

A NOTE ON WEAK- \star PERTURBATIONS OF g -MEASURES*

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SUMMARY. We observe that the standard Kemeny and Snell identity, basic in Markov Chain Perturbation Theory, generalizes to chains with complete connections. Therefrom a few elementary observations are derived for weak- \star perturbation of g -measures considered as a function of g .

This identity is applied to produce a simple proof for the phase transition phenomenon recently exhibited in Bramson and Kalikow (1993).

1. Introduction

Let $q \geq 1$ be an integer, denote $\mathcal{A} = \{0, 1, \dots, q-1\}$, $X = \mathcal{A}^{\mathbb{N}}$, and $S : X \rightarrow X$ the shift. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, and $\mathcal{X} = (X_n : \Omega \rightarrow \mathcal{A})_{n \in \mathbb{Z}}$ a stationary process.

The process is Markov if $\mathbb{P}(X_n = j | X_{n-1} = i) = p_{i,j}$, with transition matrix $P = (p_{i,j}) \in [0, 1]^{\mathcal{A} \times \mathcal{A}}$. Assume it is such, with P irreducible and aperiodic, and let π be the associated stationary distribution vector.

Let $\tilde{\mathcal{X}} = (\tilde{X}_n)_{n \in \mathbb{Z}}$ be another Markov process, having the same state space, with matrix \tilde{P} . Denote $\tilde{\pi}$ the corresponding distribution. The Kemeny and Snell identity then reads as

$$\tilde{\pi} - \pi = \sum_{k \geq 0} \tilde{\pi} (\tilde{P} - P) P^k. \quad (1)$$

For chains having finite memory, one uses the Markov case by increasing the state space. The identity (1) allows, with e.g. ergodicity coefficients (Seneta, 1988), to measure the sensitivity of the chain \mathcal{X} under perturbations – $\tilde{\mathcal{X}}$ is the perturbed chain.

The natural generalization of a Markov chain is a *chain with complete connections* (Doebelin and Fortel (1930), Iosifescu and Grigorescu (1997)) : one assumes

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there exists $g : X \rightarrow [0, 1]$ Borel such that for $x \in X$ and $i \in \mathcal{A}$, if $n \in \mathbb{Z}$ and if $\overleftarrow{X}_{n-1} = (X_{n-1}, X_{n-2}, \dots)$, then

$$\mathbb{P}(X_n = i \mid \overleftarrow{X} = x) = g(ix).$$

Such a g is a \mathcal{G} -function if it satisfies the preceding condition for a process \mathcal{X} , and if moreover it is *continuous and positive*. We shall let \mathcal{G} denote the set of \mathcal{G} -functions.

Let $\bar{\mu}$ be a stationary measure for \mathcal{X} : its projection on X , μ , is a g -measure in the sense of Keane (1972).

If $\mathcal{C}(X) = \{f : X \rightarrow \mathbb{R} : \text{continuous}\}$ is equipped with the supremum norm, then g induces a *transfer operator* $\mathcal{L}_g : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$ defined by

$$\mathcal{L}_g f(x) = \sum_{i \in \mathcal{A}} g(ix) f(ix).$$

Its dual \mathcal{L}_g^* acts continuously on the compact $\mathcal{M}(X)$, the set of Borel probability measures on X , and has at least one fixed point μ (Keane (1972)) : it is a g -measure, always S -invariant.

The irreducible aperiodic case for Markov chains corresponds to that of \mathcal{G} -functions having a unique g -measure - denoted μ_g . A g having a unique g -measure shall be referred to as *ergodic* : μ_g is then (in the usual sense) ergodic.

Several studies - (Baladi, Isola and Schmitt (1998), Froyland (1998), Murray (1998)) - concerned with the Ulam approximation scheme for the density of the invariant absolutely continuous measure associated to an expanding map of a compact manifold use the Kemeny and Snell identity - for the approximant is Markov.

Other studies are concerned with setting down additional regularity assumptions on g to grant its ergodicity together with ergodic properties of the natural extension of the system (X, S, μ_g) Berbee, 1987, Hulse, 1997 (cf. Remark 2 in the following Section).

It was surprisingly only recently in Bramson and Kalikow (1993) that a non ergodic \mathcal{G} -function was exhibited. This corresponds to the phase transition phenomenon in Thermodynamics (Zinsmeister (1996)).

In this note, we shall show how to generalize the Kemeny and Snell identity to the case of g -measures - Theorem 1, using standard technics from Operator Theory.

This generalized identity however yields information on :

- weak- \star perturbation of μ_g for an ergodic g ;
- an elementary proof of the result presented in Bramson and Kalikow (1993) (Theorem 2).

The following Section presents our results, and the others contain the proofs.

2. Results

THEOREM 1. *Let g and \tilde{g} be two \mathcal{G} -functions, μ a g -measure and $\tilde{\mu}$ a \tilde{g} -measure. Then given $f \in \mathcal{C}(\mathcal{X})$, and $n \geq 0$,*

$$\tilde{\mu}(f) - \mu(f) = \sum_{k=0}^n \tilde{\mu}((\mathcal{L}_{\tilde{g}} - \mathcal{L}_g)\mathcal{L}_g^k f) + \tilde{\mu}(\mathcal{L}_g^{n+1} f - \mu(f)).$$

If g is ergodic, then

$$\tilde{\mu}(f) - \mu(f) = \lim_P \frac{1}{P} \sum_{p < P} (P - p) \tilde{\mu}((\mathcal{L}_{\tilde{g}} - \mathcal{L}_g)\mathcal{L}_g^p f). \quad (2)$$

Furthermore, $\tilde{\mu} \rightarrow \mu$ weak-★ if $\tilde{\mu} \left(\sum_{i \in \mathcal{A}} |\tilde{g}(i \cdot) - g(i \cdot)| \right) \rightarrow 0$.

Assume that for any $f \in X$ and $x \in \mathcal{C}X$, $\mathcal{L}_g^n f(x) \rightarrow \mu(f)$. Then

$$\tilde{\mu}(f) - \mu(f) = \sum_{k \geq 0} \tilde{\mu}((\mathcal{L}_{\tilde{g}} - \mathcal{L}_g)\mathcal{L}_g^k f).$$

REMARK 1. Let us suppose g ergodic. Let $p \geq 0$ and $u \in \mathcal{A}^p$: let $[u] = \{x \in X : (x_0, \dots, x_{p-1}) = u\}$ be the cylinder associated to the pattern u , and let $|u| = p$ denote its length. Assume that $\{g \neq \tilde{g}\} \subset \cup_{i \in \mathcal{A}} [iu]$, and that $\tilde{g} \leq 1 - \lambda$ for some $\lambda > 0$. Then $\tilde{\mu} \left(\sum_{i \in \mathcal{A}} |\tilde{g}(i \cdot) - g(i \cdot)| \right) \leq \#\mathcal{A}(1 - \lambda)^p$. Hence $\tilde{\mu}$ can be weak-★ close to μ while $\|\tilde{g} - g\|_\infty$ is not to 0.

For a more regular g this holds true in the stronger \bar{d} -metric (Lacroix (1999)).

REMARK 2. For $x, y \in X$, write $x \stackrel{m}{=} y$ if $x_i = y_i$ for $i < m$, and $x \stackrel{m}{\Delta} y$ if $x_i = y_i$ for $i \neq m$. Then put

$$\text{var}_m(g) = \sup_{x \stackrel{m}{=} y} |g(x) - g(y)| \quad \text{and} \quad v_m(g) = \sup_{x \stackrel{m}{\Delta} y} |g(x) - g(y)|.$$

According to Berbee (1987), Hulse (1997), Walters (1975), if $\sum_n v_n(g) < \infty$ or if $\sum_N \prod_{n < N} (1 - \text{var}_n(g)) = \infty$, then $\mathcal{L}_g^n f(x) \rightarrow \mu(f)$ for any $f \in \mathcal{C}(\mathcal{X})$, uniformly in $x \in X$.

REMARK 3. It holds that, for $j \in \mathcal{A}$,

$$\mathcal{L}_{\tilde{g}} h(x) - \mathcal{L}_g h(x) = \sum_{\substack{i \in \mathcal{A}, \\ i \neq j}} (\tilde{g}(ix) - g(ix))(h(ix) - h(jx)).$$

If $\sum_n v_n(g) < \infty$, and $f = \mathbf{1}_{[u]}$ for some pattern u , then it follows from the computations appearing in Walters (1975) that $\sum_k \sup_{x, y \in X} |\mathcal{L}_g^k f(x) - \mathcal{L}_g^k f(y)| < \infty$, whence

$$\tilde{\mu}(f) - \mu(f) = \sum_{\substack{i \in \mathcal{A}, \\ i \neq j}} \sum_k \tilde{\mu}((\tilde{g}(ix) - g(ix))(\mathcal{L}_g^k f(ix) - \mathcal{L}_g^k f(jx))).$$

Here is the result (Bramson and Kalikow (1993)) (where it is proved using coupling techniques) :

THEOREM 2. *There exists a \mathcal{G} -function g having several g -measures.*

3. Proof of Theorem 1

We continue notations from Theorem 1. Put $\Delta(f) = \tilde{\mu}(f) - \mu(f)$. Then since $\mathcal{L}_g^* \mu = \mu$ and $\mathcal{L}_{\tilde{g}}^* \tilde{\mu} = \tilde{\mu}$, it follows that

$$\Delta(f) = \tilde{\mu}((\mathcal{L}_{\tilde{g}} - \mathcal{L}_g)f) + \Delta(\mathcal{L}_g f).$$

Substitute $\mathcal{L}_g f$ to f : the last equality above reads

$$\Delta(f) = \tilde{\mu}((\mathcal{L}_{\tilde{g}} - \mathcal{L}_g)f) + \tilde{\mu}((\mathcal{L}_{\tilde{g}} - \mathcal{L}_g)\mathcal{L}_g f) + \Delta(\mathcal{L}_g^2 f).$$

Repeated substitutions yield the first statement of Theorem 1.

It is well known (Fan (1996)) that g is ergodic if and only if for any $f \in \mathcal{C}(\mathcal{X})$, there exists a constant $c(f)$ such that uniformly in x ,

$$\frac{1}{P} \sum_{p < P} \mathcal{L}_g^p f(x) \rightarrow c(f). \quad (3)$$

Then if μ is the only g -measure, $c(f) = \mu(f)$. Let us denote $S_P f(x) = \frac{1}{P} \sum_{p < P} \mathcal{L}_g^p f(x)$.

Then let us write down the equalities (2) for $n = 0, 2, \dots, P$ one above the other, add up, take the resulting arithmetic mean, and deduce

$$\tilde{\mu}(f) - \mu(f) = \frac{P}{P+1} \left(\frac{1}{P} \sum_{p < P} (P-p) \tilde{\mu}((\mathcal{L}_{\tilde{g}} - \mathcal{L}_g)\mathcal{L}_g^p f) \right) + \tilde{\mu}(S_{P+1} f - \mu(f)). \quad (4)$$

Since g is assumed ergodic, by (3), $\|S_P f - \mu(f)\|_\infty \rightarrow 0$, therefore $\lim_P \tilde{\mu}(S_P f - \mu(f)) = 0$. With (4), the second assertion of Theorem 1 follows.

Let $f \in \mathcal{C}(\mathcal{X})$, and $\varepsilon > 0$. Using (3), let us pick a large enough P so that $\|S_P f(x) - \mu(f)\|_\infty < \varepsilon/2$. Observe that $\|\mathcal{L}_g^p f\|_\infty \leq \|f\|_\infty$, hence

$$|\tilde{\mu}((\mathcal{L}_{\tilde{g}} - \mathcal{L}_g)\mathcal{L}_g^p f)| \leq \|f\|_\infty \left(\sum_{i \in \mathcal{A}} \tilde{\mu}(|\tilde{g}(ix) - g(ix)|) \right). \quad (5)$$

Put $\Gamma = \sum_{i \in \mathcal{A}} \tilde{\mu}(|\tilde{g}(ix) - g(ix)|)$. By (2) and (5), we get

$$\begin{aligned} |\Delta(f)| &\leq \frac{\Gamma \|\mathcal{L}_g^p f\|_\infty}{P+1} \sum_{p < P} (P-p) + \tilde{\mu}(|S_{P+1} f - \mu(f)|) \\ &\leq \frac{P}{2} \Gamma \|f\|_\infty + \varepsilon/2, \end{aligned} \quad (6)$$

proving the third assertion of Theorem 1.

The last follows from the second, avoiding the use of the arithmetic mean. \square

4. Proof of Theorem 2

4.1 *The Bramson - Kalikow example.* This example was suggested by S. Kalikow and B. Weiss in Kalikow (1990) as possibly exhibiting phase transition.

Let $p_k = \frac{1}{2} \left(\frac{2}{3}\right)^k$, $k \geq 1$. For m an odd integer, define $W_m(x) = 1/4$ if $\sum_{i < m} x_i < m/2$, and $W_m(x) = 3/4$ otherwise, $x \in X$. Notice that $\sum_k p_k = 1$, and that $\sum_{t > k} p_t = 2p_k$ for $k \geq 1$.

Throughout, let $(m_k)_{k \geq 1}$ be an increasing sequence of odd integers. Define

$$g(1x) = \sum_{k \geq 1} p_k W_{m_k}(x) \text{ and } g(0x) = 1 - g(1x), \quad x \in X.$$

It is clear that $g \in \mathcal{G}$.

We shall show that if $(m_k)_{k \geq 1}$ increases rapidly enough, then g has several g -measures.

4.2 *A first reduction of the proof.* Given $x \in X$, put $\bar{x} = (1 - x_i)_{i \geq 0}$. Since for each $k \geq 1$, $W_{m_k}(\bar{x}) = 1 - W_{m_k}(x)$, it follows that

$$g(\bar{x}) = g(x), \quad x \in X,$$

which we shall refer to as g being *symmetric*. Denote, for $N \geq 1$ and $f \in \mathcal{C}(\mathcal{X})$,

$$S_N g f(x) = \frac{1}{N} \sum_{n < N} \mathcal{L}_g^n f(x).$$

Denote also $1^\infty = (1, \dots, 1, \dots)$ and $0^\infty = (0, \dots, 0, \dots)$. Since $1_{[1]} + 1_{[0]} = 1$, one has $S_N g 1_{[1]}(x) + S_N g 1_{[0]}(x) = 1$, $x \in X$.

But by symmetry, $S_N g 1_{[1]}(x) = S_N g 1_{[0]}(\bar{x})$. Therefore if there exists an increasing sequence $(N_k)_{k \geq 1}$ and a $\delta > 0$ such that

$$S_{N_k} g 1_{[1]}(1^\infty) > \frac{1}{2} + \delta,$$

then by separability of $\mathcal{C}(\mathcal{X})$, there will exist a subsequence $(N_k)_{k \geq 1}$ (which we still denote (N_k)) such that for $f \in \mathcal{C}(\mathcal{X})$, $S_{N_k} g f(1^\infty) \rightarrow \mu^+(f)$, and $S_{N_k} g f(0^\infty) \rightarrow \mu^-(f)$ where $\mu^+(f)$ and $\mu^-(f)$ are constants.

An easy observation is that μ^+ and μ^- are g -measures. To see that they differ, sufficient is to observe that by symmetry, $\mu^+([1]) = \mu^-([0]) \geq 1/2 + \delta$, hence $\mu^+([1]) \neq \mu^-([1])$. We have proved the following :

LEMMA 1. *g has several g -measures if there exists $\delta > 0$ and an increasing sequence (N_k) such that for $k \geq 1$, $S_{N_k} g 1_{[1]}(1^\infty) > 1/2 + \delta$.*

4.3 *A second reduction of the proof.* Let $n \geq 1$ be given : put $k(n) = \min\{k \geq 2 : n < m_k/2\}$. Then if $k \geq k(n)$, and v is any pattern of length $p \leq n$, $W_{m_k}(S(v1^\infty)) = 3/4$.

Define $g_k \in \mathcal{G}$ as

$$g_k(1x) = \sum_{t \leq k} p_t W_{m_t}(x) + \frac{3}{4} \sum_{t > k} p_t \left(= \sum_{t \leq k} p_t W_{m_t}(x) + \frac{3p_k}{2} \right).$$

Then g_k determines a chain with finite memory, hence has a unique g_k -measure μ_k . Moreover $\mathcal{L}_{g_k}^n f(x) \rightarrow \mu_k(f)$ uniformly in x , for any $f \in \mathcal{C}(\mathcal{X})$.

Let $\varepsilon > 0$ be given. Denote $k_0 = k(n) - 1$: as long as $k(n) = k_0 + 1$, one has $\mathcal{L}_{g_{k_0}}^n 1_{[1]}(1^\infty) = \mathcal{L}_{g_{k_0}}^n 1_{[1]}(1^\infty)$. Hence if m_{k_0+1} is large enough, there will exist some $N_{k_0} > 1$ such that $S_{N_{k_0}} g_{k_0} 1_{[1]}(1^\infty) \geq \mu_{k_0}([1]) - \varepsilon$.

Suppose that $m_2 < \dots < m_{k_0}$ have been chosen large enough for that for each $1 \leq t \leq k_0 - 1$, there exists $N_t \geq 1$ satisfying

$$S_{N_t} g_{k_0} 1_{[1]}(1^\infty) \geq \mu_t([1]) - \varepsilon.$$

Then if we choose m_{k_0+1} large enough, we shall find some $N_{k_0} > N_{k_0-1}$ such that $S_{N_{k_0}} g_{k_0} 1_{[1]}(1^\infty) \geq \mu_{k_0}([1]) - \varepsilon$.

Let us choose initial $m_1 = 1$: then g_1 is Markov, with matrix

$$M_1 = \begin{pmatrix} \hat{g}_1(00x) & \hat{g}_1(10x) \\ \hat{g}_1(01x) & \hat{g}_1(11x) \end{pmatrix} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

Hence $\mu_1([1]) = 7/10 = 1/2 + 1/5$. Choosing $\varepsilon = \delta/2 = 1/20$ (with “ δ ” from Lemma 1) and using Lemma 1, we deduce the following :

LEMMA 2. *For that, under the assumption that $(m_k)_{k \geq 1}$ increases rapidly enough, g has several g -measures, it suffices that under the same assumption, $\mu_k([1]) > 1/2 + 1/10$ for each $k \geq 1$.*

4.4 *A third reduction.* Let $k \geq 1$. Assume that $m_2 < \dots < m_k$ have been chosen large enough for that $\mu_t([1]) > 1/2 + 1/10$, $1 \leq t \leq k$. To conclude Theorem 2, using Lemma 2, sufficient is that we show that for all m_{k+1} large enough, $\mu_{k+1}([1]) > 1/2 + 1/10$.

Fix k , and denote $M = m_{k+1}$, $g_{k+1} = g_M$, $\mu_{k+1} =: \mu_M$. To prove Theorem 2, sufficient is to show that

$$\lim_{M \rightarrow \infty} \mu_M([1]) = \mu_k([1]). \tag{7}$$

Let us observe that $g_M(1x) - g_k(1x) = p_{k+1}(W_M(x) - \frac{3}{4})$. Using Remark 3 (Section 2), we have that, if $C_k = \sum_{n \geq 0} \sup_{x,y \in X} |\mathcal{L}_{g_k}^n 1_{[1]}(x) - \mathcal{L}_{g_k}^n 1_{[1]}(y)|$,

$$|\mu_k([1]) - \mu_M([1])| \leq \frac{C_k p_{k+1}}{2} \mu_M(\{W_M = 1/4\}).$$

Hence (7) will be proved if the following holds :

$$\lim_{M \rightarrow \infty} \mu_M(\{W_M = 3/4\}) = 1. \tag{8}$$

Let us introduce the following intermediate \mathcal{G} -functions g^s and \bar{g} :

$$\left\{ \begin{aligned} g^s(1x) &= \sum_{t \leq k} p_t W_{m_t}(x) + \frac{1}{2} \left(\sum_{t > k} p_t \right) = \sum_{t \leq k} p_t W_{m_t}(x) + p_k, \\ \bar{g}(1x) &= \sum_{t \leq k} p_t W_{m_t}(x) + \frac{1}{4} p_{k+1} + \frac{3}{4} \left(\sum_{t > k+1} p_t \right) \\ &= \sum_{t \leq k} p_t W_{m_t}(x) + \frac{7}{4} p_{k+1} \\ &= g^s(1x) + p_k/6. \end{aligned} \right.$$

Observe that g^s and \bar{g} depend only on $m_k + 1$ coordinates, and that g^s is symmetric. Let μ^s be the g^s -measure, and $\bar{\mu}$ the \bar{g} -measure. Since $\mu^s([1]) = \lim_n \mathcal{L}_{g^s}^n 1_{[1]}(x) = \lim_n \mathcal{L}_{g^s} 1_{[0]}(\bar{x}) = \mu^s([0])$, it follows that $\mu^s([1]) = 1/2$.

Suppose $\mu_M(\{W_M = 3/4\}) \geq \bar{\mu}(\{W_M = 3/4\})$ and that $\bar{\mu}([1]) > 1/2$. Then using ergodicity of $\bar{\mu}$, $\mathbf{1}_{\{W_M = 3/4\}} \rightarrow 1$ $\bar{\mu}$ -p.s., whence $\bar{\mu}(\{W_M = 3/4\}) \rightarrow 1$. Therefore (8) shall hold if the following does :

$$\mu_M(\{W_M = 3/4\}) \geq \bar{\mu}(\{W_M = 3/4\}) \text{ and } \bar{\mu}([1]) > 1/2. \tag{9}$$

4.5 *Conclusion of Proof of Theorem 2.* Up to now the strategy was essentially the same as in Bramson and Kalikow (1993). The rest of the proof differs from that of Bramson and Kalikow (1993), as we use Theorem 1 instead of couplings developed in Bramson and Kalikow (1993).

For $x, y \in X$, put $x \leq y$ if for each $i \geq 0$, $x_i \leq y_i$. For $f \in \mathcal{C}(\mathcal{X})$, we shall say f is *increasing* if $x \leq y$ implies $f(x) \leq f(y)$. We shall say that $\tilde{g} \in \mathcal{G}$ is *attractive* if $\tilde{g}(1 \cdot)$ is increasing.

If \tilde{g} and f are such, and if $x \leq y$, then

$$\begin{aligned} \mathcal{L}_{\tilde{g}} f(y) - \mathcal{L}_{\tilde{g}} f(x) &= \tilde{g}(1y)(f(1y) - f(1x)) + \tilde{g}(0y)(f(0y) - f(0x)) \\ &\quad + (\tilde{g}(1y) - \tilde{g}(1x))(f(1x) - f(0x)) \geq 0, \end{aligned}$$

hence \tilde{g} and $\mathcal{L}_{\tilde{g}}(f)$ are such.

Inductively, it follows that if \tilde{g} is attractive and f is increasing, then for each $n \geq 0$, and $x \in X$, $\mathcal{L}_{\tilde{g}}^n f(1x) - \mathcal{L}_{\tilde{g}}^n f(0x) \geq 0$.

Let us observe that g^s and \bar{g} are attractive, and that $1_{[1]}$ and $f_M := \mathbf{1}_{\{W_M = 3/4\}}$ are increasing. Applying Remark 3, together with the fact that $g_M(1 \cdot) \geq \bar{g}(1 \cdot)$, it follows that

$$\begin{aligned} \mu_M(f_M) - \bar{\mu}(f_M) &= \sum_{n \geq 0} \mu_M((g_M(1x) - \bar{g}(1x))(\mathcal{L}_{\bar{g}}^n f_M(1x) - \mathcal{L}_{\bar{g}}^n f_M(0x))) \geq 0, \\ \bar{\mu}([1]) - \mu^s([1]) &= \sum_{n \geq 0} \frac{p_k}{6} \bar{\mu}(\mathcal{L}_{g^s}^n 1_{[1]}(1x) - \mathcal{L}_{g^s}^n 1_{[1]}(0x)) \geq \frac{p_k}{6}, \end{aligned}$$

from which formulas the first respectively second assertions of (9) follow. □

REMARK 4. Applying Lemma 2.3, Lemma 3.1 and Theorem 3.2 of Hulse (1991), one gathers the following additional information : if $f \in \mathcal{C}(\mathcal{X})$,

- $\lim_n \mathcal{L}_g^n f(1^\infty) = \mu^+(f)$ and $\lim_n \mathcal{L}_g^n f(0^\infty) = \mu^-(f)$;
- μ^+ and μ^- have Bernoulli natural extensions.

The proof of Theorem 2 actually only uses the standard Kemeny and Snell identity (since the chains, where used, have finite memories). \square

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