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# JOINING PROPERTIES OF ERGODIC DYNAMICAL SYSTEMS HAVING SIMPLE SPECTRUM*

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*SUMMARY.* It was shown in Goodson (1995) that if $T$ is an ergodic automorphism having simple spectrum and defined on a standard Borel probability space for which $T$ and $T^{-1}$ are isomorphic, i.e., there is an automorphism $S$ satisfying $ST = T^{-1}S$, then $S^2 = I$, the identity automorphism. If $T$ and $T^{-1}$ are not isomorphic, we can no longer consider conjugations but rather joinings of $T$ and $T^{-1}$. It is shown that if $P$ is a joining (Markov intertwining) between $T$ and $T^{-1}$, then $P$ is self–adjoint. Applications of the methods are given to provide new proofs of some well know results.

## 0. **Introduction**

Let $T$ be an invertible measure–preserving transformation (*automorphism*) defined on a standard Borel probability space $(X, \mathcal{F}, \mu)$ having simple spectrum, such that $T$ is isomorphic to its inverse $T^{-1}$. If the conjugating automorphism is $S$, i.e. $ST = T^{-1}S$ then $S^2 = I$, the identity automorphism (see Goodson (1995)). We investigate what happens when $T$ and $T^{-1}$ are not isomorphic, so there is no longer a conjugating automorphism $S$. It is natural in this case to study the joininings between $T$ and $T^{-1}$, and these turn out to be most easily investigated in their equivalent format of Markov intertwining as developed by Ryzhikov (1992). The following is the main result of this paper and is proved in Section 4. Recall that an automorphism $T$ which has simple spectrum is necessarily ergodic.

Theorem 1. *If $T : X \to X$ is an automorphism having simple spectrum, then each Markov intertwining between $T$ and $T^{-1}$ is self–adjoint.*

Corollary 1. *If $T$ is an automorphism of $(X, \mathcal{F}, \mu)$ which has simple spectrum, and $S : X \to X$ is measure–preserving with $ST = T^{-1}S$, then $S^2 = I$.*

Corollary 2. *If $T$ is an automorphism of $(X, \mathcal{F}, \mu)$ which has simple spectrum, and $\nu$ is a joining of $T$ and $T^{-1}$, then*

$$\nu(A \times B) = \nu(B \times A) \quad \textit{for all} \quad A, B \in \mathcal{F}.$$

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The class of automorphisms which have simple spectrum includes the rank–one transformations, the funny rank–one transformations and local rank–one (with covering number greater than one half) transformations. Also Gaussian automorphisms and interval exchange transformations can have simple spectrum.

In Section 1 we give some preliminaries from the theory of Markov intertwinings (see Ryzhikov (1992) or Goodson (1999) for more detail). Section 2 considers joinings between maps which have an eigenvalue in common and we give a Markov intertwining version of Lemańczyk's joining proof of the discrete spectrum theorem (see Lemańczyk and Mentzen (1990)). In Section 3 we show how these methods can be applied to give new proofs to theorems of Rudolph and Berg.

In Section 4 we consider the situation when a unitary operator $U$ having simple spectrum and a power of the operator is intertwined by a Markov operator. Using the spectral theorem for unitary operators we see that the intertwining may be realized by a type of multiplication operator. If $U$ preserves real valued functions we show that this operator is self–adjoint in certain circumstances.

## 1. **Preliminaries**

By a *dynamical system* we mean a 4–tuple $(X, \mathcal{F}, \mu, T)$ consisting of an automorphism $T : (X, \mathcal{F}, \mu) \to (X, \mathcal{F}, \mu)$ defined on a non-atomic standard Borel probability space. Both the identity automorphism and the identity operator will be denoted by $I$. The group of all automorphisms $\mathrm{Aut}(X)$ of $(X, \mathcal{F}, \mu)$ becomes a completely metrizable topological group when endowed with the weak convergence of transformations ($T_n \to T$ if for all $A \in \mathcal{F}$, $\mu(T_n^{-1}(A) \triangle T^{-1}(A)) + \mu(T_n(A) \triangle T(A)) \to 0$ as $n \to \infty$). Denote by $C(T)$ the *centralizer* of $T$, i.e., the set of those measure–preserving transformations on $(X, \mathcal{F}, \mu)$ which commute with $T$. We shall assume throughout that $C(T)$ is a group. This is certainly the case if $T$ has finite maximal spectral multiplicity, so will be true for all transformations with simple spectrum.

Related to the commutant is the set

$$\mathcal{B}(T) = \{S \in \mathrm{Aut}(X) : ST = T^{-1}S\}.$$

If $S$ is measure–preserving and $ST = T^{-1}S$ then $S^2T = ST^{-1}S = TS^2$, so $\{S^2 : S \in \mathcal{B}(T)\} \subseteq C(T)$.

Definition 1. Let $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{B}, \lambda, S)$ be two dynamical systems. We use $\mathcal{MI}(T, S)$ to denote the set of all *Markov intertwinings* between $T$ and $S$, i.e., $P \in \mathcal{MI}(T, S)$ if and only if $P : L^2(X, \mu) \to L^2(Y, \lambda)$ is an intertwining Markov operator with the following properties:

(i) $P\widehat{T} = \widehat{S}P$,

(ii) $P1_X = 1_Y$ and $P^*1_Y = 1_X$,

(iii) $f(x) \geq 0$ implies $Pf(x) \geq 0$ and $g \geq 0$ implies $P^*g(x) \geq 0$.

(where $\widehat{T}$ is $\widehat{T}f(x) = f(Tx)$, and $P^*$ is the adjoint of $P$).

There is a one–to–one correspondence between $\mathcal{MI}(T,S)$ and the set $\mathrm{J}(T,S)$ of *joinings* between $T$ and $S$. A joining of $T$ and $S$ is a $T\times S$ invariant measure $\nu$ with the property that

$$\nu(A\times Y)=\mu(A), \quad \text{and} \quad \nu(X\times B)=\lambda(B),$$

for all $A\in\mathcal{F}$ and $B\in\mathcal{B}$.

Given $P\in\mathcal{MI}(T,S)$, there is a joining $\nu\in\mathrm{J}(T,S)$ defined by

$$\langle P\chi_A,\chi_B\rangle=\nu(A\times B).$$

Conversely, given a joining $\nu\in\mathrm{J}(T,S)$ with disintegration $\nu_y$ (i.e., $\nu(A\times B)=\int_B \nu_y(A)\,d\lambda(y)$), a Markov intertwining $P:L^2(X,\mu)\to L^2(Y,\lambda)$ can be defined by

$$Pf(y)=\int_X f(x)\,d\nu_y(x).$$

Generally it can be shown that $P$ is induced by a measure–preserving transformation (i.e., $P=\widehat{K}$ for some measure–preserving transformation $K:Y\to X$, $TK=KS$), if and only if $P$ is an isometry. We shall be interested mainly in the case where $S=T$, or $S=T^{-1}$. In the former case, the Markov intertwinings arising from an automorphism coincide with the centralizer $C(T)$ of $T$, and in the latter case they coincide with the set $B(T)$. Note that if $P\in\mathcal{MI}(T,T^{-1})$, then $PP^*,P^*P,P^2\in\mathcal{MI}(T,T)$. Also $P=I$ corresponds to the diagonal joining $\Delta(A\times B)=\mu(A\cap B)$, $A,B\in\mathcal{F}$.

The set of Markov intertwinings $\mathcal{MI}(T,S)$ is seen to be a non-empty compact convex set (with the topology induced from that of $\mathrm{J}(T,S)$), so the Krein–Mil'man theorem implies that the extreme points are *indecomposable*. In particular, there always exist indecomposable intertwinings. A joining $\nu$ is said to be *ergodic* if the corresponding dynamical system $(X\times Y,\mathcal{F}\times\mathcal{B},\nu,T\times S)$ is ergodic. The ergodic joinings are the extreme points of $\mathrm{J}(T,S)$ and correspond to the indecomposable intertwinings in $\mathcal{MI}(T,S)$.

Two automorphisms $T$ and $S$ are said to be *disjoint* if the only joining of $T$ and $S$ is $\mu\times\lambda$. This is equivalent to the requirement that $P\in\mathcal{MI}(T,S)$ implies $P=\Theta$, the projection operator onto the space of constant functions.

The spectral properties of $T$ are those of the induced unitary operator defined by

$$\widehat{T}:L^2(X,\mu)\to L^2(X,\mu); \qquad \widehat{T}f(x)=f(Tx), \quad f\in L^2(X,\mu).$$

Generally a unitary operator $U:H\to H$ on a separable Hilbert space $H$ is said to have simple spectrum if there exists $h\in H$ such that $Z(h)=H$, where $Z(h)$ is the closed linear span of the vectors $U^nh$, $n\in\mathbf{Z}$.

In this case there exists a finite Borel measure $\sigma_h$ defined on the unit circle $S^1$ in the complex plane for which

$$\langle U^nh,h\rangle=\int_{S^1} z^n\,d\sigma_h, \quad n\in\mathbf{Z},$$

and such that $U$ is unitarily equivalent to $V : L^2(S^1, \sigma_h) \to L^2(S^1, \sigma_h)$ defined by $Vf(z) = zf(z)$.

## 2. Joinings of Maps with Eigenvalues in Common

We give Lemańczyk's joining proof of the Discrete Spectrum Theorem (Lemańczyk and Mentzen(1990)), but our approach uses the methods of Markov intertwinings. We start with some preliminaries:

THEOREM 2. *Let $P \in \mathcal{MI}(T,S)$ be an indecomposable Markov intertwining with corresponding ergodic joining $\nu \in \mathrm{J}(T,S)$, $\nu(A \times B) = \langle P\chi_A, \chi_B \rangle$. The following are equivalent:*

*(i) $P$ is an isometry.*

*(ii) $P = \widehat{K}$ for some measure–preserving $K : Y \to X$ satisfying $TK = KS$.*

*(iii) $\nu = \lambda_K$ for some measure–preserving $K : Y \to X$ satisfying $TK = KS$.*

*(iv) $T$ is a factor of $S$.*

*(v) For all $A \in \mathcal{F}$ there exists $B \in \mathcal{B}$ such that $\nu((A \times X) \triangle (X \times B)) = 0$.*

*(vi) $P^*P = I$, the identity operator.*

The equivalence of (i) and (ii) is classical. The equivalence of (iii) and (v) is implicit in the work of Rudolph, and the equivalence of (iv) and (vi) is due to Ryzhikov.

LEMMA 1. *Let $T$ and $S$ be ergodic automorphisms and $P \in \mathcal{MI}(T,S)$ indecomposable. Suppose there exists $\lambda \in S^1$ and normalized eigenfunctions $f, g$, $f(Tx) = \lambda f(x)$, $g(Sx) = \lambda g(x)$, then $Pf = ag$ for some constant $a$ with $|a| = 1$.*

PROOF. $P \in \mathcal{MI}(T,S)$ is indecomposable and

$$P\widehat{T}f = \widehat{S}Pf \Longrightarrow \lambda Pf = \widehat{S}Pf.$$

This implies that $Pf$ is an eigenfunction for $S$ corresponding to the eigenvalue $\lambda$.

The ergodicity of $S$ implies that $Pf = cg$ (for some constant $c \neq 0$). Now $P$ is indecomposable, so $\nu(A \times B) = \langle P\chi_A, \chi_B \rangle$ is ergodic for $T \times S$. Let $h(x,y) = f(x)\overline{g}(y)$, then

$$h(T \times S)(x,y) = f(Tx)\overline{g}(Sy) = \lambda f(x)\overline{\lambda}\overline{g}(y) = f(x)\overline{g}(y) = h(x,y).$$

The ergodicity of $T \times S$ implies $h = a$, $\nu$ a.e., a constant with absolute value equal to one. Then

$$c = \langle Pf, g \rangle = \int f(x)\overline{g}(y)\, d\nu(x,y) = a,$$

so that $Pf = ag$ with $|a| = 1$ □

Following is our version of Lemańczyk's proof of the discrete spectrum theorem (see Lemańczyk and Mentzen (1990)).

Corollary 3 (Discrete Spectrum Theorem). *Let $T$ and $S$ be ergodic with discrete spectrum and the same eigenvalue group $\{\lambda_n : n \in \mathbb{Z}\}$. Then $T$ and $S$ are isomorphic.*

Proof. Let $P \in \mathcal{MI}(T, S)$ be indecomposable and let $\{f_n : n \in \mathbb{Z}\}$ and $\{g_n : n \in \mathbb{Z}\}$ be complete orthonormal bases of eigenfunctions for $\widehat{T}$ in $L^2(X, \mu)$ and $\widehat{S}$ in $L^2(Y, \lambda)$, so that $f_n(Tx) = \lambda_n f_n(x)$, $g_n(Sy) = \lambda_n g_n(y)$.

By the lemma, $Pf_n = a_n g_n$ for some constants $|a_n| = 1$, $n \in \mathbb{Z}$. It follows that $P$ is an isometry, so by Theorem 1 is induced by a (necessarily invertible) measure–preserving transformation. $\square$

Corollary 4. *If $T$ is ergodic, $P \in \mathcal{MI}(T, T^{-1})$ and $f(Tx) = \lambda f(x)$, then $Pf = P^* f$ (i.e., $P = P^*$ on the Kronecker factor).*

*Consequently, if $Pf = P^* f \implies f =$ constant, then $T$ is weakly mixing.*

Proof. If $f(Tx) = \lambda f(x)$, then $\bar{f}(T^{-1}x) = \lambda \bar{f}(x)$, so by the lemma, if $P$ is indecomposable then $Pf = a\bar{f}$ for some constant $a$ with $|a| = 1$. Now the same argument as in the proof of the lemma shows that $P^* \bar{f} = \overline{a} f$, and we deduce $Pf = P^* f$ (we also see that $P^2 f = f$). The case where $P$ is not indecomposable follows directly from the indecomposable case. $\square$

The above shows that the restriction of $P$ to the Kronecker factor is unitary and hence induced by a measure–preserving transformation $R$ with the property that $R^2 = I$.

Corollary 5. *If $ST = T^{-1}S$ where $S^2$ and $T$ are ergodic, then $S$ and $T$ are weakly mixing.*

Proof. If $P = \widehat{S}$, then $P \in \mathcal{MI}(T, T^{-1})$ with $Pf = P^* f \Longrightarrow f =$ constant, so that $T$ is weakly mixing. Since $S^2 T = T S^2$ it follows that $S^2$ and hence $S$ is weakly mixing. $\square$

## 3. Joinings of Prime Maps

Recall that the automorphism $T : X \to X$ is prime if the only $T$-invariant sub-$\sigma$-algebras (mod 0) are $\mathcal{F}$ and $\mathcal{N} = \{X, \emptyset\}$. This is equivalent to saying that if $P \in \mathcal{MI}(T, T)$ is a projection ($P^2 = P$ and $P = P^*$), then $P = I$ or $P = \Theta$ (the orthogonal projection onto the contants).

Following is an unpublished result of Berg, which appeared implicitly in Berg (1975)and was applied to show that rank one mixing transformations are prime.

Proposition 1 (Berg (1975)). *$T$ is prime if and only if for each $P \in \mathcal{MI}(T, T)$, $P \neq I$, then $Pf = f$ implies $f = c$, a constant.*

Proof. Suppose that $T$ is prime and $P \in \mathcal{MI}(T, T)$. We may assume that $P \neq \Theta$. Suppose that there exists a non–constant measurable function $f$ with the property that $Pf = f$. Then we show that $T$ has a non–trivial factor. In fact we can find $c$ such that $A_c = \{x \in X : f(x) < c\}$, satisfies $0 < \mu(A_c) < 1$. Using the fact that the subspace $L = \{f \in L^2(X,\mu) : Pf = f\}$ is a vector lattice it can be shown that $\chi_{A_c} \in L$, and hence $P\chi_{A_c} = \chi_{A_c}$.

It follows that

$$\mathcal{A} = \{A \in \mathcal{F} : P(\chi_A) = \chi_A\},$$

is a non–trivial $T$–invariant sub $\sigma$–algebra of $\mathcal{F}$. ($T$–invariant because if $A \in \mathcal{A}$, then $P\widehat{T}\chi_A = \widehat{T}P(\chi_A) = \widehat{T}\chi_A$, so $T^{-1}A \in \mathcal{A}$). Since $T$ is prime we must have $\mathcal{A} = \mathcal{F}$, so that $P = I$, and the result follows.

Conversely, let $P(f) = E(f|\mathcal{A})$ be the conditional expectation of $f$ given $\mathcal{A}$, a $T$ invariant sub $\sigma$–algebra of $\mathcal{F}$. Then it is not hard to see that $P \in \mathcal{MI}(T,T)$. If $\mathcal{A} \neq \mathcal{N} = \{X, \emptyset\}$ mod(0), then there exists a non-trivial $f \in L^2(X,\mu)$ such that $P(f) = f$ (in fact any $\mathcal{A}$–measurable function). This is only possible if $P = I$, $\mathcal{A} = \mathcal{F}$, i.e., $T$ is prime. □

Recall that a weakly mixing transformation $T$ has two–fold minimal self–joinings if the only ergodic self–joinings are $\mu \times \mu$ and *off–diagonal* joinings, i.e., those of the form $(I \times T^n)\Delta$, where $\Delta(A \times B) = \mu(A \cap B)$ (so that $(I \times T^n)\Delta(A \times B) = \mu(A \cap T^{-n}B)$). We apply the proposition of Berg to give a new proof of a theorem due to Rudolph:

THEOREM 3 (Rudolph (1979)). *If $T$ is weakly mixing with two–fold minimal self–joinings, then $T$ is prime.*

PROOF. Suppose there exists $P \in \mathcal{MI}(T,T)$ and $f \in L^2(X,\mu)$, $f \neq$ constant, satisfying $Pf = f$. Then we can find $A \in \mathcal{F}$, $0 < \mu(A) < 1$ such that $P\chi_A = \chi_A$.

Now $P$ may not be indecomposable, but since $T$ has minimal self–joinings, we can write

$$P = \alpha\Theta + (1-\alpha)\sum_{i=-\infty}^{\infty} a_i \widehat{T}^i$$

where $0 \leq \alpha \leq 1$ and $\sum_i a_i = 1$.

Now

$$P\chi_A = \chi_A = \alpha\mu(A) + (1-\alpha)\sum_{i=-\infty}^{\infty} a_i \chi_{T^{-i}A}.$$

Also, $\langle P\chi_A, \chi_{A^c}\rangle = 0$ (where $A^c$ is the complement of $A$), so that

$$0 = \alpha\mu(A)\mu(A^c) + (1-\alpha)\sum_{i=-\infty}^{\infty} a_i \mu(A^c \cap T^{-i}A).$$

This implies that $\alpha = 0$ and $a_i = 0$ for $i \neq 0$ (since $T$ is weakly mixing, $T^i$ is ergodic for all $i \neq 0$.) Hence $P = I$, so $T$ must be prime. □

REMARK. Consider the only prime maps that we are aware of. These are:

(i) Maps having minimal self–joinings, for example, mixing rank one maps, or the Chacon map (see Rudolph (1990)). (The known mixing rank one maps include those constructed by Ornstein, and the staircase transformation). Also certain interval exchange maps on 3 intervals are known to have minimal self–joinings (see del Junco (1983)).

(ii) The del Junco–Rudolph simple, rigid, prime, rank–one map (see Junco and Rudolph (1987)).

(iii) The maps $R(x, y) = (y, Tx)$ and the symmetric cartesian square, $T^{\odot 2}$, for $T$ having minimal self–joinings (see Rudolph (1990) or Goodson and Lemańczyk (1990)).

The maps in (ii) and (iii) do not have minimal self–joinings.

Certainly the maps in (i) have trivial centralizer, and for any indecomposable $P \in \mathcal{MI}(T,T)$, $P = \Theta$ or $P = \widehat{T}^n$ for some $n \in \mathbf{Z}$.

In case (ii), the simplicity of $T$ implies that for indecomposable $P$, $P = \Theta$ or $P = \widehat{S}$ for some $S \in C(T)$. In the case where $P = \widehat{S}$ for some automorphism $S \in C(T)$, $P$ has the property that $Pf = f$ implies $f =$ constant, i.e., $S$ is ergodic, and hence is weakly mixing (if $S \neq I$).

In case (iii), the map $R$ does have non–trivial self–joinings $P \neq \Theta$ with the property that $Pf = f$ implies $f =$ constant (see Goodson (1995)).

Proposition 2. *Let $T : X \to X$ be an ergodic automorphism. If $T$ is prime and $P \in \mathcal{MI}(T, T^{-1})$ then one of the following holds:*

*(i) $P = \widehat{S}$ for some automorphism $S \in \mathcal{B}(T)$ satisfying $S^2 = I$.*

*(ii) $P = \widehat{S}$ for some automorphism $S \in \mathcal{B}(T)$ which is weakly mixing.*

*(iii) $P = P^*$ and $PP^*f = f$ implies $f =$ constant.*

*(iv) $Pf = P^*f$ implies $f =$ constant, and $P^*Pf = f$ implies $f =$ constant.*

Proof. We have that $PP^* \in \mathcal{MI}(T,T)$, so that from Proposition 1 either (a) $PP^* = I$, or (b) $PP^*f = f$ implies $f =$ constant.

Also the sub $\sigma$–algebra

$$\mathcal{A} = \{A \in \mathcal{F} : P\chi_A = P^*\chi_A\}$$

is $T$ invariant. It follows as above that either (c) $P = P^*$ or (d) $Pf = P^*f$ implies $f =$ constant. If (a) and (c) hold then we deduce (i). If (a) and (d) hold we have (ii).

The other combinations give rise to (iii) and (iv). □

## 4. Markov Intertwinings between $T$ and $T^p$ when $T$ has Simple Spectrum

The next theorem concerns operators which intertwine unitary operators on a separable Hilbert space having simple spectrum. Our aim is to obtain information about the form such intertwining operators can have. The intertwining operator $P$ is not assumed to be unitary. The following exposition closely parallels that in Goodson (1995).

Lemma 2. *Let $U : H \to H$ be a unitary operator on a Hilbert space $H$ which has simple spectrum and suppose there exists an operator $P : H \to H$ satisfying*

*$U^pP = PU$, $p \in \mathbf{Z}$, $p \neq 0$. Then $P$ is unitarily equivalent to an operator $\widetilde{P} : L^2(S^1, \sigma_h) \to L^2(S^1, \sigma_h)$ defined by*

$$\widetilde{P}f(z) = f(z^p)k(z), \quad f \in L^2(S^1, \sigma_h), \tag{1}$$

*for some $k \in L^2(S^1, \sigma_h)$ and $h \in H$.*

Proof. We can represent $U$ as $Vf(z) = zf(z)$, $f \in L^2(S^1, \sigma_h)$ where $\sigma_h$ satisfies $\langle U^n h, h\rangle = \int_{S^1} z^n \, d\sigma_h$, $n \in \mathbf{Z}$; $h$ being a cyclic vector for $U$. We have $(V^p f)(z) = z^p f(z)$.

If $\Phi$ is the operator giving rise to this unitary equivalence, we may suppose that

$$V^p \widetilde{P} f(z) = \widetilde{P} V f(z), \quad f \in L^2(S^1, \sigma_h)$$

where $\widetilde{P} = \Phi P \Phi^{-1}$. Let $p_n(z) = z^n$, $n \in \mathbf{Z}$ and $k = \widetilde{P}(p_0)$, then

$$\widetilde{P}p_1(z) = \widetilde{P}Vp_0(z) = V^p\widetilde{P}p_0(z) = z^p k(z),$$

$$\widetilde{P}p_2(z) = \widetilde{P}Vp_1(z) = V^p\widetilde{P}p_1(z) = z^{2p} k(z),$$

and in general

$$\widetilde{P}p_k(z) = z^{kp}k(z), \ \ k \in \mathbf{Z}.$$

Thus

$$\widetilde{P}(\sum_{-n}^{n} a_k z^k) = k(z)(\sum_{-n}^{n} a_k z^{kp})$$

and consequently

$$\widetilde{P}f(z) = k(z)f(z^p) \quad \text{for} \quad \text{each} \quad f \in L^2(S^1, \sigma_h),$$

and this completes the proof. □

Let $(X, \mathcal{F}, \mu)$ be a standard Borel probability space. We denote $L^2(X, \mu)$ by $H$, and write $H_{\mathbf{R}} = L^2_{\mathbf{R}}(X, \mu)$, the real valued functions in $H$. In what follows, only functions with zero mean are considered.

Let $U : H \to H$ be a unitary operator. For $f \in H$, we again use $\sigma_f$ to denote its spectral measure, i.e. the unique Borel measure on the circle given by

$$\widehat{\sigma}_f(n) = \int_{S^1} z^n \, d\sigma_f(z) = \int_X U^n f \cdot \bar{f} \, d\mu. \tag{2}$$

Assume that

$$U\bar{f} = \overline{Uf}, \quad \forall \ f \in H, \tag{3}$$

that is, $U$ commutes with the operation of complex conjugation. By (2) we have

$$\widehat{\sigma}_{\bar{f}}(n) = \int_X U^n \bar{f} \cdot f \, d\mu = \int_X \bar{f} \cdot U^{-n} f \, d\mu = \widehat{\sigma}_f(-n).$$

This means that $\sigma_{\bar{f}}$ is a measure on the circle which is the image of $\sigma_f$ via the map $z \to \bar{z}$. If $f$ realizes the maximal spectral type of $U$, then $\sigma_{\bar{f}} \ll \sigma_f$ and consequently

$\sigma_f$ has to have the type of a symmetric measure. So if we select $f$ in such a way that $\sigma_f$ *is* a symmetric measure (such an $f$ does exist), then in particular we have $\widehat{\sigma}_f(n) = \overline{\int_{S^1} z^n \, d\sigma_f(z)}$, so

$$\widehat{\sigma}_f(n) \in \mathbf{R}, \quad \forall\, n \in \mathbf{Z}. \tag{4}$$

Since (3) is satisfied, then for each $f \in H_{\mathbf{R}}$, $Uf = U\bar{f} = \overline{Uf}$, so $U(H_{\mathbf{R}}) = H_{\mathbf{R}}$. Therefore if $f \in H_{\mathbf{R}}$ realizes the maximal spectral type, then (4) holds. Denote

$$\mathcal{H} = L^2(S^1, \sigma),$$

where $\sigma$ is *symmetric*, and write

$$\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}(\sigma) = \{g \in \mathcal{H} : g(\overline{z}) = \overline{g(z)}\}.$$

As before, we let $V : \mathcal{H} \to \mathcal{H}$ be the unitary operator defined by $(Vf)(z) = zf(z)$, so that $(V^{-1}f)(z) = \overline{z}f(z)$.

The following were shown in Goodson (1995):

(i) $\widetilde{\mathcal{H}}$ is a subspace over $\mathbf{R}$ which is closed and $V$-invariant, and $\widetilde{\mathcal{H}}$ is the closure of $\{\sum_{-n}^{n} a_k z^k \ : \ a_k \in \mathbf{R}, n \geq 0\}$.

(ii) If $f \in H$, then we denote by $\Phi$ the natural isomorphism

$$\Phi : (U, Z(f)) \to (V, L^2(S^1, \sigma_f)),$$

($Z(f) =$ span $\{U^n f \ : \ n \in \mathbf{Z}\}$), determined by $\Phi(U^n f) = p_n(z)$, where $p_n(z) = z^n$, $n \in \mathbf{Z}$. If $f \in H_{\mathbf{R}}$, then $\Phi(H_{\mathbf{R}} \cap Z(f)) = \widetilde{\mathcal{H}}(\sigma_f)$.

(iii) If $U : H \to H$ is unitary, and $U(H_{\mathbf{R}}) = H_{\mathbf{R}}$, then there exists $f \in H_{\mathbf{R}}$ such that $\sigma_f$ realizes the maximal spectral type of $U$.

We can now prove Theorem 1, which is the main result of the paper.

Proposition 3. *If* $\widehat{T} : L^2(X, \mu) \to L^2(X, \mu)$ *has simple spectrum, then every* $P \in \mathcal{MI}(T, T^{-1})$ *is self–adjoint, i.e.,* $P = P^*$.

Proof. Choose $f \in H_{\mathbf{R}}$ with $\sigma = \sigma_f$, so that the maximal spectral type of $\widehat{T}$ is a symmetric measure.

Let $\mathcal{H} = L^2(S^1, \sigma_f)$, and $\widetilde{\mathcal{H}} = \{g \in \mathcal{H} : g(\overline{z}) = \overline{g(z)}\}$, then we have that $\widetilde{P} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$; $\widetilde{P}(\widetilde{\mathcal{H}}) \subseteq \widetilde{\mathcal{H}}$ (since $P$ preserves positive functions it must preserve real valued functions).

Now from Lemma 2 with $p = -1$, we deduce that $\widetilde{P}g(z) = k(z)g(\overline{z})$ where $k(z) = \widetilde{P}p_0(z)$, $p_0 \in \widetilde{\mathcal{H}}$. In particular $k \in \widetilde{\mathcal{H}}$, i.e., $k(\overline{z}) = \overline{k(z)}$. Now note that

$$\langle \widetilde{P}f(z), g(z)\rangle = \langle k(z)f(\overline{z}), g(z)\rangle = \int_{S^1} k(z)f(\overline{z})\overline{g(z)}\, d\sigma(z)$$

$$= \int_{S^1} f(z)k(\overline{z})\overline{g(\overline{z})}\, d\sigma(z) = \langle f(z), \overline{k(\overline{z})}g(\overline{z})\rangle = \langle f(z), k(z)g(\overline{z})\rangle = \langle f(z), \widetilde{P}g(z)\rangle.$$

i.e., $\widetilde{P} = \widetilde{P}^*$. This implies $P = P^*$. □

Corollary 6 (Goodson (1995)). *If $T$ has simple spectrum and $P \in \mathcal{MI}(T, T^{-1})$ is unitary (i.e., $P = \widehat{S}$ for some $S \in B(T)$, so in particular $T$ is conjugate to its inverse), then $P^2 = I$.*

Proof. $P^* = P^{-1}$ since $P$ is unitary, so $P^2 = I$. □

Corollary 7. *If $T$ has simple spectrum and $\nu \in \mathrm{J}(T, T^{-1})$, then*

$$\nu(A \times B) = \nu(B \times A), \quad \textit{for all} \quad A, B \in \mathcal{F}.$$

Proof.

$$\nu(A \times B) = \langle P\chi_A, \chi_B \rangle = \langle \chi_A, P^*\chi_B \rangle = \langle \chi_A, P\chi_B \rangle = \langle P\chi_B, \chi_A \rangle = \nu(B \times A).$$

□

Suppose $T$ has simple spectrum, then it is known (Ryzhikov (1992)) that $\mathcal{MI}(T, T)$ is a commutative semi–group containing $C(T)$ (identified by joinings of the form $\mu_S(A \times B) = \mu(A \cap S^{-1}B)$ for $S \in C(T)$).

Proposition 4 (Ryzhikov (1992)). *If $T$ has simple spectrum, then $\mathcal{MI}(T, T)$ is an abelian semi–group.*

Proof. Let $P, P' \in \mathcal{MI}(T, T)$, then using Lemma 2 with $p = 1$, we see that $P$ and $P'$ have representations in the form $\widetilde{P}f(z) = k(z)f(z)$ and $\widetilde{P}'f(z) = k'(z)f(z)$, and consequently they commute. □

Remark. If there exists $P \in \mathcal{MI}(T, T^{-1})$ which is not self–adjoint, then $T$ has non–simple spectrum.

## References


Berg. K. (1975). Mixing cyclic approximation and homomorphisms, *Unpublished preprint*, University of Maryland.

Glasner, E., Host, B. and Rudolph. D. (1992). Simple systems and their higher order self-joinings, *Israel Journal of Math.* **78**, 131–142.

Goodson, G.R, Junco, A. del, Lemańczyk, M. and Rudolph, D. (1995). Ergodic transformations conjugate to their inverses by involutions, *Ergodic Theory and Dynamical Systems* **24**, 95–124.

Goodson, G.R. and Lemańczyk., M. (1996). Transformations conjugate to their inverses have even essential values, *Proc. Amer. Math. Soc.* **124**, 2703–2710.

Goodson, G.R. and Ryzhikov, V.V. (1997). Conjugations, joinings and direct products of locally rank–one dynamical systems, *J. Dynamical and Control Systems* **3**, 321–341.

Goodson, G.R. and Lemańczyk, M. (1990). On the rank of a class of bijective substitutions, *Studia Math.* **96**, 219–230.

Goodson G.R. (1999). A survey of recent results in the spectral theory of ergodic dynamical systems, *J. Dynamical and Control Systems* **5**, 173–226.

Junco, A. del and Rudolph, D.J. (1987). A rank one, rigid, simple, prime map, *Ergodic Theory and Dynamical Systems* **7**, 229–248.

Junco, A. del and Rudolph, D.J. (1987). On ergodic actions whose self-joinings are graphs, *Ergodic Theory and Dynamical Systems* **7**, 531–557.

Junco, A. del. (1983). A family of counter examples in ergodic theory, *Israel J. Math.* **44**, 160–180.
King, J. (1986). The commutant is the weak closure of the powers, for rank one transformations, *Ergodic Theory and Dynamical Systems* **6**, 363–384.
Lemańczyk, M. and Mentzen, M.K. (1990). Compact subgroups in the centralizer of natural factors of an ergodic group extension of a rotation determine all factors, *Ergodic Theory and Dynamical Systems* **10**, 763–776.
Newton, D. (1978). On canonical factors of ergodic dynamical systems, *J. London Math. Soc.* **19**, 129–136.
Oseledec, V.I. (1971). Two non-isomorphic dynamical systems with the same simple continuous spectrum, *Funktsional Anal. i Prilozen.* **5**, 75–79.
Rudolph, D.J. (1979). An example of a measure–preserving transformation with minimal self-joinings and applications, *J. d'Analyse Math.* **35**, 97–122.
– – –– (1990). *Fundamentals of Measurable Dynamics,* Oxford University Press.
Ryzhikov, V.V. (1992). Stochastic intertwinings and joinings of dynamical systems, *Mat. Zametki* **52**, 130–140.

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