Sankhyā : The Indian Journal of Statistics
Special issue on Ergodic Theory and Harmonic Analysis
2000, Volume 62, Series A, Pt. 3, pp. 291–306

ON THE SPECTRUM OF THE POWERS OF ORNSTEIN TRANSFORMATIONS*

By EL HOUCEIN EL ABDALAOUI
Université de Rouen-Mathématiques, Colbert

SUMMARY. Generalizing Bourgain's methods we prove that the singularity problem of the spectrum of rank one transformations is related to the singularity problem of the maximal spectral type of rank one transformations with respect to its translations. For the Ornstein transformations it is proved that almost surely the spectral types of these transformations are singular with respect to their rational translations, so the powers of these transformations have almost surely simple spectrum. We deduce that for any positive integer n, there exist mixing transformations of rank n with simple spectrum.

1. **Introduction**

Ornstein (1971), using a random procedure, introduced a class of rank one transformations and using some probabilistic methods showed the mixing property for a subclass of these transformations. Rank one transformations have simple spectrum. Ornstein's class of transformations would be a candidate to Banach's well-known problem whether there exists a dynamical system $(\Omega, \mathcal{A}, \mu, T)$ with simple Lebesgue spectrum. But, Bourgain (1993) proved that almost surely the Ornstein transformations have singular spectrum. Subsequently, using the same methods, Klemes (1994) and Klemes and Reinhold (1997) obtain that the spectrum of the mixing subclass of staircase transformations of Adams (1998) and Adams and Friedman (1993) have singular spectrum. They conjectured that rank one transformations have singular spectrum.

In this paper, using the techniques of Bourgain generalized in Abdalaoui (1999), it is shown that this conjecture is related to Thouvenot's problem : Do the powers of rank one transformations have simple spectrum?. In the case of certain classes of Ornstein transformations we prove that the answer is almost surely positive. It follows, by the King-Thouvenot (1988) theorem that there exists, for any $n \in \mathbb{N}^*$, a mixing rank n transformation with simple spectrum. This latter problem is also related to the question asked by Choksi and Nadkarni (1995) on the singularity of spectral types of rank one transformations with respect to their translations. In

---

AMS (1991) *subject classification.* Primary 28D05; secondary 47A35.
Key words and phrases. Dynamical system, simple spectrum, mixing maps, rank one transformation, Lebesgue spectrum.

*Dedicated to Professor M.G. Nadkarni.

fact, the authors studied the group of eigenvalues of rank one transformation and obtain an explicit formula of the Radon-Nikodym derivative of the spectral type with respect to its translation by an eigenvalue if it exists and they asked : are the measures σ and σ_α mutually singular when $\alpha \notin e(T)$? ($e(T)$ is the group of eigenvalues of T, σ is the spectral type of T and σ_α is the translation of σ by α, i.e., $\sigma_\alpha(A) = \sigma(\alpha^{-1}A)$, for any Borel set A).

In this paper, we obtain a class of mixing rank one transformations for which the spectral types are singular with respect to their translates by rational numbers. This constitutes a partial answer to the investigations of Choksi-Nadkarni (1998). Now, if a rank one transformation of spectral type σ has the property

$$\sigma\{\alpha : \sigma_\alpha \perp \sigma\} = 1, \tag{1.1}$$

then σ is singular. In fact, suppose that (1.1) obtains. Let μ and λ denote respectively the continuous part of σ and Lebesgue measure on torus $\mathbb{T}$. Let $\tau = \mu\widehat{\lambda}$, i.e., let τ be the largest measure such that $\tau \leq \mu$ and $\tau \leq \lambda$. Then we have $\tau_\alpha \leq \mu_\alpha$ and $\tau_\alpha \leq \lambda$. We also have

$$\lim_{\alpha \to 1} \tau_\alpha(A) = \tau(A), \tag{1.2}$$

for any Borel set A. Given $\epsilon > 0$, we further have

$$\mu([-\epsilon, \epsilon]) > 0. \tag{1.3}$$

Indeed, the topological support of μ, i.e., the complement of

$$C = \bigcup_{\substack{O \subset \mathbb{T}-open, \\ \mu(O)=0}} O,$$

is $\mathbb{T}$ [Lemańczyk, 1995, p. 81]. Thus, By (1.1), (1.2) and (1.3) we can find α such that

$$\mu_\alpha \perp \mu \quad \text{and} \quad ||\tau_\alpha - \tau|| < \epsilon, \tag{1.4}$$

where $||\ .\ ||$ is the total variation norm on the space $\mathcal{M}(\mathbb{T})$ of bounded Borel measures on $\mathbb{T}$. It follows from (1.4) that

$$2\tau(\mathbb{T}) < \epsilon.$$

Since ϵ was arbitrary, this proves that $\tau = 0$, i.e., that $\mu \perp \lambda$.

In section 3 we obtain a sufficient condition for (1.1) to hold.

1.1. *Rank one transformation by construction.* Using the cutting and stacking method described in Friedman (1970, 1992), one can define inductively a family of measure preserving transformations, called rank one transformations, as follows

Let B_0 be the unit interval equipped with the Lebesgue measure. At stage one we divide B_0 into p_0 equal parts, add spacers and form a stack of height h_1 in the usual fashion. At the k-th stage we divide the stack obtained at the $(k-1)$-th stage into p_{k-1} equal columns, add spacers and obtain a new stack of height h_k. If during the k-th stage of our construction the number of spacers put above the j-th column of the $(k-1)^{th}$ stack is $a_j^{(k-1)}$, $0 \leq a_j^{(k-1)} < \infty$, $1 \leq j \leq p_k$, then we have

$$h_k = p_{k-1} h_{k-1} + \sum_{j=1}^{p_{k-1}} a_j^{(k-1)}.$$

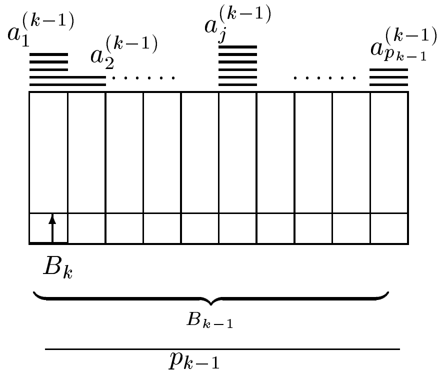

Figure 1 : k^{th}–tower

Proceeding in this way we get a rank one transformation T on a certain measure space $(X, \mathcal{B}, \nu)$ which may be finite or σ-finite depending on the number of spacers added.

The construction of any rank one transformation thus needs two parameters, $(p_k)_{k=0}^{\infty}$ (parameter of cutting and stacking), and $((a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}$ (parameter of spacers). Put

$$T \stackrel{def}{=} T_{(p_k, (a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}$$

In Choksi and Nadkarni (1994) and Klemes and Reinhold (1997) it is proved that the spectral type of this transformation is given by

$$\sigma \quad = \quad \omega^* \lim \prod_{k=1}^{n} |P_k|^2 \, d\lambda, \quad \text{where} \;\; P_k(z) = \frac{1}{\sqrt{p_k}} \left(\sum_{j=0}^{p_k - 1} z^{-(jh_k + \sum_{i=1}^{j} a_i^{(k)})} \right)$$

(λ denotes the normalized Lebesgue measure on torus $\mathbb{T}$ and ω^* is weak star convergence on space of measures on $\mathbb{T}$),

with possibly some discrete measures. The polynomials P_k appear naturally as consequences of the recursive relation between the bases B_k. In fact

$$B_k = B_{k+1} \cup T^{h_k + s_k(1)} B_{k+1} \cup \ldots \cup T^{(p_k-1)h_k + s_k(p_k-1)} B_{k+1},$$

$$\nu(B_k) = p_k \nu(B_{k+1}),$$

where $s_k(n) = a_1^{(k)} + \ldots + a_n^{(k)}$ and $s_k(0) = 0$.

Put

$$f_k = \frac{1}{\sqrt{\nu(B_k)}} 1_{B_k},$$

the characteristic function of the kth-base, normalized so that the 2-norm equals 1. So

$$f_k = P_k(U_T) f_{k+1},$$

where $U_T : L^2(X) \longrightarrow L^2(X)$ is defined by $U_T(f)(x) = f(T^{-1}x)$. Iterating this relationship, we have

$$d\sigma_k = |P_k|^2 \, d\sigma_{k+1} = \ldots = \prod_{j=0}^{m-1} |P_{k+j}|^2 \, d\sigma_{k+m},$$

σ_p being the spectral measure of f_p, $p \geq 0$.

Throughout this paper we will assume that T is weakly mixing so the weak limit above is the maximal spectral type of T.

The principal result of this paper is the subject of the following section.

2. The Powers of Subclass of Ornstein Transformations Have Almost Surely Simple Spectrum

In Ornstein's construction, the p_k's are rapidly increasing and the number of spacers, $a_i^{(k)}$, $1 \leq i \leq p_k - 1$, are chosen stochastically as follows : we choose independently, using the uniform distribution on the set $X_k = \{\frac{-h_{k-1}}{2}, \cdots, \frac{h_{k-1}}{2}\}$, the numbers $(x_{k,i})_{i=1}^{p_k-1}$, and x_{k,p_k} are chosen deterministically in $\mathbb{N}$. We put, for $1 \leq i \leq p_k$,

$$a_i^{(k)} = h_{k-1} + x_{k,i} - x_{k,i-1}, \quad \text{with } x_{k,0} = 0.$$

One sees that

$$h_{k+1} = p_k(h_k + h_{k-1}) + x_{k,p_k}.$$

So the deterministic sequence of positive integers $(p_k)_{k=0}^{\infty}$ and $(x_{k,p_k})_{k=0}^{\infty}$ completely determine the sequence of heights $(h_k)_{k=1}^{\infty}$. The total measure of the resulting measure space is finite if $\displaystyle\sum_{k=1}^{\infty} \frac{h_{k-1}}{h_k} + \sum_{k=1}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty$. We will assume that this requirement is satisfied. So we have $\displaystyle\sum_{k=0}^{\infty} \frac{1}{p_k} < \infty$. Indeed, we have

$$\begin{aligned} \frac{\frac{h_{k-1}}{h_k}}{\frac{1}{p_{k-1}}} &= \frac{p_{k-1} h_{k-1}}{p_{k-1}(h_{k-1} + h_{k-2}) + x_{k-1,p_{k-1}}} \\ &= \frac{1}{(1 + \frac{h_{k-2}}{h_{k-1}}) + \frac{x_{k-1,p_{k-1}}}{p_{k-1} h_{k-1}}} \longrightarrow 1, \quad \text{as } k \longrightarrow \infty. \end{aligned}$$

We thus have a probability space of Ornstein transformations $\prod_{l=1}^{\infty} X_l^{p_l-1}$ equipped with the natural probability measure $\otimes_{l=1}^{\infty} P_l$, where $P_l = \otimes_{l=1}^{p_l-1} \mathcal{U}_l$; $\mathcal{U}_l$ is the uniform probability on X_l. We denote this space by $(\Omega, \mathcal{A}, \mathbb{P})$. So $x_{k,i}$, $1 \leq i \leq p_k - 1$, is a projection from Ω onto the i-th co-ordinate space of Ω_k, $1 \leq i \leq p_k - 1$. Naturally each point $\omega = (\omega_k = (x_{k,i}(\omega))_{i=1}^{p_k-1})_{k=1}^{\infty}$ in Ω defines the spacers and therefore a rank one transformation which we denote by $T_{\omega,x}$ where $x = (x_{k,p_k})_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ is *admissible* i.e. $\sum_{k=1}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty$.

The definition above gives a general definition of random construction due to Ornstein. In the particular case of Ornstein's transformations constructed in [O], the probability space Ω is built step by step, continuing that for the construction of associated rank one transformations. Precisely, in order to construct the $(k+1)$-th stage we apply the probabilistic arithmetical lemma of Ornstein (1971) (see also Nadkarni (1998)) to get the cutting parameter $p_k >> h_{k-1}$ and the set of good spacers at this stage which is a subset of $X_k^{p_k-1}$. It follows that we can choose $p_k > h_k^2$. We shall henceforth assume that this requirement is satisfied. The numbers x_{k,p_k} are chosen between 1 and 4 to ensure the ergodicity of each power T^n of T, but this is not necessary as it is proved in Abdalaoui (2000). Each point $\omega = \left(\omega_k = (x_{k,i})_{i=1}^{p_k-1}\right)_{k \geq 1}$ in Ω defines the spacers and therefore a transformation T_ω. If ω is in $\Omega' \stackrel{\text{def}}{=} \{\omega : T_\omega \text{ mixing}\}$, then the spectral type of T_ω is given by

$$\sigma_{T_\omega} = \sigma_{1_{B_0}}^{(\omega)} = \sigma^{(\omega)} = \mu^{(\omega)} + \frac{1}{c}\delta_1 = \omega^* \lim \prod_{l=1}^{N} \frac{1}{p_l} \left| \sum_{p=0}^{p_l-1} z^{p(h_l+h_{l-1})+x_{l,p}} \right|^2 d\lambda,$$

$\mu^{(\omega)}$ being the continuous part of $\sigma^{(\omega)}$ and $\sqrt{c}$ the total measure of the dynamical system which doesn't depend on ω. In this section we will prove the following

Theorem 1. *For any $\alpha \in \mathbb{T}\backslash\{1\}$, the spectral types of Ornstein transformations are almost surely singular with respect to their translates by α, i.e.*

$$\mathbb{P}\{\omega : \quad \sigma_\alpha^{(\omega)} \perp \sigma^{(\omega)}\} = 1.$$

The proof of this theorem is based on the generalization of a criterion of spectral singularity of rank one transformation given by F. Bourgain (1993). But, as pointed to me by Parreau (in a private communication), the generalization of this criterion given in Abdalaoui (1999) can be used in the case of spectral type of rank one transformation and their translates. In fact, we have

Proposition 1. *If σ is the spectral type of a weak mixing rank one transformation and σ_α its translate by α then*

(a) *the sequence of measures* $\left(\prod_{n=1}^{N} \left| \frac{P_n(\alpha z)}{P_n(z)} \right|^2 d\sigma \right)_{N \in \mathbb{N}}$ *converge weakly to σ_α.*

(b) *the two sequences of measures* $\left(\left| \frac{P_m(\alpha z)}{P_m(z)} \right|^2 d\sigma \right)_{m \in \mathbb{N}}$ *and* $\left(\frac{d\sigma}{|P_m(z)|^2} \right)$ *converge weakly to σ.*

The proof of the above proposition can be obtained easily by computation of Fourier coefficients. From Proposition 1 we have the following proposition as proved in Abdalaoui (1999).

PROPOSITION 2. *The following are equivalent*

(i) $\sigma_\alpha \perp \sigma$,

(ii) $\displaystyle\int \prod_{n=1}^{N} \left| \frac{P_n(\alpha z)}{P_n(z)} \right| d\sigma \longrightarrow_{N\to\infty} 0.$

Now, using Lebesgue's dominated convergence theorem and the above proposition, we obtain :

PROPOSITION 3. *The following are equivalent*

(i) $\sigma_\alpha^{(\omega)} \perp \sigma^{(\omega)} \qquad \mathbb{P}\ a.s.$

(ii) $\displaystyle\int\int \prod_{n=1}^{N} \left| \frac{P_n(\alpha z)}{P_n(z)} \right| d\sigma^{(\omega)} d\mathbb{P} \underset{N\to\infty}{\longrightarrow} 0.$

Thus, as in Abdalaoui (1999), by Cauchy-Schwarz inequality one can prove the following

PROPOSITION 4. *The following are equivalent*

(i) $\sigma_\alpha^{(\omega)} \perp \sigma^{(\omega)} \qquad \mathbb{P}\ a.s.$

(ii) $\displaystyle\inf \left\{ \int\int \prod_{l=1}^{k} \left| \frac{P_{n_l}(\alpha z)}{P_{n_l}(z)} \right| d\sigma^{(\omega)} d\mathbb{P},\ k \in \mathbb{N},\ n_1 < n_2 < \ldots < n_k \right\} = 0.$

Now fix some subsequence $\mathcal{N} = \{n_1 < n_2 < \ldots < n_k\}$, $k \in \mathbb{N}$, $m > n_k$ and put

$$Q(z) = \prod_{i=1}^{k} \frac{P_{n_i}(\alpha z)}{P_{n_i}(z)}, \qquad P(z) = \frac{P_m(\alpha z)}{P_m(z)}.$$

Using the same arguments as in Abdalaoui (1999), one can show the following lemma and propositions.

LEMMA 1.

$$\int |Q|\,|P|\ d\sigma^{(\omega)}(z) d\mathbb{P} \quad \le \quad \frac{1}{2}\left(\int |Q| + \int |Q|\,|P|^2\right) - \frac{1}{8}\left(\int |Q| \left||P|^2 - 1\right|\right)^2.$$

PROPOSITION 5. $\displaystyle\lim_{m\to\infty} \int |Q| \left| \frac{P_m(\alpha z)}{P_m(z)} \right|^2 d\sigma^{(\omega)}(z) d\mathbb{P} \ = \ \int |Q|\ d\sigma^{(\omega)}(z) d\mathbb{P}.$

PROPOSITION 6.

$$\overline{\lim} \int |Q| \left| \frac{P_m(\alpha z)}{P_m(z)} \right| d\sigma^{(\omega)}(z) d\mathbb{P} \quad \le \quad \int |Q| - \frac{1}{8}\left(\underline{\lim} \int |Q| \left| \left| \frac{P_m(\alpha z)}{P_m(z)} \right|^2 - 1 \right| \right)^2.$$

In order to estimate $\int |Q| \left|\frac{P_m(\alpha z)}{P_m(z)}\right| d\sigma^{(\omega)}(z)d\mathbb{P}$, it is sufficient to estimate the quantity

$$\int |Q| \left| \left|\frac{P_m(\alpha z)}{P_m(z)}\right|^2 - 1\right| d\sigma^{(\omega)}(z)d\mathbb{P},$$

which is the limit of the following sequence

$$\int \prod_{j\in\mathcal{N}} |P_j(\alpha z)P_j(z)| \left||P_m(\alpha z)|^2 - |P_m(z)|^2\right| \prod_{j=1, j\notin\mathcal{N}, j\neq m}^{M} |P_j(z)|^2 d\lambda d\mathbb{P}.$$

But, as in Abdalaoui (1999), the contribution of Dirac measure at 1 can create some difficulties. Here, we have

$$\begin{aligned}\int |Q|\,(1)d\mathbb{P} &= \int \prod_{i=1}^{k} \left|\frac{P_{n_i}(\alpha)}{P_{n_i}(1)}\right| d\mathbb{P} = \int \prod_{i=1}^{k} \frac{1}{p_{n_i}} \left| \sum_{p=0}^{p_{n_i}-1} \alpha^{p(h_{n_i}+h_{n_i-1})+x_{n_i,p}} \right| d\mathbb{P} \\ &\leq \int \left| \frac{1}{p_{n_k}} \sum_{p=0}^{p_{n_k}-1} \alpha^{p(h_{n_k}+h_{n_k-1})+x_{n_k,p}} \right|^2 d\mathbb{P},\end{aligned}$$

and the last expression converges to 0 as k goes to $+\infty$. Now, the estimation of $\overline{\lim} \int |Q| \left|\frac{P_m(\alpha z)}{P_m(z)}\right| d\sigma^{(\omega)}(z)d\mathbb{P}$ will be controlled once we show the following proposition.

PROPOSITION 7.

$$\underline{\lim} \int\int |Q| \left| \left|\frac{P_m(\alpha z)}{P_m(z)}\right|^2 - 1\right| d\sigma^{(\omega)}d\mathbb{P} \geq K\left(\int\int |Q|\, d\sigma^{(\omega)}d\mathbb{P} - 4\int |Q|(1)d\mathbb{P}\right),$$

where K is positive constant.

The proof of this proposition is divided in two steps and will be the subject of the following section.

2.1 *Khintchine-Bonami inequality and Fejér kernel.*

2.1.1 *Step* 1. (Khintchine- Bonami inequality). Fix z and m, ω is in Ω. Define τ by :

$$\tau : \mathbb{Z} \longrightarrow \mathbb{T}, \qquad \tau(s) = z^s.$$

So,

$$|P_m(z)|^2 - 1 = \sum\nolimits_{p\neq q} a_{pq}\ \tau_p(\omega)\ \overline{\tau_q}(\omega), \quad \text{where} \ \ a_{pq} = \frac{z^{(p-q)(h_m+h_{m-1})}}{p_m},$$

where τ_p is defined by $\tau_p = \tau \circ x_{m,p}$ and $x_{m,p}$ is the $p-$th projection on $\Omega_m = X_m^{p_m-1}$.

By the definition of $\mathbb{P}$ the random variables $(\tau_p)_{p=1}^{p_m-1}$ are independent. Put

$$\tau^\circ = \tau - \frac{1}{h_{m-1}+1} \sum_{s=-\frac{h_{m-1}}{2}}^{\frac{h_{m-1}}{2}} z^s. \tag{2.1}$$

Then

$$\tau_p^\circ = \tau_p - \int \tau \; d\mathbb{P}$$

and

$$\sum a_{pq} \; \tau_p \; \overline{\tau_q} = (\sum a_{pq}) \left| \int \tau_1 \right|^2 + \sum a_{pq} (\int \overline{\tau_1} \; \tau_p^\circ + \int \tau_1 \; \overline{\tau_q^\circ}) + \sum a_{pq} \tau_p^\circ \; \overline{\tau_q^\circ}. \tag{2.2}$$

Now, using the same arguments as Bourgain (1993), let us consider a random sign $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_{p_m-1}\} \in \{-1,1\}^{p_m-1}$, and the probability space

$$Z_m = \Omega_m \times \{-1,1\}^{p_m-1} \;, \text{ where } \; \Omega_m = \left\{ -\frac{h_{m-1}}{2}, ..., \frac{h_{m-1}}{2} \right\}^{p_m-1}.$$

Taking the conditional expectation of the following quantity

$$\sum a_{pq} \left(\int \overline{\tau_1} \quad \tau_p^\circ + \int \tau_1 \quad \overline{\tau_q^\circ} \right) + \sum a_{pq} \tau_p^\circ \quad \overline{\tau_q^\circ}$$

with respect to the $\sigma-$algebra $\mathcal{B}$ given by the cylindrical sets $A(I,x)$ where $I \subset \{0, \ldots, p_m - 1\}$, $x \in \Omega_m$ and

$$A(I,x) = \prod_{i \in I} \{x_i\} \times \left\{ -\frac{h_{m-1}}{2}, ..., \frac{h_{m-1}}{2} \right\}^{|I^c|} \times \{1\}^{|I|} \times \{-1\}^{|I^c|}.$$

(I corresponds to $\varepsilon_i = 1$, $\forall i \in I$ and $\varepsilon_i = -1$, $\forall i \notin I$). In other words, taking conditional expectation with respect to the random variables τ_p for which $\varepsilon_p = 1$, one finds the following polynomial expression in ε of degree 2

$$\sum a_{pq} \left(\frac{1+\varepsilon_p}{2} \int \overline{\tau_1} \quad \tau_p^\circ + \frac{1+\varepsilon_q}{2} \int \tau_1 \quad \overline{\tau_q^\circ} \right) + \sum a_{pq} \frac{1+\varepsilon_p}{2} \frac{1+\varepsilon_q}{2} \tau_p^\circ \quad \overline{\tau_q^\circ}. \tag{2.3}$$

So,

$$\begin{aligned} &\int \left| |P_m(\alpha z)|^2 - |P_m(z)|^2 \right| d\mathbb{P} \\ = &\int \int \mathbb{E} \left(\left| |P_m(\alpha z)|^2 - |P_m(z)|^2 \right| \;\Big|\; \mathcal{B} \right) \; d\mathbb{P} d\varepsilon \qquad (2.4) \\ \geq &\int \int \left| \mathbb{E} \left(|P_m(\alpha z)|^2 - |P_m(z)|^2 \;\Big|\; \mathcal{B} \right) \right| \; d\mathbb{P} d\varepsilon. \end{aligned}$$

It follows, by the Khintchine-Bonami inequality, [1] Bourgain (1993), that there exists a positive constant K such that

$$\begin{aligned}
&\int\int\left|\mathbb{E}\Big(|P_m(\alpha z)|^2-|P_m(z)|^2 \ \Big|\ \mathcal{B}\Big)\right| d\mathbb{P}d\varepsilon\\
\geq\ & K\int\left[\int\left|\mathbb{E}\Big(|P_m(\alpha z)|^2-|P_m(z)|^2\ \Big|\ \mathcal{B}\Big)\right|^2 d\varepsilon\right]^{\frac{1}{2}} d\mathbb{P}\\
=\ & K\int\Big(\sum |\ a_{pq}(\alpha z)\tau_p^{\circ}(\alpha z)\overline{\tau_q^{\circ}(\alpha z)}-a_{pq}(z)\tau_p^{\circ}(z)\overline{\tau_q^{\circ}(z)}\ |^2\Big)^{\frac{1}{2}} d\mathbb{P}\\
=\ & K\int\left(\frac{1}{p_m^2}\sum |\ \alpha^{(p-q)(h_m+h_{m-1})}\tau_p^{\circ}(\alpha z)\overline{\tau_q^{\circ}}(\alpha z)-\tau_p^{\circ}(z)\overline{\tau_q^{\circ}(z)}\ |^2\right)^{\frac{1}{2}} d\mathbb{P}.
\end{aligned} \tag{2.5}$$

But all these random variables are bounded by 2. It follows that

$$\begin{aligned}
&\int\left||P_m(\alpha z)|^2-|P_m(z)|^2\right| d\mathbb{P}\\
\geq\ & K'\int\frac{1}{p_m^2}\sum |\ \alpha^{(p-q)(h_m+h_{m-1})}\tau_p^{\circ}(\alpha z)\overline{\tau_q^{\circ}}(\alpha z)-\tau_p^{\circ}(z)\overline{\tau_q^{\circ}(z)}\ |^2 d\mathbb{P}\\
=\ & K'\tfrac{1}{p_m^2}\sum\Bigg(\Big(\int|\tau_p^{\circ}(z)|^2 d\mathbb{P}\Big)^2+\Big(\int|\tau_p^{\circ}(\alpha z)|^2 d\mathbb{P}\Big)^2\\
&-\ \left|\int\overline{\tau}_p^{\circ}(z)\tau_p^{\circ}(\alpha z)d\mathbb{P}\right|^2\Big(\alpha^{(p-q)(h_m+h_{m-1})}+\alpha^{(q-p)(h_m+h_{m-1})}\Big)\Bigg)\\
\geq\ & K'\tfrac{p_m-1}{p_m}\left(\Big(\int|\tau_p^{\circ}(z)|^2 d\mathbb{P}\Big)^2-2\left|\int\overline{\tau}_p^{\circ}(z)\tau_p^{\circ}(\alpha z)d\mathbb{P}\right|^2\right).
\end{aligned} \tag{2.6}$$

Since

$$\int|\tau_p^{\circ}(z)|^2 d\mathbb{P}=var(\tau_p(z))=1-\left|\frac{1}{k_m}\sum_{s=0}^{k_m-1}z^s\right|^2 \tag{2.7}$$

where $k_m=h_{m-1}+1$, and

$$\begin{aligned}
&\left|\int\overline{\tau}_p^{\circ}(z)\tau_p^{\circ}(\alpha z)d\mathbb{P}\right|\\
=\ &\left|\int\tau_p(\alpha)d\mathbb{P}-\int\overline{\tau_p(z)}d\mathbb{P}\int\tau_p(\alpha z)d\mathbb{P}\right|\ \leq\ \left|\frac{1}{k_m}\sum_{s=0}^{k_m-1}\alpha^s\right|+\left|\frac{1}{k_m}\sum_{s=0}^{k_m-1}z^s\right|.
\end{aligned} \tag{2.8}$$

One checks easily from (2.8) the following

$$\left|\int\overline{\tau}_p^{\circ}(z)\tau_p^{\circ}(\alpha z)d\mathbb{P}\right|^2\ \leq\ 3\left|\tfrac{1}{k_m}\textstyle\sum_{s=0}^{k_m-1}\alpha^s\right|\ +\ \left|\tfrac{1}{k_m}\textstyle\sum_{s=0}^{k_m-1}z^s\right|^2. \tag{2.9}$$

[1] One can extend easily this inequality to the bounded sequences of independent real random variables, with vanishing expectation.

But $\alpha \neq 1$, therefore we have

$$\left|\frac{1}{k_m}\sum_{s=0}^{k_m-1}\alpha^s\right| \leq \frac{1}{k_m}\frac{2}{|1-\alpha|}\longrightarrow_{m\to+\infty}0.$$

Now, the inequalities (2.5), (2.6) and (2.9) combined with (2.7) imply that

$$\begin{aligned}\int \left||P_m(\alpha z)|^2 - |P_m(z)|^2\right| d\mathbb{P} \geq K'\left(1-\frac{1}{p_m}\right)\\ \left(\left(1-\left|\frac{1}{k_m}\sum_{s=0}^{k_m-1} z^s\right|^2\right)^2 - \left|\tfrac{6}{k_m}\textstyle\sum_{s=0}^{k_m-1}\alpha^s\right| - \left|\tfrac{2}{k_m}\textstyle\sum_{s=0}^{k_m-1} z^s\right|^2\right) \\ \geq K'\left(1-\tfrac{1}{p_m}\right)\left(1-\left|\tfrac{4}{k_m}\textstyle\sum_{s=0}^{k_m-1} z^s\right|^2 - \left|\tfrac{6}{k_m}\textstyle\sum_{s=0}^{k_m-1}\alpha^s\right|\right).\end{aligned} \tag{2.10}$$

Finally, multiply (2.10) by

$$\int \prod_{j\in\mathcal{N}} |P_j(\alpha z)P_j(z)| \prod_{j=1, j\notin\mathcal{N}, j\neq m}^{M} |P_j(z)|^2 \, d\mathbb{P}, \tag{2.11}$$

using the independence of (2.11) and $\mid |P_m(z)|^2 - |P_m(\alpha z)|^2|$. Integrating with respect to the Lebesgue measure and letting M go to $+\infty$, one sees that

$$\begin{aligned}\int\!\!\int |Q| \left|\left|\frac{P_m(\alpha z)}{P_m(z)}\right|^2 - 1\right| d\sigma^{(\omega)} d\mathbb{P} \ \geq \ K'\left(1-\frac{1}{p_m}\right)\left(\int\!\!\int |Q|\frac{d\sigma^{(\omega)}}{|P_m(z)|^2}d\mathbb{P}\right.\\ \left. -\ 4\int\!\!\int |Q|\frac{1}{(k_m)^2}\left|\sum_{s=0}^{k_m-1} z^s\right|^2 \frac{d\sigma^{(\omega)}}{|P_m(z)|^2}d\mathbb{P} \ - \ \left|\frac{6}{k_m}\sum_{s=0}^{k_m-1}\alpha^s\right| \int\!\!\int |Q|\frac{d\sigma^{(\omega)}}{|P_m(z)|^2}d\mathbb{P}\right).\end{aligned}$$

2.1.2 *Step* 2. (an estimation using the Fejér kernel). In order to complete the proof of Proposition 7, in what follows we will estimate the limit of the following sequence (or the limit of some subsequence of it)

$$\int |Q|\frac{d\sigma^{(\omega)}}{|P_m(z)|^2}d\mathbb{P} \ - \ 4\int |Q|\left|\frac{1}{k_m}\sum_{s=0}^{k_m-1} z^s\right|^2 \frac{d\sigma^{(\omega)}}{|P_m(z)|^2}d\mathbb{P}.$$

Considering the (b) of the Proposition 1 it is sufficient to estimate

$$\int |Q|\left|\frac{1}{k_m}\sum_{s=0}^{k_m-1} z^s\right|^2 \frac{d\sigma^{(\omega)}}{|P_m(z)|^2}d\mathbb{P},$$

by proving the following lemma.

LEMMA 2. *The sequence of measures*

$$\left|\frac{1}{k_m}\sum_{s=0}^{k_m-1} z^s\right|^2 \frac{d\sigma^{(\omega)}}{|P_m(z)|^2}$$

converge weakly to $\frac{1}{c}\delta_1$.

PROOF. Put

$$d\sigma_m^{(\omega)} = \frac{\sigma^{(\omega)}}{|P_m(z)|^2} = \omega^* \lim_{N\to\infty} \prod_{l<N,\ l\neq m} |P_l(z)|^2\, d\lambda$$

and

$$d\rho_m^{(\omega)} = \prod_{l=1}^{m-1} |P_l(z)|^2\, d\lambda.$$

Observe that the coefficients of all these polynomials are positive. It follows that for any $k \in \mathbb{Z}$, $\widehat{\rho_m^{(\omega)}}(k) \leq \widehat{\sigma_m^{(\omega)}}(k) \leq \widehat{\sigma^{(\omega)}}(k)$. Since $\left|\frac{1}{k_m}\sum_{s=0}^{k_m-1} z^s\right|^2 d\sigma^{(\omega)}$ converges weakly to $\frac{1}{c}\delta_1$, it is sufficient to prove that $\left|\frac{1}{k_m}\sum_{s=0}^{k_m-1} z^s\right|^2 d\rho_m^{(\omega)}$ converges weakly to $\frac{1}{c}\delta_1$.

In fact, let $n \in \mathbb{Z}$ and m be positive integer such that $k_m - 1 > |n|$. Since $p_{m-1} > (k_m-1)^2$, we have

$$\begin{aligned}
&\left|\int z^n \left|\frac{1}{k_m}\sum_{s=0}^{k_m-1} z^s\right|^2 d\rho_m^{(\omega)} - \int z^n \left|\frac{1}{k_m}\sum_{s=0}^{k_m-1} z^s\right|^2 d\sigma^{(\omega)}\right| \\
&\leq \frac{1}{{k_m}^2}\sum_{s,t} |\widehat{\rho_m^{(\omega)}}(n+s-t) - \widehat{\sigma^{(\omega)}}(n+s-t)| \\
&\leq \frac{1}{{k_m}^2}\sum_{s,t} 2\frac{(n+s-t)^2}{p_0\cdots p_{m-1}} \leq \frac{8}{p_0\cdots p_{m-2}} \underset{m\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

and this completes the proof of the lemma.[2]

PROOF OF PROPOSITION 7. Lemma 2 combined with the (b) of the Proposition 1 imply

$$\underline{\lim} \int\int |Q| \left|\left|\frac{P_m(\alpha z)}{P_m(z)}\right|^2 - 1\right| d\sigma^{(\omega)} d\mathbb{P} \;\geq\; K\Big(\int\int |Q|\, d\sigma^{(\omega)} d\mathbb{P} - \frac{4}{c}\int |Q|(1) d\mathbb{P}\Big),$$

and this completes the proof.

PROOF OF THEOREM 1. First, let us choose the good subsequence $\mathcal{N}$. Observe that from the Propositions 7 and 8 one can write

[2] the proof of the middle inequality can be found in Klemes and Reinhold (1997).

$$\begin{aligned}&\overline{\lim}\int |Q| \left|\frac{P_m(\alpha z)}{P_m(z)}\right| d\sigma^{(\omega)}(z) d\mathbb{P}\\ &\quad\leq \int |Q| - \frac{1}{8}K^2\left(\int\int |Q|\, d\sigma^{(\omega)} d\mathbb{P} - \frac{4}{c}\int |Q|(1) d\mathbb{P}\right)^2,\end{aligned}$$

and from this last inequality we shall construct $\mathcal{N}$. In fact, suppose we have chosen the k first elements of the subsequence $\mathcal{N}$. We wish to define the $(k+1)^{\text{th}}$ element. Let $m > n_k$ such that

$$\begin{aligned}&\int\int |Q| \left|\frac{P_m(\alpha z)}{P_m(z)}\right| d\sigma^{(\omega)}(z) d\mathbb{P}\\ &\quad\leq \int\int |Q|\, d\sigma^{(\omega)} d\mathbb{P} - \frac{1}{8}K^2\left(\int\int |Q|\, d\sigma^{(\omega)} d\mathbb{P} - \frac{4}{c}\int |Q|(1) d\mathbb{P}\right)^2,\end{aligned}$$

and put $n_{k+1} \stackrel{\text{def}}{=} m$. It follows that the elements of the subsequence $\mathcal{N}$ verify

$$\begin{aligned}\int\int \prod_{i=1}^{k+1} \left|\frac{P_{n_i}(\alpha z)}{P_{n_i}(z)}\right| d\sigma^{(\omega)} d\mathbb{P} \ &\leq \ \int\int \prod_{i=1}^{k} \left|\frac{P_{n_i}(\alpha z)}{P_{n_i}(z)}\right| d\sigma^{(\omega)} d\mathbb{P}\\ &- \frac{1}{8}K^2\left(\int\int \prod_{i=1}^{k} \left|\frac{P_{n_i}(\alpha z)}{P_{n_i}(z)}\right| d\sigma^{(\omega)} d\mathbb{P} \ - \ \frac{4}{c}\int \prod_{i=1}^{k} \left|\frac{P_{n_i}(\alpha)}{P_{n_i}(1)}\right| d\mathbb{P}\right)^2.\end{aligned}$$

We deduce that the sequence $\left(\int\int \prod_{i=1}^{k} \left|\frac{P_{n_i}(\alpha z)}{P_{n_i}(z)}\right| d\sigma^{(\omega)} d\mathbb{P}\right)_{k\geq 1}$ is decreasing and converges to the limit l which satisfies

$$l \ \leq \ l - \frac{1}{8}K^2 l^2,$$

and this implies that $l = 0$ and completes the proof. □

We shall give now some corollaries. Putting $\alpha_n = e^{2i\pi\frac{1}{n}}$, $n \in \mathbb{N}^*$, we have the following.

Corollary 1. $\mathbb{P}\{\omega : \ \sigma^{\omega}_{\alpha_n} \perp \sigma^{\omega}\} = 1$.

Corollary 2. $\mathbb{P}\{\omega :$ *The spectrum of* T^n_ω *is simple* $\} = 1$.

The proof of the above corollaries is based on the following lemma (Lemańczyk, Parreau and Thouvenot (1997)).

Lemma 3. *If T is a transformation with simple spectrum then the following conditions are equivalent*

(i) *The spectral type of T is singular with respect to its translates by α_n*

(ii) *The spectra of the powers of T are simple.*

3. On the Problem of the Singularity of Spectral Types of Rank One Transformations with Respect to their Translates

In this section we discuss the problem of the singularity of spectral types of rank one transformations with respect to their translates. We note that transformations with $\alpha-$mixing property satisfy property (1) (T is called $\alpha-$mixing, $0 < \alpha < 1$, if for some sequence $n_k \to \infty$ and for any Borel set A, $\lim_{k\to\infty} \mu(T^{n_k}A \cap A) = (1-\alpha)\mu(A) + \alpha(\mu(A))^2$). In fact, if σ is the spectral type of such transformation then $\sigma \perp \sigma{*}\sigma$. M. Lemańcyzk and A. del Junco constructed in Junco and Lemaćzyk (1992) rank one transformations with $\alpha-$mixing property. Recently, Prikhold'ko and Ryzhikov (1999), proved that the spectral type of a Chacon transformation ($\tau \in \mathcal{M}(\mathbb{T})$ has independent powers, if $\tau^{(m)} = \underbrace{\tau * \cdots * \tau}_{m} \perp \tau^{(n)}$ whenever $0 \leq m < n < \infty$).

It follows that a Chacon transformation satisfies property (1). Subsequently, F. Parreau obtains a class of rank one transformations which contain Chacon transformations and has the property considered by Choksi-Nadkarni.

In the following, we give a sufficient condition in order to obtain a positive answer to the problem of the singularity of spectral types of rank one transformations with respect to their translates. In fact we have the following propositions which follow from the above section.

Proposition 8. *The two following conditions are equivalent :*

(i) $\sigma_\alpha \perp \sigma \qquad \sigma - a.s \text{ in } \alpha$.

(ii) $\displaystyle\int \prod_{n=1}^{N} \left|\frac{P_n(\alpha z)}{P_n(z)}\right| d\sigma(z)\ d\sigma(\alpha) \longrightarrow_{N\to\infty} 0.$

Proposition 9. *The two following conditions are equivalent :*

(i) $\sigma_\alpha \perp \sigma$.

(ii) $\displaystyle\inf\{\int \prod_{l=1}^{N} \left|\frac{P_{n_l}(\alpha z)}{P_{n_l}(z)}\right| d\sigma, N \in \mathbb{N}, n_1 < n_2 < \ldots < n_N\} = 0.$

We deduce the following corollary.

Corollary 3. *The two following conditions are equivalent :*

(i) $\sigma\{\alpha : \sigma_\alpha \perp \sigma\} = 1$.

(ii) $\displaystyle\inf\{\int \prod_{l=1}^{N} \left|\frac{P_{n_l}(\alpha z)}{P_{n_l}(z)}\right| d\sigma(z)\ d\sigma(\alpha), N \in \mathbb{N}, n_1 < n_2 < \ldots < n_N\} = 0.$

Now, fix some subsequence $\mathcal{N} = \{n_1 < n_2 < \ldots < n_N\}$, $N \in \mathbb{N}$, $m > n_N$ and put

$$Q\,(\alpha, z) = \prod_{i=1}^{N} \frac{P_{n_i}(\alpha z)}{P_{n_i}(z)}, \qquad P(z) = \frac{P_m(\alpha z)}{P_m(z)}.$$

One can show easily the following lemma.

LEMMA 4.

$$\int\int |Q|\,|P|\,d\sigma(z)d\sigma(\alpha)$$
$$\leq \frac{1}{2}\left(\int\int |Q| + \int\int |Q|\,|P|^2\right) - \frac{1}{8}\left(\int\int |Q|\,\left||P|^2 - 1\right|\right)^2.$$

PROPOSITION 10. $\lim_{m\to\infty}\int |Q|\left|\frac{P_m(\alpha z)}{P_m(z)}\right|^2 d\sigma = \int |Q|\ d\sigma.$

PROOF. Let τ be the probability measure given by

$$\tau = \omega^* \lim_{M\to\infty}\prod_{l\notin\mathcal{N}, l\leq M} |P_l|^2$$

and note that we have

$$d\tau = \frac{1}{\prod_{n\in\mathcal{N}} |P_n|^2}\, d\sigma.$$

Since

$$d\sigma = \prod_{l=1}^{n_N} |P_l|^2\ d\sigma_{n_N+1}, \qquad \text{we have } d\tau = \prod_{n\notin\mathcal{N}, n\leq n_k} |P_n|^2\, d\sigma_{n_k+1}.$$

We obtain from (b) of Proposition 1 that the sequence of measures $\left(\left|\frac{P_m(\alpha z)}{P_m(z)}\right|^2 d\tau\right)_{m\geq 0}$ converges weakly to τ and this completes the proof.

From Lemma 4 and Proposition 10, we obtain, using the superior limit the following proposition.

PROPOSITION 11.

$$\overline{\lim}\int |Q|\left|\frac{P_m(\alpha z)}{P_m(z)}\right| d\sigma(z)d\sigma(\alpha) \leq \int |Q| - \frac{1}{8}\left(\underline{\lim}\int |Q|\left|\left|\frac{P_m(\alpha z)}{P_m(z)}\right|^2 - 1\right|\right)^2.$$

Finally, we shall prove the following proposition.

PROPOSITION 12. *Suppose that we have*

$$\underline{\lim}\int\int |Q|\left|\left|\frac{P_m(\alpha z)}{P_m(z)}\right|^2 - 1\right| d\sigma(z)d\sigma(\alpha) \geq K\int\int |Q|\,d\sigma(z)d\sigma(\alpha),$$

where K is a positive constant. Then

$$\inf\left\{\int\prod_{l=1}^{N}\left|\frac{P_{n_l}(\alpha z)}{P_{n_l}(z)}\right| d\sigma, \quad N\in\mathbb{N}, \quad n_1 < n_2 < \ldots < n_N\right\} = 0.$$

PROOF OF PROPOSITION 12. Put

$$\beta = \inf \left\{ \|Q\|_{L^1(\sigma\otimes\sigma)} : \; Q = \prod_{i=1}^{k} \left| \frac{P_{n_i}(\alpha z)}{P_{n_i}(z)} \right| ; \quad k \in \mathbb{N}, \quad n_1 < \; n_2 < \ldots < n_k \right\}.$$

If $Q(\alpha, z) = \prod_{i=1}^{k} \left| \frac{P_{n_i}(\alpha z)}{P_{n_i}(z)} \right|$, then we have

$$\underline{\lim} \int |Q(\alpha, z)| \left| \left| \frac{P_m(\alpha z)}{P_m(z)} \right|^2 - 1 \right| d\sigma(z) \; d\sigma(\alpha) \quad \geq \quad K \, \|Q\|_{L^1(\sigma\otimes\sigma)} \geq K\beta$$

and by Proposition 11, it follows that

$$\overline{\lim} \int |Q| \left| \frac{P_m(\alpha z)}{P_m(z)} \right| d\sigma \otimes \sigma \quad \leq \quad \|Q\|_{L^1(\sigma\otimes\sigma)} - \frac{1}{8} K^2 \beta^2.$$

But the left hand side is bounded by β since $m > n_k$ as $n \to +\infty$. Hence

$$\beta \quad \leq \quad \|Q\|_{L^1(\sigma\otimes\sigma)} - \frac{1}{8} K^2 \beta^2.$$

Taking the infimum over all Q now gives

$$\beta \quad \leq \quad \beta - \frac{1}{8} K^2 \beta^2$$

and this implies that $\beta = 0$. □

Acknowledgments. The author would like to express thanks to J-P. Thouvenot who posed to him the problem of simplicity of the spectrum of the powers of Ornstein transformations, and also to J. De Sam Lazaro, B. Host and F. Parreau for many conversations on the subject he had had with them.

References

ABDALAOUI, E.H.EL (1999). La singularité mutuelle presque sûre du spectre des transformations d'Ornstein, *Israel J. Math.* **112** , 135–156.

ABDALAOUI, E.H.EL (2000). A large class of Ornstein transformations with mixing property, *Canad. Math. Bull.* **43**, 157–161.

ADAMS, T.R. (1998). On Smorodinsky conjecture, *Proc. Amer. Math. Soc.* **126**, 739–744.

ADAMS, T.R. AND FRIEDMAN, N.A. (1993). Staircase mixing, to appear in *Ergodic Theory Dynam. Systems.*

BOURGAIN, J. (1993). On the spectral type of Ornstein class one transformations, *Israel J. Math.* **84**, 53–63.

BONAMI, A. (1968). Ensembles $\Lambda(p)$ dans le dual de D^∞, *Ann. Inst. Fourier* (Grenoble) **18**, 293–304.

CHOKSI , J.R. AND NADKARNI, M.G. (1994). The maximal spectral type of rank one transformation, *Canad. Math. Bull.* **37**, 29–36.

CHOKSI, J.R. AND NADKARNI, M.G. (1995). The group of eigenvalues of a rank one transformation, *Canad. Math. Bull.* **38**, 42–54.

CHOKSI, J.R. AND NADKARNI, M.G. (1996). On the question of transformations with simple Lebesgue spectrum, In: *Lie Groups and Ergodic Theory, Proceedings of International Colloquium*, Tata Inst. of Fundamental Research, Mumbai, Narosa Publ. Co. (1998), 33–57.
FRIEDMAN, N.A. (1970). *Introduction to Ergodic Theory*, van Nostrand Reinhold, New York.
– – –– (1992). Replication and stacking in ergodic theory, *Amer. Math. Monthly* **99**, 31–34.
JUNCO, A. DEL AND LEMAŃCZYK, M. (1992). Generic spectral properties of measure preserving maps and applications, *Proc. Amer. Math. Soc.* **115**, 725–736.
KING, J.L. AND THOUVENOT, J-P. (1988). A canonical structure theorem for joining-rank maps, *J. Analyse Math.* **51**, 182–227.
KLEMES, I. (1994). The spectral type of staircase transformations, *Tohoku Math. J.* **48**, p 247–258.
KLEMES I. AND REINHOLD, K. (1997). Rank one transformations with singular spectre type, *Israel J. Math.* **98**, 1–14.
LEMAŃCZYK, M. (1995). *Introduction to Ergodic Theory from the Point of View of Spectral Theory,* Lecture Notes of Tenth Kaisk Mathematical Workshop, Geon Ho Choe (ed.), Korea Adv. Inst. Sci. Technology, Math. Res. Center, Taejon, Korea.
LEMAŃCZYK, M., PARREAU, F. AND THOUVENOT, J-P. (1997). Gaussian automorphisms whose ergodic self-joinings are Gaussian, *preprint.*
NADKARNI, M.G. (1998). *Spectral Theory of Dynamical Systems*, Hindustan Book Agency, New Delhi; Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser-Verlag, Basel.
ORNSTEIN, D.S. (1971). On the root problem in ergodic theory, In: *Proc. Sixth Berkeley Symposium Math. Statist. Probab.,* University of California Press, 347–356.
PARREAU, F. private communication.
PRIKHOD'KO, A. A. AND RYZHIKOV, V.V. (1999). Disjointness of the convolutions for Chacon transformation, *Preprint.*

EL HOUCEIN EL ABDALAOUI
UNIVERSITÉ DE ROUEN-MATHÉMATIQUES
UPRES-A 60 85, SITE COLBERT
MONT SAINT AIGNAN 76821, FRANCE.
E-mail : ElHocein.Elabdalaoui@univ-rouen.fr