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ON THE LORENZ ORDER WITHIN PARAMETRIC FAMILIES OF INCOME DISTRIBUTIONS*

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SUMMARY. This paper provides a unified approach to Lorenz ordering within almost all commonly considered families of income distributions. For the most general model, the four-parameter generalized beta of the second kind (GB2), I derive a new sufficient condition. Unlike previous results, this criterion is empirically relevant; I find that inequality was lower in Germany than in the US in 1985 when both empirical distributions are approximated by the GB2.

1. **Introduction**

Atkinson's (1970) classic paper has created much interest in stochastic orders for the comparison of income distributions such as the Lorenz order. It is therefore quite surprising that only recently attempts have been made to characterize the Lorenz order within common parametric families of income distributions. For one- and two-parameter models this is straightforward, and it is well-known that the Lorenz order is linear within the Pareto and log-normal families (Arnold, 1987). Within three- and four-parameter families the Lorenz order is no longer linear, however. For the most popular three-parameter model, the Singh-Maddala, and the even closer fitting Dagum distributions the problem has been settled by Wilfling and Krämer (1993) and Kleiber (1996), respectively. Both distributions are special cases of McDonald's (1984) four-parameter generalized beta of the second kind (hereafter: GB2). For this family, Wilfling (1996) was able to derive a necessary and also a sufficient condition (but not, as Maasoumi, 1997, p. 224, states, a necessary *and* sufficient condition). However, the latter is so restrictive that it never applies in practice.

By a different technique, I find a much weaker sufficient condition below. The present framework also provides a unified approach to Lorenz ordering within all subfamilies of the GB2. The new condition is conclusive in some applications.

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2. **An Almost Necessary Condition for the GB2**

Let $\mathcal{L}$ denote the set of all non-negative random variables with finite and positive means. For $X \in \mathcal{L}$ with distribution function F_X and quantile function $F_X^{-1}(u) = \sup\{x | F_X(x) \leq u\}$, $u \in [0,1]$, the Lorenz curve is given by

$$L_X(p) = \frac{1}{E(X)} \int_0^p F_X^{-1}(u)\, du\ , \quad p \in [0,1].$$

For $X_1, X_2 \in \mathcal{L}$, the random variable X_1 is said to be at least as unequal as X_2 in the Lorenz sense if $L_{X_1}(p) \leq L_{X_2}(p)$ for all $p \in [0,1]$. This is denoted as $X_1 \geq_L X_2$.

Arnold (1987) provides a number of necessary and sufficient conditions for Lorenz ordering. A sufficient condition required below is in terms of star-shaped ordering (see, e.g., Barlow and Proschan, 1975) which is defined as follows: X_1 is said to be at least as large as X_2 in the sense of the star-shaped order, symbolically: $X_1 \geq_* X_2$, if $F_1^{-1}F_2$ is star-shaped, i.e. $F_1^{-1}F_2(x)/x$ is increasing. Like Lorenz ordering, star-shaped ordering is a scale-invariant concept; for distributions in $\mathcal{L}$ it implies Lorenz ordering. The latter result is frequently attributed to Taillie (1981) or Moothathu (1991), but is already contained in a little-known paper of Yanagimoto and Sibuya (1976).

Now, McDonald (1984) suggested the generalized beta of the second kind (GB2) as a model for the size distribution of incomes. This four-parameter family has density

$$f_{GB2}(x) = \frac{a\, x^{ap-1}}{b^{ap}\, B(p,q)\, [1+(x/b)^a]^{p+q}}\ , \qquad \ldots(1)$$

where $B(\cdot,\cdot)$ is the beta function, x is non-negative and all four parameters a, b, p, q are positive. The parameter b is a scale parameter, the others are shape parameters. Note that a GB2 distribution has finite mean - and, therefore, is in $\mathcal{L}$ - if and only if $aq > 1$. This model nests most of the previously considered functions as special or limiting cases, such as the Singh-Maddala (for $p = 1$), Dagum (for $q = 1$) or the beta of the second kind (B2, for $a = 1$). Here, a three-parameter limiting case is required, the generalized gamma (GG) distribution, with density

$$f_{GG}(x) = \frac{a}{b^{ap}\, \Gamma(p)}\, x^{ap-1}\, e^{-(x/b)^a}\ , \qquad \ldots(2)$$

where $\Gamma(\cdot)$ is the gamma function. For further details and more than twenty other distributions related to the GB2, see McDonald and Xu (1995).

For X_1, X_2 in $\mathcal{L}$, $X_i \sim \text{GB2}(a_i, b_i, p_i, q_i)$, i=1,2, Wilfling (1996) found that $X_1 \geq_L X_2$ implies both $a_1p_1 \leq a_2p_2$ and $a_1q_1 \leq a_2q_2$, whereas $a_1 \leq a_2$, $p_1 \leq p_2$, and $q_1 \leq q_2$ together imply $X_1 \geq_L X_2$. Note that this result already contains a complete characterization for the B2 subfamily, but not for the Singh-Maddala and Dagum distributions. More precisely, Wilfling's result leaves open the following four parameter constellations:

(i) $a_1 \leq a_2$, $p_1 \leq p_2$, and $q_1 \geq q_2$, but $a_1 q_1 \leq a_2 q_2$;

(ii) $a_1 \leq a_2$, $p_1 \geq p_2$, and $q_1 \leq q_2$, but $a_1 p_1 \leq a_2 p_2$;

(iii) $a_1 \leq a_2$, $p_1 \geq p_2$, and $q_1 \geq q_2$, but $a_1 p_1 \leq a_2 p_2$ and $a_1 q_1 \leq a_2 q_2$;

(iv) $a_1 \leq a_2$, $p_1 \geq p_2$, and $q_1 \geq q_2$, but $a_1 p_1 \geq a_2 p_2$ and $a_1 q_1 \geq a_2 q_2$.

The following theorem settles all but the last.

THEOREM 1. *Let X_1, X_2 be in $\mathcal{L}$, $X_i \sim GB2(a_i, b_i, p_i, q_i)$, i=1,2. Then $a_1 \leq a_2$, $a_1 p_1 \leq a_2 p_2$, and $a_1 q_1 \leq a_2 q_2$ imply $X_1 \geq_L X_2$.*

PROOF: By scale-invariance of the star-shaped and Lorenz orders, b_i, i=1,2, may be set equal to one without loss of generality. Let $Y_i \sim \mathrm{GG}(a_i, 1, p_i)$, i=1,2, and $Z_i \sim \mathrm{GG}(a_i, 1, q_i)$, i=1,2, be jointly independent. Moreover, assume $a_1 \leq a_2$, $a_1 p_1 \leq a_2 p_2$, and $a_1 q_1 \leq a_2 q_2$. From Taillie (1981, Theorem 7) $Y_1 \geq_* Y_2$ and $Z_1 \geq_* Z_2$, hence $Y_1 \geq_L Y_2$ and $Z_1 \geq_L Z_2$. From Taillie (1981, Theorem 6) one also has $Z_1 \geq_* Z_2 \iff 1/Z_1 \geq_* 1/Z_2$, implying $1/Z_1 \geq_L 1/Z_2$. Now Whitt (1980, Lemma 2) shows that products of Lorenz-ordered random variables are also Lorenz ordered, thus

$$Y_1 \cdot \frac{1}{Z_1} \quad \geq_L \quad Y_2 \cdot \frac{1}{Z_2} \, .$$

(Note that Whitt's result is derived for 'convex ordering'. As Lorenz ordering is equivalent to convex ordering of the mean-scaled random variables, it is applicable here.) But from Ahuja (1969, Theorem 3), the Y_i/Z_i are distributed as $\mathrm{GB2}(a_i, 1, p_i, q_i)$ and the Theorem is proved. □

This theorem not only contains Wilfling's sufficient condition as a special case, but also yields Lorenz ordering results for all subfamilies of the GB2. Therefore, it provides a unified approach to Lorenz ordering within almost all commonly considered income distribution models.

For an empirical illustration, consider the following parameter estimates for the US and Germany (McDonald and Xu, 1995, and Brachmann, Stich, and Trede, 1996, respectively).

TABLE 1. GB2 ESTIMATES FOR THE US AND GERMANY, 1985

	a	p	q	ap	aq
USA	2.4890	0.5732	1.8870	1.4267	4.6967
Germany	3.2460	0.8227	1.4910	2.6705	4.8398

From the Theorem, the US income distribution of 1985 was at least as unequal as the corresponding German distribution. As this is a case (i) situation, this conclusion is only possible by the new criterion.

However, a complete characterization of the Lorenz order within the GB2 family is still not available. Evidently, the present approach cannot handle case (iv) above. Unfortunately, this last case is also relevant in applications: McDonald (1984) fitted the GB2 to US income data of 1970, 1975, and 1980. For any two pairs, the necessary conditions for Lorenz ordering are met, but not the sufficient conditions derived above. Hence, the corresponding Lorenz curves may or may not intersect. I conjecture that they do not and, therefore, that the necessary conditions are also sufficient. But this requires yet another proof.

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