Sankhyā: The Indian Journal of Statistics
Special issue on Ergodic Theory and Harmonic Analysis
2000, Volume 62, Series A, Pt. 3, pp. 318–330

ON BERNSTEIN'S EXAMPLE OF THREE PAIRWISE INDEPENDENT RANDOM VARIABLES*

By YVES DERRIENNIC
Université de Bretagne Occidentale, Brest, France
and
ANDRZEJ KŁOPOTOWSKI
Université Paris XIII, Institut Galilée, Villetaneuse, France

SUMMARY. We give here a complete description of triplets and quadruplets of pairwise independent binary random variables with a special emphasis on the role of Bernstein's example.

1. **Introduction**

A central part of the theory of probability is devoted to the study of sequences of random variables, which are mutually independent. It is almost complete and considered classical. There are many different ways to study such sequences, when some dependences between random variables appear. Traditionally, some aspects of dependence are expressed in terms of conditioning with respect to the past, like for Markov processes or for martingales. A different type of dependence is given by exchangeability and others by "separated" forms of independence like $m-$dependence or by "asymptotic" independence described by various types of mixing.

Here we propose another point of view on the notion of dependence. We suppose that we lose the mutual independence, when the number of observed variables exceeds some fixed level; for example, if any two variables are independent, but some triplets (or more numerous groups of them) are dependent, then this case is called *pairwise independent*. One can find an introduction to this subject in Stoyanov's very interesting book (Stoyanov, 1987, Sections 3 and 7). The main disadvantage of this notion is that there is no general method of the construction of pairwise independent sequences, so a really new example of such a sequence is always an interesting mathematical object. Moreover, in Derriennic and Kłopotowski (1991) we have shown that a pairwise independence is not constructive in the sense, that laws of corresponding processes can not be constructed by induction like *for ex.*

AMS (1991) *subject classification.* Primary 60A05, 60G10; secondary 47A35, 28D20.
Key words and phrases. Pairwise independence, one-dependence, stationarity, exchangeability, Markov chain.

*Dedicated to Professor M.G. Nadkarni.

Markov chains. All known pairwise independent sequences are constructed by induction, so it would be interesting to find an essentially non-constructive example of it. We present here a very beginning of the theory giving a complete description of triplets and quadruplets of pairwise independent binary random variables with a special emphasis on the role of Bernstein's example. (Let us remark that in Derriennic and Kłopotowski (1991) we have given a very partial description for stationary sequences of five pairwise independent binary random variables.)

Paradoxically, a great number of articles were devoted to a very particular case of mutually independent random variables and very few are concerned with a wider class of pairwise independent sequences. We can quote (non-exhaustively) the articles of Robertson (1988a, 1988b, 1988c) Robertson and Womack (1985), Etemadi (1981) and Janson (1988), as the most interesting ones. Let us mention that in Bretagnolle and Kłopotowski (1995) we have given the first known example of pairwise independent non–symmetric coin tossing. In Kłopotowski and Robertson (1999) we give some new examples of sequences of pairwise independent random variables and we present some open problems related to Banach's famous question about the existence of a dynamical system with simple Lebesgue spectrum (see also Kłopotowski and Nadkarni (1999) for a more advanced approach).

2. **Preliminary Notions**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space. We recall that two *events* $A, B \in \mathcal{F}$ are said to be *independent* (with respect to $\mathbb{P}$), if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Two *σ–fields* $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$ are independent, if any two events $A_1 \in \mathcal{F}_1,\ A_2 \in \mathcal{F}_2$, are independent.

We say that the *events* $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are *mutually independent*, if for each $2 \leq k \leq n$ and for each $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ we have

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \ldots \mathbb{P}(A_{i_k}).$$

A *finite family* of σ–fields $\{\mathcal{F}_i \subset \mathcal{F};\ 1 \leq i \leq n\}$ is *mutually independent*, if every choice of events $A_i \in \mathcal{F}_i,\ 1 \leq i \leq n$, is independent.

An *infinite family* of σ–fields $\{\mathcal{F}_i \subset \mathcal{F};\ i \in I\}$ is *mutually independent*, if each finite subfamily $\{\mathcal{F}_{i_k},\ i_k \in I;\ 1 \leq k \leq n\}$ is independent.

The *mutual independence of random variables* $X_i,\ i \in I$, is defined using generated σ–fields $\mathcal{F}_i := X_i^{-1}(\mathcal{B}^1) \subset \mathcal{F},\ i \in I$.

A family of σ–fields $\{\mathcal{F}_i \subset \mathcal{F};\ i \in I\}$ (or of random variables $\{X_i;\ i \in I\}$) is *pairwise independent*, if every two of them are independent.

A sequence of random variables $X_n,\ n \in \mathbb{Z}$, is *stationary* if and only if for each $n \in \mathbb{Z}$ and for each $k, m \in \mathbb{N}$ the random vector $(X_n, X_{n+1}, \ldots, X_{n+m})$ has the same joint probability law as the random vector $(X_{n+k}, X_{n+k+1}, \ldots, X_{n+k+m})$; it is *exchangeable* if and only if for each $n \in \mathbb{Z}$ and for each $m \in \mathbb{N}$ the random vector

$(X_n, X_{n+1}, \ldots, X_{n+m})$ has the same joint probability law as the random vector $(X_{\sigma(n)}, X_{\sigma(n+1)}, \ldots, X_{\sigma(n+m)})$ for every permutation σ of $\{n, n+1, \ldots, n+m\}$. ($\mathbb{N}$ and $\mathbb{Z}$ denote the sets of naturals and integers.)

Obviously exchangeability implies stationarity, but the converse implication is not true.

3. Bernstein's Examples

The fact that pairwise independence does not imply mutual independence was mentioned for first time in the correspondence in the years 1910 to 1917 between A.A. Chuprov (1874-1926) and A.A. Markov (1856-1922). The following example is attributed traditionally to S.N. Bernstein (1946).

Let us consider $\Omega := \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\mathcal{F} := 2^\Omega$, $\mathbb{P}(\{\omega_k\}) := \frac{1}{4}$; $1 \le k \le 4$. We define random variables $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \{0,1\}$; $i = 1, 2, 3$, by:

$$X_i(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_1, \omega_{1+i}; \\ \\ 0 & \text{if } \omega \neq \omega_1, \omega_{1+i}. \end{cases}$$

We see at once that X_1, X_2, X_3 are pairwise independent:

$$\mathbb{P}[X_1 = 0, X_2 = 0] = \mathbb{P}(\{\omega_4\}) = \frac{1}{4} = \mathbb{P}[X_1 = 0] \cdot \mathbb{P}[X_2 = 0],$$

$$\mathbb{P}[X_1 = 0, X_3 = 0] = \mathbb{P}(\{\omega_3\}) = \frac{1}{4} = \mathbb{P}[X_1 = 0] \cdot \mathbb{P}[X_3 = 0],$$

$$\mathbb{P}[X_2 = 0, X_3 = 0] = \mathbb{P}(\{\omega_2\}) = \frac{1}{4} = \mathbb{P}[X_2 = 0] \cdot \mathbb{P}[X_3 = 0].$$

On the other hand, X_1, X_2, X_3 are not mutually independent:

$$\mathbb{P}[X_1 = 1, X_2 = 1, X_3 = 1] = \mathbb{P}(\{\omega_1\}) = \frac{1}{4} \neq \mathbb{P}[X_1 = 1] \cdot \mathbb{P}[X_2 = 1] \cdot \mathbb{P}[X_3 = 1].$$

Moreover,

$$\begin{aligned} & \mathbb{P}[X_1 = 1, X_2 = 0, X_3 = 0] = \mathbb{P}[X_1 = 0, X_2 = 1, X_3 = 0] \\ = \; & \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 1] = \tfrac{1}{4}, \end{aligned}$$

$$\begin{aligned} & \mathbb{P}[X_1 = 0, X_2 = 1, X_3 = 1] = \mathbb{P}[X_1 = 1, X_2 = 0, X_3 = 1] \\ = \; & \mathbb{P}[X_1 = 1, X_2 = 1, X_3 = 0] = \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 0] = 0. \end{aligned}$$

Interchanging 0 and 1 we obtain a symmetric example. These examples can be obtained in a different way found by Markov. Suppose that we have two independent and identically distributed random variables Y_1, Y_2 taking the values $-1, +1$ with the probability $\frac{1}{2}$. Then the probability law of the random vector $(Y_1, Y_2, Y_1 Y_2)$ is (essentially) the Bernstein's one.

4. General Case of Three Binary Variables

A natural question is: are Bernstein's examples the only one's possible? FELLER has considered them as curiosities and has said that *"...It still takes some search to find a plausible natural example..."* (Feller, 1968, p. 126). In this section we describe *completely* all triplets of pairwise independent binary random variables and then we discuss the exceptional character of Bernstein's examples.

Let us suppose that the values of the considered random variables are 0 and 1, the probabilities

$$p := \mathbb{P}[X_1 = 0],\ q := \mathbb{P}[X_2 = 0],\ r := \mathbb{P}[X_3 = 0],$$

are given and they determine all two–dimensional marginal laws as products, so that the random variables X_1, X_2, X_3 are pairwise independent. Without loss of generality we can suppose that:

$$0 \leq p \leq q \leq r \leq 1. \tag{1}$$

It is easy to prove:

THEOREM 1. *Let $X_i : \Omega \longrightarrow \{0,1\}$; $i = 1,2,3$, be random variables described above. They are pairwise independent if and only if:*

$$\begin{aligned}
&\mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 0] = \alpha,\\
&\mathbb{P}[X_1 = 1, X_2 = 0, X_3 = 0] = -\alpha + qr,\\
&\mathbb{P}[X_1 = 0, X_2 = 1, X_3 = 0] = -\alpha + pr,\\
&\mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 1] = -\alpha + pq,\\
&\mathbb{P}[X_1 = 1, X_2 = 1, X_3 = 0] = \alpha + r(1 - p - q),\\
&\mathbb{P}[X_1 = 1, X_2 = 0, X_3 = 1] = \alpha + q(1 - p - r),\\
&\mathbb{P}[X_1 = 0, X_2 = 1, X_3 = 1] = \alpha + p(1 - q - r),\\
&\mathbb{P}[X_1 = 1, X_2 = 1, X_3 = 1] = 1 - \alpha - p - q - r + pq + qr + pr,
\end{aligned} \tag{2}$$

for each value of the parameter α satisfying

$$\Phi(p,q,r) \leq \alpha \leq \Psi(p,q,r), \tag{3}$$

where

$$\begin{aligned}
\Phi(p,q,r) &:= \max\ \{0,\ p(q + r - 1),\ q(p + r - 1),\ r(p + q - 1)\},\\
\Psi(p,q,r) &:= \min\ \{pq,\ pr,\ qr,\ 1 - p - q - r + pq + pr + qr\}.
\end{aligned}$$

Equivalently,

$$\begin{array}{rlllll}
0 & \leq \alpha \leq pq, & \mathit{if} & p + q \leq 1 & \mathit{and} & q + r \leq 1;\\
p(q + r - 1) & \leq \alpha \leq pq, & \mathit{if} & p + q \leq 1 & \mathit{and} & q + r > 1;\\
p(q + r - 1) & \leq \alpha \leq pq - (p + q - 1)(1 - r), & \mathit{if} & p + q > 1 & \mathit{and} & q + r > 1.
\end{array}$$

The domains of α given by (3), (4), (5), always contain the point $\alpha_0 = pqr$, which represents the mutual independence of X_1, X_2, X_3. They are reduced to one point, when at least one of the values of p, q, r, is equal to 0 or 1, i.e. at least one variable considered is constant, which evidently gives the mutual independence.

In the stationary case $p = q = r$ the parameter α must satisfy

$$0 \le \alpha \le p^2, \qquad \text{if} \qquad 0 \le p \le \frac{1}{2}, \tag{4}$$

$$2p^2 - p \le \alpha \le 3p^2 - 3p + 1, \qquad \text{if} \qquad \frac{1}{2} \le p \le 1, \tag{5}$$

(see Fig.1). Observe that three stationary pairwise independent variables are always exchangeable.

It is easy to verify that for each k, $4 \le k \le 8$, there exist three pairwise independent binary random variables defined on the discrete probability space of k points. If $k = 4$, then Bernstein's examples are the only possible ones, which is surprising. In the nondegenerate case one can not have less than 4 atoms; in this sense Bernstein's examples are optimal and unique.

5. **General Case of Four Binary Variables**

Now, we ask if there exist four binary pairwise independent random variables such that at least one three–dimensional marginal is equal to one of Bernstein's examples. The answer is trivial; it suffices to take X_4 independent of the vector (X_1, X_2, X_3). In this case we lose stationarity and exchangeability of the random vector (X_1, X_2, X_3, X_4), so the next question is, how to do it keeping at least one of these properties. As we shall see below, it is also not possible.

For it, firstly we give a general description of four pairwise independent random variables $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \{0, 1\}$, $i = 1, 2, 3, 4$, such that their probability laws are given by:

$$\mathbb{P}[X_1 = 0] = p,\ \mathbb{P}[X_2 = 0] = q,\ \mathbb{P}[X_3 = 0] = r,\ \mathbb{P}[X_4 = 0] = s,$$

where

$$0 \le p \le q \le r \le s \le 1. \tag{6}$$

This is equivalent to characterising all probability measures concentrated in the corners of the four-dimensional cube $[0,1]^4$, such that all their marginal laws of dimension 2 are product measures of those of dimension 1.

Let us suppose that the common law of every triplet (X_i, X_j, X_k) is already chosen, i.e. one has:

$$\begin{aligned} \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 0] = \alpha_1, \\ \mathbb{P}[X_1 = 0, X_2 = 0, X_4 = 0] = \alpha_2, \\ \mathbb{P}[X_1 = 0, X_3 = 0, X_4 = 0] = \alpha_3, \\ \mathbb{P}[X_2 = 0, X_3 = 0, X_4 = 0] = \alpha_4, \end{aligned} \tag{7}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy the inequalities analogous to (3).

As in the three-dimensional case we put some mass into one corner; the measures of the others are completely determined. First we put

$$\mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0] := A. \tag{8}$$

The equalities analogous to

$$\alpha_1 = \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 0] = A + \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 1]$$

imply

$$\begin{aligned} \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 1] &:= \alpha_1 - A, \\ \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 0] &:= \alpha_2 - A, \\ \mathbb{P}[X_1 = 0, X_2 = 1, X_3 = 0, X_4 = 0] &:= \alpha_3 - A, \\ \mathbb{P}[X_1 = 1, X_2 = 0, X_3 = 0, X_4 = 0] &:= \alpha_4 - A. \end{aligned} \tag{9}$$

Now we want to have, for example,

$$\begin{aligned} \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 1] \ + \ \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 1] \\ = \ \mathbb{P}[X_1 = 0, X_2 = 0, X_4 = 1] \ = \ -\alpha_2 + pq, \end{aligned}$$

so we must define

$$\mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 1] := A + pq - \alpha_1 - \alpha_2. \tag{10}$$

We have

$$\begin{aligned} &\mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 0] + \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 1] \\ &= \alpha_2 - A + A + pq - \alpha_1 - \alpha_2 = -\alpha_1 + pq = \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 1], \end{aligned}$$

so the measure of the corner $(0,0,1,1)$ does not depend on the way by which we arrive from $(0,0,0,0)$ to $(0,0,1,1)$. In the same manner, we define:

$$\begin{aligned} \mathbb{P}[X_1 = 0, X_2 = 1, X_3 = 0, X_4 = 1] &:= A + pr - \alpha_1 - \alpha_3, \\ \mathbb{P}[X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 0] &:= A + ps - \alpha_2 - \alpha_3, \\ \mathbb{P}[X_1 = 1, X_2 = 0, X_3 = 0, X_4 = 1] &:= A + qr - \alpha_1 - \alpha_4, \\ \mathbb{P}[X_1 = 1, X_2 = 0, X_3 = 1, X_4 = 0] &:= A + qs - \alpha_2 - \alpha_4, \\ \mathbb{P}[X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 0] &:= A + rs - \alpha_3 - \alpha_4. \end{aligned} \tag{11}$$

The equality

$$\begin{aligned} &\mathbb{P}[X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 1] + \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 1] \\ &= \mathbb{P}[X_1 = 0, X_3 = 1, X_4 = 1] = \alpha_3 + p(1 - r - s), \end{aligned}$$

implies

$$\mathbb{P}[X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 1] := -A + \alpha_1 + \alpha_2 + \alpha_3 + p(1 - q - r - s) \tag{12}$$

and similarly we put

$$\begin{aligned}\mathbb{P}[X_1=1,X_2=1,X_3=0,X_4=1] &:= -A+\alpha_1+\alpha_3+\alpha_4+r(1-p-q-s),\\ \mathbb{P}[X_1=1,X_2=0,X_3=1,X_4=1] &:= -A+\alpha_1+\alpha_2+\alpha_4+q(1-p-r-s),\\ \mathbb{P}[X_1=1,X_2=1,X_3=1,X_4=0] &:= -A+\alpha_2+\alpha_3+\alpha_4+s(1-p-q-r).\end{aligned} \tag{13}$$

Finally, we define the measure of the last corner as the difference between 1 and the sum of measures of all other corners

$$\begin{aligned}&\mathbb{P}[X_1=1,X_2=1,X_3=1,X_4=1] := 1-p-q-r-s\\ &-\alpha_1-\alpha_2-\alpha_3-\alpha_4+pq+pr+ps+qs+qr+rs+A.\end{aligned} \tag{14}$$

All the numbers above are well defined, so they are non-negative if and only if the parameter A satisfies the following inequalities

$$F(\alpha_1,\alpha_2,\alpha_3,\alpha_4;p,q,r,s) \le A \le G(\alpha_1,\alpha_2,\alpha_3,\alpha_4;p,q,r,s), \tag{15}$$

where

$$\begin{aligned}&F(\alpha_1,\alpha_2,\alpha_3,\alpha_4;p,q,r,s) := \max\{0,\ \alpha_1+\alpha_2-pq,\ \alpha_1+\alpha_3-pr,\\ &\alpha_2+\alpha_3-ps,\ \alpha_1+\alpha_4-qr,\ \alpha_2+\alpha_4-qs,\ \alpha_3+\alpha_4-rs,\\ &p+q+r+s-1+\alpha_1+\alpha_2+\alpha_3+\alpha_4-pq-pr-ps-qr-qs-rs\},\end{aligned}$$

$$\begin{aligned}&G(\alpha_1,\alpha_2,\alpha_3,\alpha_4;p,q,r,s) := \min\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\\ &s(1-p-q-r)+\alpha_2+\alpha_3+\alpha_4,\ r(1-p-q-s)+\alpha_1+\alpha_3+\alpha_4,\\ &q(1-p-r-s)+\alpha_1+\alpha_2+\alpha_4,\ p(1-q-r-s)+\alpha_1+\alpha_2+\alpha_3\}.\end{aligned}$$

Theorem 1 implies:

THEOREM 2. *Let $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \{0,1\}$, $i = 1,2,3,4$, be random variables satisfying* (6) *and* (7)*. They are pairwise independent if and only if their common probability law is given by the equalities* (8) – (14)*, where the parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy the inequalities analogous to* (3) *and the parameter A satisfies the inequality* (15). □

If $0 < p \le q \le r \le s \le 1$ and $r + s \le 1$, then $\alpha_i,\ 1 \le i \le 4$, must satisfy

$$0 \le \alpha_1 \le pq,\ 0 \le \alpha_2 \le pq,\ 0 \le \alpha_3 \le pr,\ 0 \le \alpha_4 \le qr,$$

so choosing

$$\alpha_1 = pq,\ \alpha_2 = pq,\ \alpha_3 = pr,\ \alpha_4 = 0,$$

we have

$$F(\alpha_1,\alpha_2,\alpha_3,\alpha_4;p,q,r,s) \ge pq > 0$$

$$G(\alpha_1,\alpha_2,\alpha_3,\alpha_4;p,q,r,s) = 0,$$

and hence the searched four–dimensional measure does not exist.

COROLLARY 1. *Let $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \{0,1\}$, $i = 1,2,3,4$, be pairwise independent random variables satisfying* (6) *and such that one of their three–dimensional* *marginal laws is of Bernstein's type. Then such a random vector is independent from the remaining fourth variable.* □

It is easy to see that in this case each triplet containing the fourth variable is mutually independent, all four variables are not mutually independent and are not exchangeable. The vector (X_1, X_2, X_3, X_4) can be stationary only if the laws of (X_1, X_2, X_3) and (X_2, X_3, X_4) are not of the Bernstein type.

COROLLARY 2. *There is no four random variables such that:*

• *all marginal three–dimensional laws are equal to the same law given by one of Bernstein's examples;*

or

• *every common law of dimension* 3 *is equal to the one of the Bernstein's laws.*

□

6. Case of Four Exchangeable Variables

Let us suppose that

$$p = q = r = s,\ \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 := \alpha,$$

where the parameter α satisfies (4) or (5).

If we choose some value of parameter A satisfying (15), then the corresponding measure is exchangeable, because the measures of corners depend only of the number of their zeros.

The inequality (15) takes the form

$$\max\ \{0, 2\alpha - p^2, 4\alpha - 1 + 4p - 6p^2\} \leq A \leq \min\ \{\alpha, 3\alpha + p(1-3p)\}.$$

We see immediately that

1° $0 \leq A \leq 3\alpha + p(1-3p)$, if $0 \leq \alpha \leq \frac{p^2}{2}$ and $\alpha \leq \frac{1}{2}p(3p-1)$;

2° $0 \leq A \leq \alpha$, if $0 \leq \alpha \leq \frac{p^2}{2}$ and $\frac{1}{2}p(3p-1) \leq \alpha$;

3° $2\alpha - p^2 \leq A \leq 3\alpha + p(1-3p)$, if $\frac{p^2}{2} \leq \alpha \leq \frac{p^2}{2} + 2(p-\frac{1}{2})^2$ and $\alpha \leq \frac{1}{2}p(3p-1)$;

4° $2\alpha - p^2 \leq A \leq \alpha$, if $\frac{p^2}{2} \leq \alpha \leq \frac{p^2}{2} + 2(p-\frac{1}{2})^2$ and $\frac{1}{2}p(3p-1) \leq \alpha$;

5° $4\alpha - 1 + 4p - 6p^2 \leq A \leq 3\alpha + p(1-3p)$, if $\frac{p^2}{2} + 2(p-\frac{1}{2})^2 \leq \alpha$ and $\alpha \leq \frac{1}{2}p(3p-1)$;

6° $4\alpha - 1 + 4p - 6p^2 \leq A \leq \alpha$, if $\frac{p^2}{2} + 2(p-\frac{1}{2})^2 \leq \alpha$ and $\frac{1}{2}p(3p-1) \leq \alpha$.

In the case 1° one has $0 \leq 3\alpha + p(1-3p)$ if and only if $\alpha \geq \frac{1}{3}p(3p-1)$. Hence if $\alpha < \frac{1}{3}p(3p-1)$ (for ex. $\alpha = 0,\ p = \frac{1}{2}$), the parameter A (i.e. the searched measure) does not exist.

Similarly in the case 6° we have $4\alpha - 1 + 4p - 6p^2 \leq \alpha$ if and only if $\alpha \leq 2p^2 - \frac{4}{3}p + \frac{1}{3}$, so for $\alpha > 2p^2 - \frac{4}{3}p + \frac{1}{3}$ the parameter A does not exist (for ex. $\alpha = \frac{1}{4},\ p = \frac{1}{2}$). It is easy to verify that in the cases 2°-5° all the intervals for "admissible" values of A are non-empty.

The picture below shows, which three–dimensional laws can be extended to the exchangeable four–dimensional law.

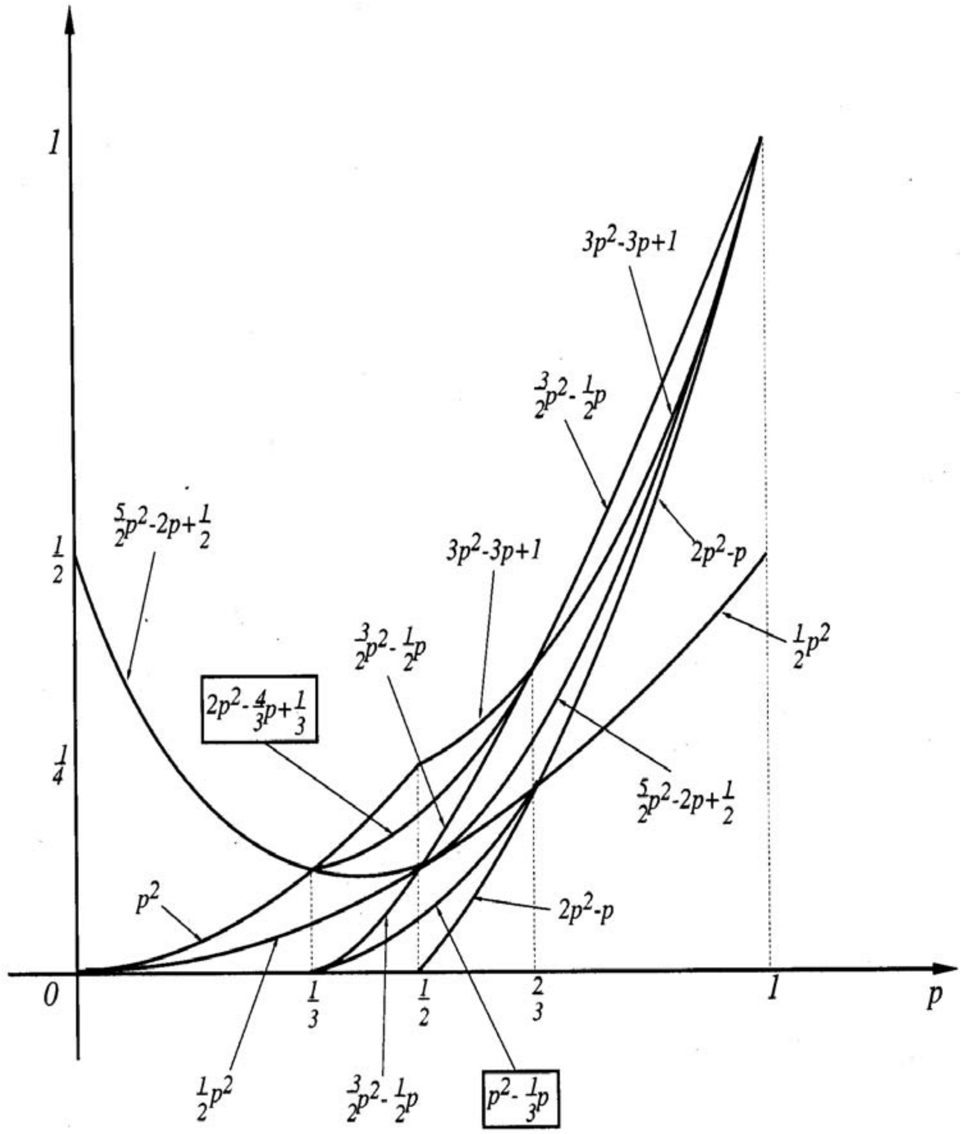

Figure 1

7. Case of Four Stationary Variables

Assume that

$$p = q = r = s,$$

$$\alpha_1 = \mathbb{P}[X_1 = 0, X_2 = 0, X_3 = 0] = \mathbb{P}[X_2 = 0, X_3 = 0, X_4 = 0] = \alpha_4 := \alpha.$$

In this case the parameter A must satisfy the inequality

$$\Phi(\alpha,\alpha_2,\alpha_3;p) \le A \le \Psi(\alpha,\alpha_2,\alpha_3;p), \tag{16}$$

where

$$\Phi(\alpha,\alpha_2,\alpha_3;p) = \max \quad \{0, 2\alpha - p^2, \alpha+\alpha_2-p^2, \alpha+\alpha_3-p^2, \alpha_2+\alpha_3-p^2, \\ 2\alpha+\alpha_2+\alpha_3-6p^2+4p-1\},$$

$$\Psi(\alpha,\alpha_2,\alpha_3;p) = \min \quad \{\alpha,\alpha_2,\alpha_3,\alpha+\alpha_2+\alpha_3+p(1-3p), 2\alpha+\alpha_2+p(1-3p), \\ 2\alpha+\alpha_3+p(1-3p)\}.$$

This interval is non-empty if and only if the following inequalities are satisfied:

$$\max\,\{0,(2p-1)p\} \le \alpha \le \min\,\{p^2, 3p^2-3p+1\}, \tag{17}$$

$$\max\,\{0,(2p-1)p\} \le \alpha_2 \le \min\,\{p^2, 3p^2-3p+1\}, \tag{18}$$

$$\max\,\{0,(2p-1)p\} \le \alpha_3 \le \min\,\{p^2, 3p^2-3p+1\}, \tag{19}$$

$$(3p-1)p \le \alpha+\alpha_2+\alpha_3 \le 6p^2-4p+1 = (3p^2-3p+1)+(3p-1)p, \tag{20}$$

$$(3p-1)p \le 2\alpha+\alpha_2 \le 6p^2-4p+1, \tag{21}$$

$$(3p-1)p \le 2\alpha+\alpha_3 \le 6p^2-4p+1, \tag{22}$$

$$(2p-1)p \le 2\alpha-\alpha_2 \le p^2, \tag{23}$$

$$(2p-1)p \le 2\alpha-\alpha_3 \le p^2, \tag{24}$$

$$(2p-1)p \le \alpha+\alpha_2-\alpha_3 \le p^2, \tag{25}$$

$$(2p-1)p \le \alpha-\alpha_2+\alpha_3 \le p^2, \tag{26}$$

$$(2p-1)p \le -\alpha+\alpha_2+\alpha_3 \le p^2. \tag{27}$$

Each inequality above means that the set of admissible parameters $(\alpha,\alpha_2,\alpha_3)$ is contained between two parallel planes and obviously it is equal to the intersection of all domains given by (17) – (27).

For $p = \frac{1}{2}$ we obtain the following set:

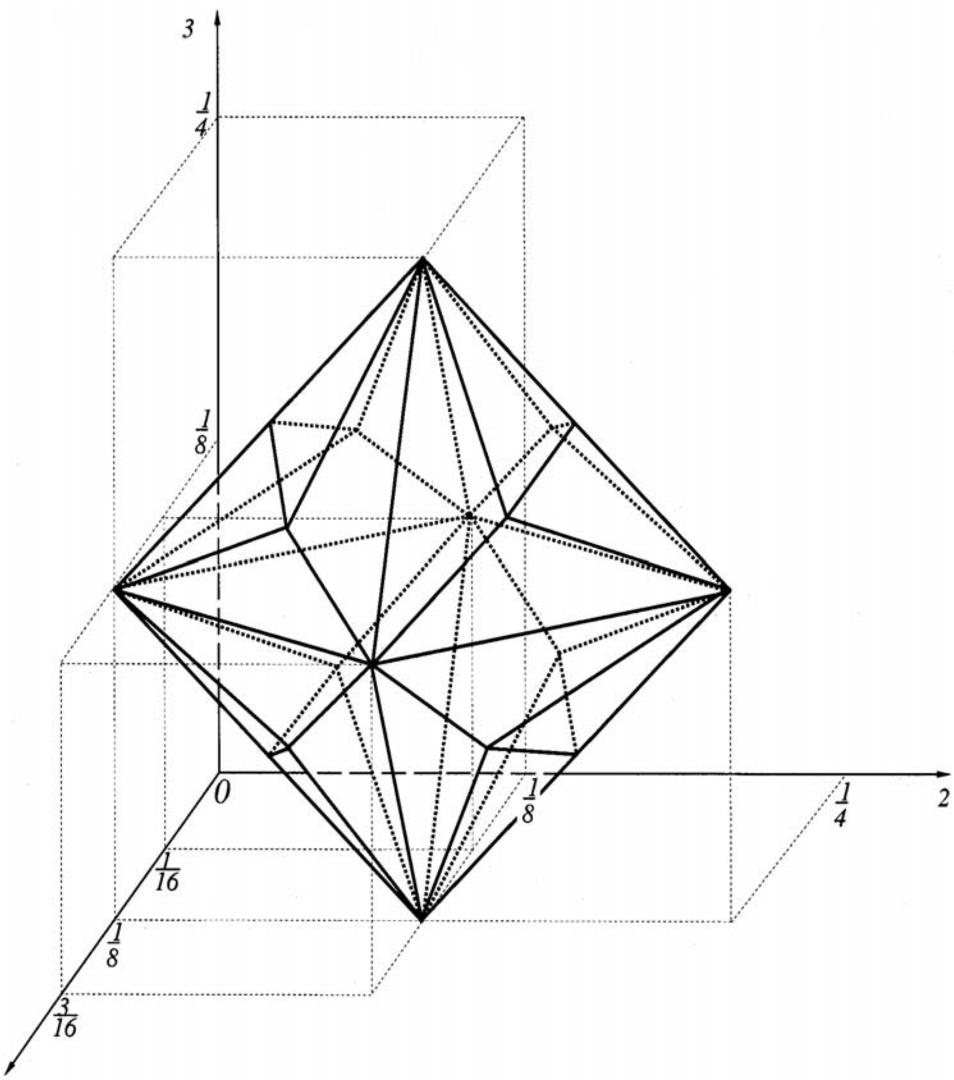

Figure 2

Then for $p = \frac{1}{2}$ and for $(\alpha, \alpha_2, \alpha_3)$ outside of the set A the searched measure does not exist. In particular, we can see once more that it is not possible to extend Bernstein's examples $(\alpha = 0, \frac{1}{4})$ to the stationary and pairwise independent four–dimensional measure. In fact, we have already seen that the only possibility of extending one of Bernstein's laws to four pairwise independent random variables is in a trivial manner, i.e. when the variable X_4 is independent of the vector (X_1, X_2, X_3).

It does not mean, that there is no four stationary random variables such that one of its marginal three–dimensional laws is equal to some Bernstein's example.

Fig. 2 shows that there are exactly four such measures, represented by "extremal" points of the set A. It can be proved that these measures can be extended (keeping stationarity and pairwise independence) only up to dimension seven, every step being unique.

8. Multidimensional Bernstein's Example

The following example can be found in Stoyanov (1987).

Let us consider n–dimensional product $\{0,1\}^n$, $n \geq 3$. First we define

$$\nu(x) := \text{number of zeros in } x \in \{0,1\}^n,$$

and then we define a measure $\mathbb{P}$ on $\{0,1\}^n$ by:

$$\mathbb{P}(x) := \begin{cases} \frac{1}{2^{n-1}}, & \text{if } \nu(x) \text{is an odd number;} \\ 0, & \text{if not.} \end{cases}$$

For $n = 3$ it gives the example of Bernstein. One can prove:

THEOREM 3.

• *All n canonical projections are (under $\mathbb{P}$) pairwise independent and exchangeable.*

• *Moreover, every $n-1$ canonical projections are mutually independent.*

• *This measure realises the* one–dependence, *i.e. for every $1 < k < n$ the vectors composed by the $k-1$ first projections and the $n-k$ last projections are independent.*

• *The measure $\mathbb{P}$ has an unique stationary extension $\hat{\mathbb{P}}$ to $\{0,1\}^{n+1}$, for which the pairwise independence is not verified.* □

REMARK. Let us consider the random vector (X, Y, Z, XYZ) , where X, Y, Z, are mutually independent random variables taking the values 1 with probability $\frac{1}{2}$. It is triplewise, then also pairwise independent and it has (at least) three non-trivial pairwise independent extensions: (X, Y, Z, XYZ, XY), (X, Y, Z, XYZ, XZ) and (X, Y, Z, XYZ, YZ), which evidently can not be stationary. It would be interesting to verify, if there exist other non-trivial pairwise independent extensions of (X, Y, Z, XYZ).

References

BERNSTEIN, S.N. (1946). *Theory of Probability*, Gostechizdat, Moscow-Leningrad, 4th ed. (in Russian).

BRETAGNOLLE, J. AND KŁOPOTOWSKI, A. (1995). Sur l'existence des suites de variables aléatoires *s á s* indépendantes échangeables ou stationnaires, *Ann. Inst. Henri Poincaré* **31**, 325–350.

CUESTA, J.A. AND MATRÁN C. (1991). On the asymptotic behavior of sums of pairwise independent random variables, *Statist. Prob. Letters* **11**, 201–210.

DERRIENNIC, Y. AND KŁOPOTOWSKI, A. (1991a). Cinq variables aléatoires binaires stationnaires deux à deux indépendantes, *Prépubl. Institut Galilée, Université Paris XIII,* November, 1–38.

– – –– (1991b) Sur les hypothèses constructibles concernant des suites de variables aléatoires binaires, *idem,* December, 1–10.

ETEMADI, N. (1981). An elementary proof of the strong law of large numbers, *Z. Wahrsch. verw. Gebiete* **55**, 119–122.

FELLER, W. (1968). *An Introduction to Probability Theory and Its Applications, vol. I* John Wiley.

JANSON, S. (1988). Some pairwise independent sequences for which the central limit theorem fails, *Stochastics* **23**, 439–448.

KŁOPOTOWSKI, A. AND NADKARNI, M.G. (1999). On the existence of automorphisms with simple Lebesgue spectrum, *Proc. Indian Acad. Sciences, Math. Sci.* **109**, 47–56.

KŁOPOTOWSKI, A. AND ROBERTSON, J.B. (1999). Some new examples of pairwise independent random variables, *Sankhyā Ser. A* **61**, 72–88.

ROBERTSON, J.B. (1985). Independence and fair coin-tossing, *Math. Scientist* **10**, 109–117.

– – –– (1988). A two state pairwise independent stationary process for which $X_1X_3X_5$ is dependent, *Sankhyā Ser. A* **50**, 171–183.

– – –– (1988). Another pairwise independent stationary chain, In: *Approximation, Probability and Related Fields.* Edited by G. Anastassiou and S.T. Rachev, Plenum Press, New York, 423–434.

ROBERTSON, J.B. AND WOMACK, J.M. (1985). A pairwise independent stationary stochastic process, *Statist. Probab. Lett.* **3**, 195–199.

STOYANOV, J.M. (1987). *Counterexamples in Probability*, John Wiley.

YVES DERRIENNIC
UNIVERSITÉ DE BRETAGNE OCCIDENTALE
DÉPARTEMENT DE MATHÉMATIQUES
ET INFORMATIQUE
6, AV. VICTOR LE GORGEU
29287 BREST CEDEX
FRANCE
E-mail:Yves.Derriennic@univ-brest.fr

ANDRZEJ KŁOPOTOWSKI
UNIVERSITÉ PARIS XIII
INSTITUT GALILÉE
99, AV. J.-B. CLÉMENT
93430 VILLETANEUSE CEDEX
FRANCE
E-mail: klopot@zeus.math.univ-paris13.fr