_2 \approx g_2$ *and* $g_1 \star g_2$ *exists, then* $f_1 \star f_2$ *exists and* $f_1 \star f_2 \approx g_1 \star g_2$;
(*e*) $\chi_{\mathbf{s}} \chi_{\mathbf{s}'} = \chi_{\mathbf{s}+\mathbf{s}'}, \chi_{\mathbf{s}}^{-1} = \chi_{-\mathbf{s}}$;
(*f*) *If* $\mathbf{s} = (s_1, s_2, s_1 + s_2)$, *then* $\chi_{\mathbf{s}}(x,y) = \psi_{s_1}(x)\, \psi_{s_2}(y)$ *for all* $(x,y) \in \mathbb{R}^2$.

For $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^3$ we write $\mathbf{s} \le \mathbf{s}'$ if $s_i \le s_i'$ for all $i = 1,2,3$. Then $\le$ is a partial order on $\mathbb{R}^3$ and induces a partial ordering of the functions (3), which the following lemma shows coincides with domination. Note first that part (*e*) of Lemma 1 implies $\chi_{\mathbf{s}} \succeq \chi_{\mathbf{s}'} \Leftrightarrow \chi_{\mathbf{s}} \chi_{\mathbf{s}'}^{-1} \succeq \chi_{\mathbf{0}} \equiv 1$.

LEMMA 2. $\mathbf{s}' \le \mathbf{s} \Leftrightarrow \chi_{\mathbf{s}}(x,y) \le \chi_{\mathbf{s}'}(x,y)\ \forall x,y \Leftrightarrow \chi_{\mathbf{s}} \preceq \chi_{\mathbf{s}'}$.

PROOF. By the preceding remarks it suffices to consider $\mathbf{s}' = \mathbf{0}$. But $\mathbf{s} \ge \mathbf{0}$ implies that $\chi_{\mathbf{s}}(x,y) \le 1$ and this implies $\chi_{\mathbf{s}} \preceq 1$ trivially. Conversely, if $\chi_{\mathbf{s}} \preceq 1$ then putting $y = 0$ we get $(1+x^2)^{-s_1/2} \le 1$ which implies that $s_1 \ge 0$ and, similarly $s_2 \ge 0$. Putting $y = \gamma x$ for some $\gamma \ne 0$ we see $\chi_{\mathbf{s}}(x, \gamma x) \approx (1+x^2)^{-s_3/2}$ and hence also $s_3 \ge 0$. □

The last part of the proof shows that, if there is a $\hat{\mathbf{s}} \in \mathbb{R}^3$, such that $f \approx \chi_{\hat{\mathbf{s}}}$ for some given function f, then the $\hat{\mathbf{s}}$ is determined by the behaviour of f in neighbourhoods of the x-axis, the y-axis and any chosen line $y = \gamma x$ with $\gamma \ne 0$. All functions equivalent to $\chi_{\mathbf{s}}$ show the same behaviour on these lines, and together form an equivalence class defined by $\chi_{\mathbf{s}}$.

LEMMA 3. (*a*) *If* $f \approx \chi_{\mathbf{s}}$, *then for all* $\mathbf{s}' \ne \mathbf{s}$, $f \not\approx \chi_{\mathbf{s}'}$;
(*b*) *If* $f(x,y) = \chi_{\mathbf{s}}(ax+b, cy+d)$ *for all* $(x,y) \in \mathbb{R}^2$ *and for some constants* $a \ne 0, b, c \ne 0$ *and* d, *then* $f \approx \chi_{\mathbf{s}}$.

PROOF. (*a*) By consideration of the same three lines as in the proof of Lemma 2, it is clear that $\mathbf{s}' \ne \mathbf{s} \Leftrightarrow \chi_{\mathbf{s}} \chi_{\mathbf{s}'}^{-1} \not\approx 1$.

(*b*) follows from the fact that $\{1 + (ax+b)^2\}(1+x^2)^{-1}$, $\{1 + (cy+d)^2\}(1+y^2)^{-1}$ and $\{1 + (ax+b)^2 + (cy+d)^2\}(1+x^2+y^2)^{-1}$ are all bounded. □

Part (*a*) shows that the equivalence classes of the $\chi_{\mathbf{s}}$ functions are all distinct (and partially ordered by Lemma 2). Part (*b*) shows further that for these functions equivalence is independent of arbitrary location and scale shifts in the two axes. (We do not consider rotating the axes, and note that except in special circumstances a rotated f will not be equivalent to any $\chi_{\hat{\mathbf{s}}}$.) This is in accordance with our intention that the $\chi_{\mathbf{s}}$ functions should characterize density functions with different forms of tail thickness. If $f \succeq g$ then f has tails at least as thick in all directions as g. However, the possibility of different tail thicknesses in different directions is what causes the dominance relation to produce only a partial ordering. If $f \not\succeq g$ and $f \not\preceq g$ then f has thicker tails than g in some direction(s) but thinner tails in some other direction(s).

3. Marginal Densities and Propriety

If f is a joint probability density function for two random variables X and Y, it is natural to enquire about its marginal and conditional densities. We now answer that question when $f \approx \chi_{\mathbf{s}}$.

Conditional densities are straightforward. Regarding (3) as a function of x for fixed y, and using Lemma 3(b), the conditional densities of X given $Y = y$ are clearly equivalent to ψ_{s_1} for all y. Similarly, the conditional densities of Y given $X = x$ are equivalent to ψ_{s_2} for all x. These results correspond to looking along the two axes in the proof of Lemma 2, while looking along any other line $y = \gamma x$ shows that the conditional densities of X or Y given $aX + bY = z$ are equivalent to ψ_{s_3} for all z and all $a, b \neq 0$. Therefore the components of s can be seen as defining tail thickness in the various directions appropriate to these conditional densities.

Now define the marginal density f_X of X by

$$f_X(x) \equiv \int_{-\infty}^{\infty} f(x,y)\,dy \approx \int_{-\infty}^{\infty} \chi_{\mathbf{s}}(x,y)\,dy.$$

From the preceding discussion, the integral exists for all x provided $s_2 > 1$. With this assumption, the following theorem characterizes f_X.

THEOREM 1. (a) *If* $s_1 < s_3 - 1$, *then* $f_X \approx \psi_{s_1}$.
(b) *If* $s_1 = s_3 - 1$, *then* $f_X \approx \psi^*_{s_1}$, *where* $\psi^*_d(x) = (1+x^2)^{-d/2}\log(2+x^2)$ *for* $d \in \mathbb{R}$.
(c) *If* $s_1 > s_3 - 1$, *then* $f_X \approx \psi_{s_3-1}$.

PROOF. Using the form (2) for $\chi_{\mathbf{s}}$, $f_X \approx \psi_{c_1}\phi$, where

$$\phi(x) = \int_{-\infty}^{\infty} (1+x^2+y^2)^{-\frac{c}{2}}(1+y^2)^{-\frac{c_2}{2}}\,dy$$

(a) To show $\phi \succeq \psi_c$, for any $\epsilon > 0$

$$\begin{aligned}\phi(x) &\geq \int_{-\epsilon}^{\epsilon} (1+x^2+y^2)^{-\frac{c}{2}}(1+y^2)^{-\frac{c_2}{2}}\,dy \\ &\geq (1+x^2+\epsilon^2)^{-\frac{c}{2}}(1+\epsilon^2)^{-\frac{c_2}{2}}\int_{-\epsilon}^{\epsilon} dy.\end{aligned}$$

To show $\phi \preceq \psi_c$,

$$\phi(x) \le \left(1+x^2\right)^{-\frac{c}{2}} \int_{-\infty}^{\infty} \left(1+y^2\right)^{-\frac{c_2}{2}} dy$$

and $s_1 < s_3 - 1 \Leftrightarrow c_2 > 1$ so the integral converges.

(*b*) $s_1 = s_3 - 1 \Leftrightarrow c_2 = 1$. To show $\phi \succeq \psi_c^*$,

$$\begin{aligned}
\phi(x) &\ge \int_0^{\sqrt{1+x^2}} \left(1+x^2+y^2\right)^{-\frac{c}{2}} \left(1+y^2\right)^{-\frac{1}{2}} dy \\
&\ge 2^{-\frac{c}{2}} \left(1+x^2\right)^{-\frac{c}{2}} \int_0^{\sqrt{1+x^2}} \left(1+y^2\right)^{-\frac{1}{2}} dy \\
&\ge 2^{-\frac{c}{2}} \left(1+x^2\right)^{-\frac{c}{2}} \int_0^{\sqrt{1+x^2}} y\left(1+y^2\right)^{-1} dy \\
&= 2^{-\frac{c}{2}} \left(1+x^2\right)^{-\frac{c}{2}} 2^{-1} \left[\log\left(1+y^2\right)\right]_0^{\sqrt{1+x^2}} \\
&= 2^{-1-\frac{c}{2}} \psi_c^*(x).
\end{aligned}$$

To show $\phi \preceq \psi_c^*$, note that $\phi(x) = 2\{\phi_1(x) + \phi_2(x)\}$, where

$$\phi_1(x) = \int_0^{\sqrt{1+x^2}} \left(1+x^2+y^2\right)^{-\frac{c}{2}} \left(1+y^2\right)^{-\frac{c_2}{2}} dy$$

and

$$\phi_2(x) = \int_{\sqrt{1+x^2}}^{\infty} \left(1+x^2+y^2\right)^{-\frac{c}{2}} \left(1+y^2\right)^{-\frac{c_2}{2}} dy.$$

We will prove that $\phi_i \preceq \psi_c^*$ for $i = 1, 2$. First,

$$\phi_1(x) \le \left(1+x^2\right)^{-\frac{c}{2}} \int_0^{\sqrt{1+x^2}} \left(1+y^2\right)^{-\frac{1}{2}} dy.$$

Now for $y \ge 3^{-1/2}$, $\left(1+y^2\right)^{-1/2} \le 2y\left(1+y^2\right)^{-1}$, and the preceding argument then shows $\phi_1 \preceq \psi_c^*$. Next, for general c_2 we have

$$\begin{aligned}
\phi_2(x) &\le \int_{\sqrt{1+x^2}}^{\infty} \left(1+x^2+y^2\right)^{-\frac{c}{2}} \left\{1+\left(1+x^2\right)/2+y^2/2\right\}^{-\frac{c_2}{2}} dy \\
&\le 2^{\frac{c_2}{2}} \int_{\sqrt{1+x^2}}^{\infty} \left(1+x^2+y^2\right)^{-\frac{c+c_2}{2}} dy \\
&= 2^{\frac{c_2}{2}} \left(1+x^2\right)^{-\frac{c+c_2-1}{2}} \int_1^{\infty} \left(1+z^2\right)^{-\frac{c+c_2}{2}} dz
\end{aligned}$$

and therefore since $c + c_2 = s_2 > 1$ is assumed, $\phi_2 \preceq \psi_{c+c_2-1}$. In this case $c_2 = 1$ and $\phi_2 \preceq \psi_c \preceq \psi_c^*$.

(c) To prove $\phi \succeq \psi_{c+c_2-1}$,

$$\phi(x) \geq \int_{-\infty}^{\infty} \left(1+x^2+y^2\right)^{-\frac{c+c_2}{2}} dy = \left(1+x^2\right)^{-\frac{c+c_2-1}{2}} \int_{-\infty}^{\infty} \left(1+z^2\right)^{-\frac{c+c_2}{2}} dz.$$

To prove $\phi \preceq \psi_{c+c_2-1}$, write $\phi = 2\{\phi_1 + \phi_2\}$ as in (b), and $\phi_2 \preceq \psi_{c+c_2-1}$ is already proved above. Finally, since $s_2 > s_3 - 1 \Leftrightarrow c_2 < 1$,

$$\begin{aligned} \phi_1(x) &\leq \left(1+x^2\right)^{-\frac{c}{2}} \int_0^{\sqrt{1+x^2}} \left(1+y^2\right)^{-\frac{c_2}{2}} dy \\ &\leq \left(1+x^2\right)^{-\frac{c}{2}} \int_0^{\sqrt{1+x^2}} y^{-c_2} dy \\ &= (1-c_2)^{-1} \left(1+x^2\right)^{-\frac{c}{2}} \left(1+x^2\right)^{\frac{1-c_2}{2}}, \end{aligned}$$

therefore $\phi_1 \preceq \psi_{c+c_2-1}$. □

The tail thickness of the marginal density of Y, which exists if $s_1 > 1$, may be determined simply by interchanging subscripts 1 and 2 in Theorem 1. The appearance of ψ_d^* in Theorem 1 is interesting because $\psi_d \preceq \psi_d^* \preceq \psi_{d+\epsilon}$ for any $\epsilon > 0$. Thus ψ_d^* lies between 'adjacent' ψ densities and introduces a finer gradation of tail thickness.

COROLLARY 1. *$f \approx \chi_{\mathbf{s}}$ and f is a proper density function on $\mathbb{R}^2 \Leftrightarrow s_1 > 1$, $s_2 > 1$, $s_3 > 2$.*

PROOF. f can be proper if and only if $\chi_{\mathbf{s}}$ is integrable over $\mathbb{R}^2$. $s_2 > 1$ is needed to integrate with respect to y, whereupon Theorem 1 applies. In cases (a) and (b), f_X is integrable iff $s_1 > 1$, and then $s_3 \geq s_1 + 1 > 2$. In case (c), f_X is integrable iff $s_3 > 2$, and then $s_1 > s_3 - 1 > 1$. □

In the remainder of this paper we shall be concerned primarily with characterising tail behaviour of proper density functions, and so will in general impose the conditions of Corollary 1. Furthermore, we will assume $c \geq 0$, $c_1 \geq 0$, $c_2 \geq 0$ in the formulation (2), equivalent to $\max\{s_1, s_2\} \leq s_3 \leq s_1 + s_2$. These conditions ensure that $\chi_{\mathbf{s}}$ is unimodal, and simplify many proofs without always being necessary. (For instance, they were assumed implicitly in the proof of Theorem 1, although the proof is easily adapted to the case of negative c, c_1 or c_2.) The restriction is unlikely to be important in practice.

We therefore define the class of functions

$$\mathbf{X} = \{\chi_{\mathbf{s}} : \mathbf{s} = (s_1, s_2, s_3) > (1,1,2), \max\{s_1, s_2\} \leq s_3 \leq s_1 + s_2\}.$$

For the remainder of this paper we shall assume that $\chi_{\mathbf{s}}$ and $\chi_{\mathbf{s}'}$ are in $\mathbf{X}$. Since $\chi_{\mathbf{s}} \star \chi_{\mathbf{s}'}$ has similar behaviour on each of the four quadrants we may, when relevant, assume that $x \geq 0$ and $y \geq 0$.

4. Posterior Distribution and Modes

Let (U,V) have prior density $f \approx \chi_{\mathbf{S}}$ and $(X-u, Y-v)$ given $(U,V)=(u,v)$ have density $g \approx \chi_{\mathbf{S}'}$. Then the posterior density of (U,V) given $(X,Y)=(x,y)$ is (by Lemma 1)

$$q_{x,y}(u,v) = \frac{f(u,v)\,g(x-u,y-v)}{\{f \star g\}(x,y)} \approx \frac{\chi_{\mathbf{S}}(u,v)\,\chi_{\mathbf{S}'}(x-u,y-v)}{\{\chi_{\mathbf{S}} \star \chi_{\mathbf{S}'}\}\,(x,y)}.$$

Treating the denominator $\{\chi_{\mathbf{S}} \star \chi_{\mathbf{S}'}\}\,(x,y)$ as a constant, we first note that $q_{x,y} \approx \chi_{\mathbf{S}+\mathbf{S}'}\ \forall (x,y)$.

This scenario represents our prime motivation in this paper. Bayes' theorem combines two sources of information, namely the prior information represented by the prior density $f \approx \chi_{\mathbf{S}}$, and the data represented by the likelihood $g \approx \chi_{\mathbf{S}'}$. We are interested in the posterior density $q_{x,y}$, and particularly in its behaviour as conflict arises between the two sources of information. The prior density asserts that (U,V) is very unlikely to be far from $(0,0)$, whereas the likelihood suggests that (U,V) should not be far from (x,y). These claims will conflict when $|x| \to \infty$ and/or $|y| \to \infty$. If f and g were both normal densities then, as discussed in section 1, the conflict would always be resolved by a kind of compromise. The posterior distribution would be normal (and hence unimodal) and centered at $(x,y)A$, where A is a fixed 2×2 matrix. As a result, the posterior probability that (U,V) is in any fixed neibourhood of $(0,0)$ or of (x,y) will tend to zero as $|x| \to \infty$ or $|y| \to \infty$. So in the limit the posterior distribution effectively denies both its constituent sources of information.

Very different kinds of resolution of conflict will occur with heavy-tailed distributions. In this section we consider the existence of modes. It is known (c.f. Dawid (1973)) that, in the one-dimensional case, the posterior density will be bimodal for sufficiently large x. In the two-dimensional case we are considering here, the same result applies directly if x becomes large while y is fixed. Then regarding f and g as functions of u for fixed v, the one-dimensional theory applies and we find that, for sufficiently large x, $q_{x,y}$ will have two modes in neighbourhoods of $(0,0)$ and $(x,,0)$. (Note that, since y is not large, we could equally well say that these modes are in neighbourhoods of $(0,y)$ and (x,y) respectively.) Similarly, for large y, $q_{x,y}$ will have modes in neighbourhoods of $(0,0)$ and $(0,y)$. The following theorem shows that as both x and y become large we find the possibility of 4 modes.

THEOREM 2. *If x and y are both sufficiently large, $q_{x,y}$ has a mode in the neighbourhood of $(0,0)$ and a mode in the neighbourhood of (x,y). If, in addition, $s_3 > s_1$ and $s_3' > s_2'$ (respectively $s_3 > s_2$ and $s_3' > s_1'$), $q_{x,y}$ also has a mode in the neighbourhood of $(x,0)$ (respectively $(0,y)$).*

PROOF. Since the proofs for all cases are similar, we consider the possible mode of $q_{x,y}$ in the neighbourhood of $(x,0)$ as an example. We will show that, under the given conditions, $q_{x,y}$ will be greater at $(x,0)$ than at all points on some curve C around $(x,0)$. Now

$$\frac{q_{x,y}(u,v)}{q_{x,y}(x,0)} \leq K\,(1+x^2)^{\frac{s_1}{2}}\,(1+y^2)^{\frac{s'_2}{2}}\,\chi_{\mathrm{s}}(u,v)\,\chi_{\mathrm{s}'}(x-u,y-v)$$

$$\leq K\left\{\frac{1+x^2}{1+u^2}\right\}^{\frac{s_1}{2}}\left\{\frac{1+y^2}{1+(y-v)^2}\right\}^{\frac{s'_2}{2}}\left(1+(x-u)^2\right)^{-\frac{s'_3-s'_2}{2}}\left(1+v^2\right)^{-\frac{s_3-s_1}{2}}$$

for some constant $K>0$. We choose the curve C to be the boundary of the region $\{(u,v):\ |x-u|<a,\ |v|<b\}$ for some $a>0$ and $b>0$. Then the ratio of $q_{x,y}$ at all points on C to its value at $(x,0)$ is bounded by

$$K\left\{\frac{1+x^2}{1+(x-a)^2}\right\}^{\frac{s_1}{2}}\left\{\frac{1+y^2}{1+(y-b)^2}\right\}^{\frac{s'_2}{2}}\times\max\left\{\left(1+a^2\right)^{-\frac{s'_3-s'_2}{2}},\left(1+b^2\right)^{-\frac{s_3-s_1}{2}}\right\}.$$

It is clear that, under the given conditions on the parameters s and s′, when x and y are both sufficiently large this ratio will be strictly less than unity provided we choose a and b large enough. □

To study more detailed behaviour of $q_{x,y}$ requires an understanding of $\chi_{\mathrm{s}}\star\chi_{\mathrm{s}'}$, which is considered in the next section.

5. Convolution

LEMMA 4. *$\chi_{\mathrm{s}}\star\chi_{\mathrm{s}'}\succeq\chi_{\mathrm{s}}$ and $\chi_{\mathrm{s}}\star\chi_{\mathrm{s}'}\succeq\chi_{\mathrm{s}'}$.*

PROOF.

$$\begin{aligned}
&\left\{\chi_{\mathrm{s}}^{-1}\left(\chi_{\mathrm{s}}\star\chi_{\mathrm{s}'}\right)\right\}(x,y)\\
>\ &\chi_{\mathrm{s}}^{-1}(x,y)\int_x^{x+1}\int_y^{y+1}\chi_{\mathrm{s}}(u,v)\,\chi_{\mathrm{s}'}(x-u,y-v)\,du\,dv\\
\geq\ &K\int_x^{x+1}\int_y^{y+1}\chi_{\mathrm{s}'}(x-u,y-v)\,du\,dv\\
=\ &K\int_0^1\int_0^1\chi_{\mathrm{s}'}(u,v)\,du\,dv
\end{aligned}$$

for some constant K, since χ_{s} is decreasing in each variable. Thus $\chi_{\mathrm{s}}\star\chi_{\mathrm{s}'}\succeq\chi_{\mathrm{s}}$. Similarly $\chi_{\mathrm{s}}\star\chi_{\mathrm{s}'}$ dominates $\chi_{\mathrm{s}'}$. □

LEMMA 5. $\chi_{\mathbf{s}} \star \chi_{\mathbf{s}} \approx \chi_{\mathbf{s}}$.

PROOF. By Lemma 4 we only need to prove that $\chi_{\mathbf{s}} \star \chi_{\mathbf{s}} \preceq \chi_{\mathbf{s}}$. Write $\{\chi_{\mathbf{s}} \star \chi_{\mathbf{s}}\}(x,y) = I_1 + I_2 + I_3 + I_4$, where

$$\begin{aligned} I_1 &= \int_{-\infty}^{\frac{x}{2}} \int_{-\infty}^{\frac{y}{2}} \chi_{\mathbf{s}}(u,v)\, \chi_{\mathbf{s}}(x-u, y-v)\, du\, dv \\ I_2 &= \int_{-\infty}^{\frac{x}{2}} \int_{\frac{y}{2}}^{\infty} \chi_{\mathbf{s}}(u,v)\, \chi_{\mathbf{s}}(x-u, y-v)\, du\, dv \end{aligned}$$

with I_3 and I_4 defined similarly on the other two quadrants. Note first that changing variables in I_3 and I_4 reduces them to I_2 and I_1. Note also that, for $x \geq 0$, the function $(x-u)^2$ is decreasing on $(-\infty, x/2]$ and so achieves its minimum $x^2/4$ at $u = x/2$. Thus, for $(u,v) \in (-\infty, x/2] \times (-\infty, y/2]$,

$$\chi_{\mathbf{s}}(x-u, y-v) \leq \chi_{\mathbf{s}}(x/2, y/2) \leq K \chi_{\mathbf{s}}(x,y)$$

for some constant K, since $\chi_{\mathbf{s}} \in \mathbf{X}$. Hence

$$I_1 \leq K \chi_{\mathbf{s}}(x,y) \int_{-\infty}^{\frac{x}{2}} \int_{-\infty}^{\frac{y}{2}} \chi_{\mathbf{s}}(u,v)\, du\, dv$$

so that $I_1 \preceq \chi_{\mathbf{s}}$. To show that $I_2 \preceq \chi_{\mathbf{s}}$, we rewrite $\chi_{\mathbf{s}}$ in its original form (2). Since $u^2 \leq (x-u)^2$ for $u \in (-\infty, x/2]$ and $(y-v)^2 \leq v^2$ for $v \in [y/2, \infty)$

$$\begin{aligned} & \left(1 + u^2 + (y-v)^2\right)\left(1 + (x-u)^2 + v^2\right) \\ \leq\ & \left(1 + u^2 + v^2\right)\left(1 + (x-u)^2 + (y-v)^2\right) \end{aligned}$$

over the range of the integral. Hence, as we also have $(x-u)^2 \geq x^2/4$ and $v^2 \geq y^2/4$, we get that, for some constant K,

$$\begin{aligned} I_2 \ \leq\ & \left(1 + \frac{x^2}{4} + \frac{y^2}{4}\right)^{-\frac{c}{2}} \left(1 + \frac{x^2}{4}\right)^{-\frac{c_1}{2}} \left(1 + \frac{y^2}{4}\right)^{-\frac{c_2}{2}} \\ & \times \int_{-\infty}^{\frac{x}{2}} \int_{\frac{y}{2}}^{\infty} \left(1 + u^2 + (y-v)^2\right)^{-\frac{c}{2}} \left(1 + u^2\right)^{-\frac{c_1}{2}} \left(1 + (y-v)^2\right)^{-\frac{c_2}{2}} du\, dv \\ \leq\ & K \chi_{\mathbf{s}}(x,y) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{\mathbf{s}}(u, y-v)\, du\, dv. \end{aligned}$$

□

Analogously to the one-dimensional case, we consider the triple $\mathbf{s}^*$ where $s_i^* = \min\{s_i, s_i'\}$. Then the following is immediate.

LEMMA 6. *If* $\chi_{\mathbf{s}}, \chi_{\mathbf{s}'} \in \mathbf{X}$, *then* $\chi_{\mathbf{s}^*} \in \mathbf{X}$ *if and only if* $s_3^* \leq s_1^* + s_2^*$.

Lemma 2 and Lemma 5 together imply the following result.

LEMMA 7. *If* $\chi_{\mathrm{s}^*} \in \mathbf{X}$, *then* $\chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \preceq \chi_{\mathrm{s}^*}$.

THEOREM 3. (*i*) *If* $\mathrm{s} = \mathrm{s}^*$ *or* $\mathrm{s}' = \mathrm{s}^*$ *or* $s_3 = s_3' < s_1^* + s_2^*$, *then* $\chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \approx \chi_{\mathrm{s}^*}$.
(*ii*) *If* $s_3^* \geq s_1^* + s_2^*$, *then* $\chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \approx \chi_{(s_1^*, s_2^*, s_1^* + s_2^*)}$.

PROOF. (*i*) If $\mathrm{s} = \mathrm{s}^*$ or $\mathrm{s}' = \mathrm{s}^*$ the result follows from Lemmas 4 and 7.
If $s_3 = s_3' < s_1^* + s_2^*$ then, given the previous cases, we may assume without loss of generality that $s_1 > s_1'$ and $s_2 < s_2'$ so that

$$\left\{\chi_{\mathrm{s}^*}^{-1} \chi_{\mathrm{s}}\right\}(x, y) = \left\{\frac{1 + y^2}{1 + x^2 + y^2}\right\}^{\frac{s_1 - s_1'}{2}}$$

which has a positive lower bound if $|x| \leq |y|$. If $|x| \geq |y|$, $\chi_{\mathrm{s}^*}^{-1} \chi_{\mathrm{s}'}$ is similarly bounded below. It follows that so too is $\chi_{\mathrm{s}^*}^{-1}(\chi_{\mathrm{s}} \star \chi_{\mathrm{s}})$ for all x, y. Thus $\chi_{\mathrm{s}^*} \preceq \chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \preceq \chi_{\mathrm{s}^*}$ as required.

(*ii*) We may assume without loss of generality that $s_1^* = s_1'$ and $s_2^* = s_2$, and then we have $s_3, s_3' \geq s_1' + s_2$. Thus $\mathrm{s}, \mathrm{s}' \geq \hat{\mathrm{s}} = (s_1, s_2, s_1' + s_2) \equiv (s_1^*, s_2^*, s_1^* + s_2^*)$. On the other hand, since $s_1 + s_2 \geq s_3$, $\chi_{\tilde{\mathrm{s}}} \preceq \chi_{\mathrm{s}}$ where $\tilde{\mathrm{s}} = (s_1, s_2, s_1 + s_2)$, and a similar result holds for $\chi_{\mathrm{s}'}$. Hence the one-dimensional result gives $\chi_{\hat{\mathrm{s}}} \preceq \chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \preceq \chi_{\hat{\mathrm{s}}}$. □

In the remainder of this section we consider two questions left open by Theorem 3. Firstly, what is the role of χ_{s^*} when it lies in $\mathbf{X}$?

THEOREM 4. *Suppose that* $\chi_{\mathrm{s}^*} \in \mathbf{X}$. *Then* χ_{s^*} *is the greatest lower bound, with respect to the partial order* $\preceq$, *in* $\mathbf{X}$ *of all* χ *such that* $\chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \preceq \chi$. *Moreover either* $\chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \approx \chi_{\mathrm{s}^*}$ *or there is no* $\chi \in \mathbf{X}$ *such that* $\chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \approx \chi$.

PROOF. Let $\hat{\mathbf{X}}$ be $\{\chi \in \mathbf{X} : \chi \succeq \chi_{\mathrm{s}} \star \chi_{\mathrm{s}'}\}$ and $\chi_{\hat{\mathrm{s}}} \in \hat{\mathbf{X}}$. Then $\chi_{\hat{\mathrm{s}}} \succeq \chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \succeq \chi_{\mathrm{s}}, \chi_{\mathrm{s}'}$ implies $\hat{\mathrm{s}} \leq \mathrm{s}, \mathrm{s}'$ by Lemma 2. Hence $\hat{\mathrm{s}} \leq \mathrm{s}^*$, so $\chi_{\mathrm{s}^*} \preceq \chi_{\hat{\mathrm{s}}}$ which shows that χ_{s^*} is a lower bound of $\hat{\mathbf{X}}$. However by Lemma 7 χ_{s^*} is in $\hat{\mathbf{X}}$, so it must be the greatest lower bound.

If there exists a $\chi_{\hat{\mathrm{s}}} \in \mathbf{X}$ such that $\chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \approx \chi_{\hat{\mathrm{s}}}$, then we also have by Lemma 7 that $\chi_{\hat{\mathrm{s}}} \preceq \chi_{\mathrm{s}^*}$ and so by Lemma 2 that $\hat{\mathrm{s}} = \mathrm{s}^*$. □

We now consider the circumstances under which all the alternative hypotheses of Theorem 3 fail. For our result we shall need to make further minor restrictions on the triples s and s' as follows.

(C) If $s_i < s_i'$, for $i = 1$ or 2, then $s_3 - s_i > 1$; if $s_i' < s_i$, for $i = 1$ or 2, then $s_3' - s_i' > 1$.

THEOREM 5. *Assuming that condition* (C) *holds and* $\mathrm{s}^* \neq \mathrm{s}$, $\mathrm{s}^* \neq \mathrm{s}'$, $s_3^* < s_1^* + s_2^*$ *and* $s_3 \neq s_3'$, *then there is no* $\chi \in \mathbf{X}$ *such that* $\chi_{\mathrm{s}} \star \chi_{\mathrm{s}'} \approx \chi$.

PROOF. It follows from the condition that $s_3^* < s_1^* + s_2^*$ and Lemma 6 that $\chi_{\mathrm{s}^*} \in \mathbf{X}$. Thus by Theorem 4, if χ exists it must be χ_{s^*}. Without loss of generality, we may assume that $s_3 > s_3'$, $s_2 \geq s_2'$ and hence $s_1^* = s_1 < s_1'$ from which, by condition (C), we get $s_3 > s_1 + 1$. We also have $s_3^* < s_1^* + s_2^* = s_1 + s_2'$ and $s_3^* = s_3' < s_3' + s_1 - 1$ as $s_1 > 1$. Hence $s_3^* < \min\{s_3,\ s_2' + s_1,\ s_3' + s_1 - 1\}$. We now consider the function $H = \chi_{\mathbf{s}^*}^{-1}\{\chi_{\mathbf{s}} \star \chi_{\mathbf{s}'}\}$. Since for some constant K

$$\begin{aligned}
&\chi_{\mathbf{s}} \star \chi_{\mathbf{s}'}(x,y) \\
= &\left\{\int_{-\infty}^{\frac{x}{2}}\int_{-\infty}^{\infty} + \int_{\frac{x}{2}}^{\infty}\int_{-\infty}^{\infty}\right\} \chi_{\mathbf{s}}(x-u, y-v)\, \chi_{\mathbf{s}'}(u,v)\, du\, dv \\
\leq &\int_{-\infty}^{\frac{x}{2}}\int_{-\infty}^{\infty} \left(1 + (x-u)^2\right)^{-\frac{s_1}{2}} \left(1 + (y-v)^2\right)^{-\frac{s_3 - s_1}{2}} \chi_{\mathbf{s}'}(u,v)\, du\, dv \\
&+ \int_{-\infty}^{\frac{x}{2}}\int_{-\infty}^{\infty} \left(1 + (x-u)^2\right)^{-\frac{s_1'}{2}} \left(1 + (y-v)^2\right)^{-\frac{s_3' - s_1'}{2}} \chi_{\mathbf{s}}(u,v)\, du\, dv \\
\preceq &\ K\left(1 + x^2\right)^{-\frac{s_1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{\mathbf{s}}(u,v) \left(1 + (y-v)^2\right)^{-\frac{s_3 - s_1}{2}} du\, dv \\
&+ K\left(1 + x^2\right)^{-\frac{s_1'}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{\mathbf{s}}(u,v) \left(1 + (y-v)^2\right)^{-\frac{s_3' - s_1'}{2}} du\, dv,
\end{aligned}$$

we have, noting that $s_1^* = s_1 < s_1'$,

$$\begin{aligned}
0 \ \leq\ &\overline{\lim_{x\to\infty}}\, H(x,y) \\
\leq\ &K\left(1 + y^2\right)^{\frac{s_3^* - s_1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{\mathbf{s}'}(u,v) \left(1 + (y-v)^2\right)^{-\frac{s_3 - s_1}{2}} du\, dv.
\end{aligned}$$

¿From Theorem 1,

$$\int_{-\infty}^{\infty} \chi_{\mathbf{s}'}(u,v)\, du \approx \begin{cases} \left(1 + v^2\right)^{-\frac{\min\{s_2', s_3' - 1\}}{2}} & \text{if } s_3' - s_2' \neq 1 \\ \left(1 + v^2\right)^{-\frac{s_2'}{2}} \log\left(2 + v^2\right) & \text{if } s_3' - s_2' = 1. \end{cases}$$

Thus, if $s_3' - s_2' \neq 1$ and $s_3 - s_1 > 1$, $\left(1 + y^2\right)^{-(s_3^* - s_1)/2}\ \overline{\lim_{x\to\infty}}\, H(x,y)$, as a function of y, is bounded above by

$$\begin{aligned}
&K\int_{-\infty}^{\infty} \left(1 + v^2\right)^{-\frac{\min\{s_2', s_3' - 1\}}{2}} \left(1 + (y-v)^2\right)^{-\frac{s_3 - s_1}{2}} dv \\
\leq\ &K'\left\{\left(1 + y^2\right)^{-\frac{s_3 - s_1}{2}} + \left(1 + y^2\right)^{-\frac{\min\{s_2', s_3' - 1\}}{2}}\right\},
\end{aligned}$$

where K and K' are some constants. Similarly, if $s_3' - s_2' = 1$ and $s_3 - s_1 > 1$, then the function $\left(1 + y^2\right)^{-(s_3^* - s_1)/2}\ \overline{\lim_{x\to\infty}}\, H(x,y)$ is bounded above by

$$K\int_{-\infty}^{\infty} \log\left(2 + v^2\right) \left(1 + v^2\right)^{-\frac{s_2'}{2}} \left(1 + (y-v)^2\right)^{-\frac{s_3 - s_1}{2}} dv$$

$$\begin{aligned}
&\leq K\int_{-\infty}^{\infty}(1+v^2)^{-\frac{s_2'-\epsilon}{2}}(1+(y-v)^2)^{-\frac{s_3-s_1}{2}}\,dv\\
&\leq K'\left\{(1+y^2)^{-\frac{s_3-s_1}{2}}+(1+y^2)^{-\frac{s_2'-\epsilon}{2}}\right\}
\end{aligned}$$

for any $s_2' - 1 > \epsilon > 0$, where K and K' are some constants. It follows from $s_3^* < \min\{s_3, s_2' + s_1, s_3' + s_1 - 1\}$ that

$$\begin{aligned}
0 &\leq \overline{\lim_{y\to\infty}}\ \overline{\lim_{x\to\infty}}\ H(x,y)\\
&\leq K'\lim_{y\to\infty}(1+y^2)^{\frac{s_3^*-s_1}{2}}\left\{(1+y^2)^{-\frac{s_3-s_1}{2}}+(1+y^2)^{-\frac{\min\{s_2',s_3'-1\}}{2}}\right\}=0
\end{aligned}$$

if $s_3' - s_2' \neq 1$, and, similarly, that $\overline{\lim}_{y\to\infty}\ \overline{\lim}_{x\to\infty}\ H(x,y) = 0$ if $s_3' - s_2' = 1$ as we can always choose $\epsilon > 0$ such that $s_3^* < \min\{s_3, s_2' + s_1 - \epsilon\}$ whenever $s_3^* < \min\{s_3, s_2' + s_1\}$. Thus there is no constant $K > 0$ such that, $\forall (x,y) \in \mathbb{R}^2$, $\{\chi_{\mathbf{s}} \star \chi_{\mathbf{s}'}\}(x,y) \geq K\chi_{\mathbf{s}^*}(x,y)$, i.e. $\chi_{\mathbf{s}^*}$ is not dominated by $\chi_{\mathbf{s}} \star \chi_{\mathbf{s}'}$. □

6. Conflict

In this section we return to studying the behaviour of $q_{x,y}$ as (x,y) moves far from $(0,0)$, in particular using the information we have gained above about the relevant convolutions. As in the one-dimensional case, the properties of the posterior distribution, which we are going to prove in the following, correspond to outlier rejection problems. Without loss of generality, we may assume that $x > 0$ and $y > 0$.

For any $a > 0$, we have by Lemma 4 that, for some constant K,

$$\begin{aligned}
&\mathbf{P}\{U < -ax \mid x, y\}\\
&\leq K\left\{\int_{-\infty}^{-ax}\int_{|v|>\frac{y}{2}}\chi_{\mathbf{s}'}(x-u, y-v)\,du\,dv+\int_{-\infty}^{-ax}\int_{|v|\leq\frac{y}{2}}\chi_{\mathbf{s}}(u,v)\,du\,dv\right\}\\
&\leq K\left\{\int_{(1+a)x}^{\infty}\int_{-\infty}^{\infty}\chi_{\mathbf{s}'}(u,v)\,du\,dv+\int_{-\infty}^{-ax}\int_{-\infty}^{\infty}\chi_{\mathbf{s}}(u,v)\,du\,dv\right\}.
\end{aligned}$$

Hence $\mathbf{P}\{U < -ax \mid x, y\}$ will vanish as $x \to \infty$. Similarly, for any $a > 1$, $\lim_{x\to\infty} \mathbf{P}\{U > ax \mid x, y\} = 0$. That the random variable V has similar properties can be seen by interchanging the roles of x and U with those of y and V respectively.

For the remainder of this section, we shall assume that $u, v : \mathbb{R}_+ \to \mathbb{R}_+$ are any functions such that, as $x \to \infty$, $u(x) \to \infty$, $v(x) \to \infty$ and the limits of

$u(x)/x$ and $v(x)/x$ both exist, possibly infinite. In the case when $u(x) \leq ax$ for some $a > 0$, we always have that, for any $b < 1$,

$$\begin{aligned}
& \mathbf{P}\{u(x) \leq |U| \leq ax,\ V \leq by \,|\, x, y\} \\
\leq\ & K \{\chi_{\mathbf{s}} \star \chi_{\mathbf{s}'}\}^{-1}(x,y) \{ \{\chi_{\mathbf{s}} \star \chi_{\mathbf{s}'}\}(x - ax, y - by) \\
& + \{\chi_{\mathbf{s}} \star \chi_{\mathbf{s}'}\}(x + u(x), y - by)\} \int_{u(x)}^{\infty} \int_{-\infty}^{\infty} \chi_{\mathbf{s}}(u,v)\, du\, dv \\
\leq\ & K' \int_{u(x)}^{\infty} \int_{-\infty}^{\infty} \chi_{\mathbf{s}}(u,v)\, du\, dv,
\end{aligned}$$

for some constants K and K', so that $\mathbf{P}\{u(x) \leq |U| \leq ax, V \leq by \,|\, x, y\}$ will tend to zero as $x \to \infty$.

The above facts will be taken for granted without further mention in the proofs of the following three theorems. For simplicity, we shall write $\mathbf{t} = (t_1, t_2, t_3)$ where $t_i = s_i - s_i'$, $i = 1, 2, 3$.

THEOREM 6. *If* $\mathbf{t} > \mathbf{0}$, *then*

$$\lim_{x\to\infty} \mathbf{P}\{|U| \leq u(x) \,|\, x, y\} = 1, \qquad \lim_{y\to\infty} \mathbf{P}\{|V| \leq v(y) \,|\, x, y\} = 1,$$

and

$$\lim_{x,y\to\infty} \mathbf{P}\{|U| \leq u(x),\ |V| \leq v(y) \,|\, x, y\} = 1.$$

If $\mathbf{t} < \mathbf{0}$, *then*

$$\lim_{x\to\infty} \mathbf{P}\{|U - x| \leq u(x) \,|\, x, y\} = 1, \qquad \lim_{y\to\infty} \mathbf{P}\{|V - y| \leq v(y) \,|\, x, y\} = 1,$$

and

$$\lim_{x,y\to\infty} \mathbf{P}\{|U - x| \leq u(x),\ |V - y| \leq v(y) \,|\, x, y\} = 1.$$

PROOF. Suppose that $\mathbf{t} > \mathbf{0}$. Then $\chi_{\mathbf{s}} \star \chi_{\mathbf{s}'} \approx \chi_{\mathbf{s}'} \succeq \chi_{\mathbf{s}}$ and $\lim_{x\to\infty} \chi_{\mathbf{t}}(x,y) = \lim_{y\to\infty} \chi_{\mathbf{t}}(x,y) = 0$. Thus, for any $0 < a$, $b < 1/2$, since $\mathbf{P}\{U > ax,\ V > by \,|\, x, y\}$ is bounded above by a constant multiple of $\chi_{\mathbf{t}}(x,y)$ it follows that $\lim_{x\to\infty} \mathbf{P}\{U > ax,\ V > by \,|\, x, y\} = 0$. The fact that the limit as $x \to \infty$ of $\mathbf{P}\{U > ax,\ |V| \leq$

$by \mid x, y\}$ is equal to zero is implied by the following inequality relation.

$$\begin{aligned} & \mathbf{P}\{U > ax,\ |V| \le by \mid x, y\} \\ \le\ & K\left(1+x^2\right)^{-\frac{t_3-t_2}{2}} \int_{ax}^{\infty} \int_{|v|\le by} \left(1+(x-u)^2\right)^{-\frac{s'_3-s'_2}{2}} \\ & \times \left(1+v^2\right)^{-\frac{s_3-s_1}{2}} \left(1+u^2+v^2\right)^{-\frac{t_1+t_2-t_3}{2}} \\ & \times \left(1+(x-u)^2+v^2\right)^{-\frac{s'_1+s'_2-s'_3}{2}} du\, dv \\ \le\ & K\left(1+x^2\right)^{-\frac{\min\{t_1,t_3\}}{2}} \int_{ax}^{\infty} \int_{|v|\le by} \chi_{\mathbf{s}'}(x-u, v)\, du\, dv \\ \le\ & K'\left(1+x^2\right)^{-\frac{\min\{t_1,t_3\}}{2}}, \end{aligned} \qquad \ldots(4)$$

where K and K' are some constants. Hence, we have $\lim_{x\to\infty} \mathbf{P}\{|U| \le ax \mid x, y\} = 1$. To show that $\lim_{x\to\infty} \mathbf{P}\{|U| \le u(x) \mid x, y\} = 1$, we now only need to consider the case that the limit of $u(x)/x$ is zero as $x \to \infty$. For, otherwise, there exists a constant $0 < a < 1/2$ such that $u(x) \ge ax$ when x is sufficiently large from which we may deduce what we require. If the limit of $u(x)/x$ is zero as $x \to \infty$ then, for any $0 < a < 1/2$, there exists an x_0 such that $x > x_0$ implies that $u(x) < ax$. Thus, without loss of generality, we may assume that $u(x) \le ax$ for all x. Then, by applying a similar technique to that in (4) we have, for any $0 < b < 1$, that

$$\begin{aligned} & \mathbf{P}\{u(x) \le |U| \le ax,\ V > by \mid x, y\} \\ \le\ & K\left(1+y^2\right)^{-\frac{\min\{t_2,t_3\}}{2}} \chi_{\mathbf{s}'}^{-1}(x, y) \\ & \times \left\{ \chi_{\mathbf{s}'}(x-ax, by) \int_{u(x)}^{ax} \int_{|v|>by} \chi_{\mathbf{s}'}(u, y-v)\, du\, dv \right. \\ & \left. + \chi_{\mathbf{s}'}(x+u(x), by) \int_{-ax}^{-u(x)} \int_{|v|>by} \chi_{\mathbf{s}'}(u, y-v)\, du\, dv \right\} \\ \le\ & K' \int_{u(x)}^{\infty} \int_{-\infty}^{\infty} \chi_{\mathbf{s}'}(u, v)\, du\, dv, \end{aligned}$$

where K and K' are two constants, so that $\lim_{x\to\infty} \mathbf{P}\{|U| \le u(x) \mid x, y\} = 1$ as required. Similarly, we have $\lim_{y\to\infty} \mathbf{P}\{|V| \le v(y) \mid x, y\} = 1$. These two results, together with a similar argument to that above, imply that $\lim_{x,y\to\infty} \mathbf{P}\{|U| \le u(x),\ |V| \le v(y) \mid x, y\} = 1$.

The proof for the case when $\mathbf{t} < \mathbf{0}$ is similar. □

This theorem shows the simplest kind of outlier rejection behaviour. The condition $\mathbf{t} > \mathbf{0}$ implies not only that $\chi_{\mathbf{s}} \preceq \chi_{\mathbf{s}'}$, or equivalently $f \preceq g$, but also that the ratio $\chi_{\mathbf{s}}/\chi_{\mathbf{s}'}$, or equivalently f/g, tends to zero as $x \to \infty$ or $y \to \infty$. Then as conflict arises between the prior density f and the likelihood g, it is resolved by rejecting the data. This conclusion becomes clear if we consider the strongest case of the theorem, in which $u(x)/x \to 0$ and $v(y)/y \to 0$ as $x, y \to \infty$. Then the posterior probability concentrates in the region $|U| \leq u(x)$, $|V| \leq v(y)$. This is in agreement with the prior information, but constitutes a rejection of the data (as expressed in the likelihood g) which suggested that (U, V) should lie in a neighbourhood of (x, y). (As shown in section 4, there is a posterior mode in the neighbourhood of (x, y), but we now see that its probability content is ultimately vanishing.) It contrasts also with the posterior compromise which results in the case of normal distributions, for then the posterior probability both in this region and in a corresponding neighbourhood of (x, y) would tend to zero.

Conversely, if the prior dominates, i.e. if $\mathbf{t} < \mathbf{0}$, then the conflict is always resolved by rejecting the prior information. The remainder of this section deals with the more complex resolutions of conflict when neither prior nor likelihood deminates the other. In particular, in Theorem 8 we see the posterior probability concentrating around either $(0, y)$ or $(x, 0)$ as $x, y \to \infty$. Then we have the prior information dominating in one dimension but the data in the other.

THEOREM 7. *Suppose that* $s_3 = s_3' < s_1^* + s_2^*$ *and that* $s_3 - s_i > 1$, $s_3' - s_i' > 1$, $i = 1, 2$, *and* $t_1 t_2 < 0$. *If* $t_1 > 0$, *then*

$$\lim_{x\to\infty} \mathbf{P}\{|U| \leq u(x) \,|\, x, y\} = 1$$

and

$$\lim_{x,y\to\infty} \mathbf{P}\{|U| \leq u(x),\ |V| \leq v(y) \,|\, x, y\} = 1;$$

while if $t_1 < 0$, *then*

$$\lim_{y\to\infty} \mathbf{P}\{|V| \leq v(y) \,|\, x, y\} = 1$$

and

$$\lim_{x,y\to\infty} \mathbf{P}\{|U| \leq u(x),\ |V| \leq v(y) \,|\, x, y\} = 1.$$

PROOF. Without loss of generality, we may assume that $t_1 > 0$ and $t_1 + t_2 \leq 0$. Since $t_3 = 0$, $\mathbf{s}^* = (s_1', s_2, s_3')$. Thus, for any $0 < a$, $b < 1/2$, we have

$$\begin{aligned} &\mathbf{P}\{U > ax,\ |V| \leq by \,|\, x, y\} \\ \leq\ & K\left(1 + x^2 + y^2\right)^{-\frac{t_1}{2}} \int_{ax}^{\infty} \int_{|v|\leq by} \left(1 + (x-u)^2\right)^{-\frac{s_3'-s_2'}{2}} \left(1 + v^2\right)^{-\frac{s_3-s_1}{2}} \\ & \times \left(1 + (x-u)^2 + v^2\right)^{-\frac{s_1+s_2-s_3}{2}} \left(1 + (x-u)^2 + (y-v)^2\right)^{\frac{t_1+t_2-t_3}{2}} du\, dv \end{aligned}$$

$$\begin{aligned} &\leq K\left(1+x^2+y^2\right)^{-\frac{t_1}{2}} \int_{ax}^{\infty} \int_{|v|\leq by} \chi_{\mathbf{S}^*}(x-u,v)\,du\,dv \\ &\leq K'\left(1+x^2+y^2\right)^{-\frac{t_1}{2}}, \end{aligned}$$

where K and K' are two constants, while the vanishing of $\mathbf{P}\{U > ax,\, V > by \,|\, x,y\}$ as $x \to \infty$ follows from $\{\chi_{\mathbf{S}^*}^{-1}\chi_{\mathbf{S}}\} = \chi_{(t_1,0,0)}$. Therefore, we obtain $\lim_{x\to\infty} \mathbf{P}\{|U| \leq ax \,|\, x,y\} = 1$. Since for some constants K and K'

$$\begin{aligned} &\mathbf{P}\{|U| \leq ax,\, V > by \,|\, x,y\} \\ \leq\;& K\chi_{\mathbf{S}^*}^{-1}(x,y) \int_{by}^{\infty} \int_{|u|\leq ax} \left(1+u^2\right)^{-\frac{s_3-s_2}{2}} \left(1+(x-u)^2\right)^{-\frac{s_1'}{2}} \\ &\times \left(1+v^2\right)^{-\frac{s_2}{2}} \left(1+(y-v)^2\right)^{-\frac{s_3'-s_1'}{2}}\,du\,dv \qquad \ldots(5) \\ \leq\;& K'\left\{\chi_{\mathbf{S}^*}^{-1}\chi_{(s_1',s_2,s_1'+s_2)}\right\}(x,y) \\ =\;& K'\chi_{(0,0,s_1'+s_2-s_3')}(x,y), \end{aligned}$$

we deduce that $\lim_{x,y\to\infty} \mathbf{P}\{|U| \leq ax,\, |V| \leq by \,|\, x,y\} = 1$. To obtain the required results, we now consider the case that $u(x) \leq ax$ for some $0 < a < 1/2$ as before. Following the lines of (5) and the relevant part of the proof of Theorem 6, we have that for some constant K

$$\begin{aligned} &\mathbf{P}\{u(x) \leq |U| \leq ax,\, V > by \,|\, x,y\} \\ \leq\;& K \int_{u(x)}^{\infty} \int_{-\infty}^{\infty} \chi_{(s_3-s_2,s_3'-s_1',s_3+s_3'-s_2-s_2')}(u,v)\,du\,dv, \end{aligned}$$

so that $\lim_{x\to\infty} \mathbf{P}\{|U| \leq u(x) \,|\, x,y\} = 1$. Similarly, $\lim_{x,y\to\infty} \mathbf{P}\{|U| \leq u(x),\, |V| \leq v(y) \,|\, x,y\} = 1$.

The remaining cases follow by a similar argument. □

THEOREM 8. *Suppose that* $s_3^* > s_1^* + s_2^*$. *If* $t_1 \geq 0$, *then*

$$\lim_{x\to\infty} \mathbf{P}\{|U| \leq u(x) \,|\, x,y\} = 1, \qquad \lim_{y\to\infty} \mathbf{P}\{|V-y| \leq v(y) \,|\, x,y\} = 1$$

and

$$\lim_{x,y\to\infty} \mathbf{P}\{|U| \leq u(x),\, |V-y| \leq v(y) \,|\, x,y\} = 1.$$

If $t_2 \geq 0$, *then*

$$\lim_{x\to\infty} \mathbf{P}\{|U-x| \leq u(x) \,|\, x,y\} = 1, \qquad \lim_{y\to\infty} \mathbf{P}\{|V| \leq v(y) \,|\, x,y\} = 1$$

and

$$\lim_{x,y\to\infty} \mathbf{P}\{|U-x| \le u(x),\, |V| \le v(y) \,|\, x,y\} = 1.$$

PROOF. The conditions $s_3^* > s_1^* + s_2^*$ and $t_1 \ge 0$ together imply that $t_2 \le 0$, and so $s_3 - s_2 > s_1' = s_1^*$ and $s_3' - s_1' > s_2 = s_2^*$. Thus, $\chi_{\mathbf{s}} \preceq \chi_{\hat{\mathbf{s}}}$ and $\chi_{\mathbf{s}'} \preceq \chi_{\hat{\mathbf{s}}}$ where $\hat{\mathbf{s}} = (s_1^*, s_2^*, s_1^* + s_2^*)$. This implies that, for any $0 < a,\, b < 1$,

$$\begin{aligned} & \mathbf{P}\{U > ax,\, V > -by \,|\, x,y\} \\ \le\ & K\left(1+x^2\right)^{-\frac{s_3-s_2-s_1^*}{2}} \left\{ \int_{ax}^{\infty} \int_{|v|\le by} \left(1+(x-u)^2\right)^{-\frac{s_1^*}{2}} \left(1+v^2\right)^{-\frac{s_2^*}{2}} du\, dv \right. \\ & \left. + \int_{ax}^{\infty} \int_{by}^{\infty} \left(1+(x-u)^2\right)^{-\frac{s_1^*}{2}} \left(1+(y-v)^2\right)^{-\frac{s_2^*}{2}} du\, dv \right\} \\ \le\ & K'\left(1+x^2\right)^{-\frac{s_3-s_2-s_1^*}{2}}, \end{aligned}$$

where K and K' are two constants, so that $\lim_{x\to\infty} \mathbf{P}\{|U| \le ax \,|\, x,y\} = 1$. Similarly, we have $\lim_{y\to\infty} \mathbf{P}\{|V-y| \le by \,|\, x,y\} = 1$. Since $\mathbf{P}\{|U| \le ax,\, |V-y| > by \,|\, x,y\}$ is dominated by $\left(1+y^2\right)^{-(s_3'-s_1'-s_2^*)/2}$, we have $\lim_{x,y\to\infty} \mathbf{P}\{|U| \le ax,\, |V-y| \le by \,|\, x,y\} = 1$. As in the proofs of the last two theorems, we now suppose that $u(x) \le ax$ for $0 < a < 1/2$. Then it follows from

$$\begin{aligned} & \mathbf{P}\{u(x) \le |U| \le ax,\, V > by \,|\, x,y\} \\ \le\ & K\chi_{\hat{\mathbf{s}}}^{-1}(x,y) \left\{ \chi_{\hat{\mathbf{s}}}(x-ax, by) \int_{u(x)}^{ax} \int_{v>by} \chi_{\hat{\mathbf{s}}}(u, y-v)\, du\, dv \right. \\ & \left. + \chi_{\hat{\mathbf{s}}}(x+u(x), by) \int_{-ax}^{-u(x)} \int_{v>by} \chi_{\hat{\mathbf{s}}}(u, y-v)\, du\, dv \right\} \\ \le\ & K' \int_{u(x)}^{\infty} \int_{-\infty}^{\infty} \chi_{\hat{\mathbf{s}}}(u,v)\, du\, dv, \end{aligned}$$

where K and K' are two constants, that $\lim_{x\to\infty} \mathbf{P}\{|U| \le u(x) \,|\, x,y\} = 1$. Similarly, we have $\lim_{y\to\infty} \mathbf{P}\{|V-y| \le v(y) \,|\, x,y\} = 1$, and so $\lim_{x,y\to\infty} \mathbf{P}\{|U| \le u(x),\, |V-y| \le v(y) \,|\, x,y\} = 1$.

Interchanging the roles of x, t_1 and U with those of y, t_2 and V respectively will give the required results when $t_2 \ge 0$. □

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