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SOFIC GROUPS AND DYNAMICAL SYSTEMS*

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SUMMARY. Sofic groups were first defined by M. Gromov as a common generalization of amenable groups and residually finite groups. We discuss this new class and especially its relationship to an old problem in topological dynamics of W. Gottschalk on surjunctive groups.

The basic objects of study in topological dynamics are a pair (X, G) with X a topological space and G a group, together with a homomorphism from G into the group of homeomorphisms of X. In classical dynamics $G = \mathbb{R}$ and the action of $\mathbb{R}$ is defined by the solutions of some system of differential equations. Our interest here will be in countable groups G, for which there is a very natural action defined on the compact space $\{1, 2, \cdots a\}^G = \Omega$ called the **shift** σ given by:

$$(\sigma_g \omega)(h) = \omega(hg).$$

More precisely this is the right shift, the left shift τ is defined by $\tau_g \omega(h) = \omega(g^{-1}h)$. This generalizes the classical a-shifts of symbolic dynamics – in which $G = \mathbb{Z}$, and one can think of σ as shifting the linear sequence $\{\omega(n)\}$, either to the right – or to the left.

The general question which we shall discuss is – what are the connections between **group theoretic** properties of G and **dynamical properties** of actions of G in general, and the shift $(\{1, 2, \cdots a\}^G, \sigma_g)$ in particular. Here is an example of such a connection. Together with any system (X, G) we can associate the set of probability measures on X that are invariant under the action. Denote this set by $\mathcal{P}_i(X)$. If X is compact then it is easily seen that $\mathcal{P}_i(X)$ is a compact (in the weak* topology) convex set. There is no general reason for $\mathcal{P}_i$ to be non empty. Indeed, here is our first connection between a class of groups and dynamical properties of (X, G):

A group G is amenable if and only if for all compact systems $(X, G), \mathcal{P}_i(X)$ **is non empty**. However, if we look at the shift $(\{1, 2, \cdots a\}^G, \{\sigma_g\})$, then invariant measures always exist. For example, one can take any probability distribution $P = (p_1, p_2, \cdots p_a)$ on the symbol set $\{1, 2, \cdots a\}$ and then the product measures of P at each coordinate, which makes the coordinate functions $w(h)$, independent, identically distributed random variables, is clearly invariant under the shift.

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*To Mahendra Nadkarni – with best wishes for many more fruitful and enjoyable years.

Amenable groups can be distinguished from non amenable groups by certain properties of these measure preserving actions – but we shall not go into further detail here, see for example Zimmer (1984) for more information. For the shift, the geometric structure of $\mathcal{P}_i(\Omega)$ also can be seen from the nature of the group. In fact there is a sharp dichotomy: either the extreme points of $\mathcal{P}_i(\Omega)$ are dense in $\mathcal{P}_i(\Omega)$ or the extreme points from a closed subset of $\mathcal{P}_i(\Omega)$. The first alternative holds if G fails to have property T and the second holds if G has property T. Furthermore, up to affine equivalence there is a unique simplex with the first property — extreme points being dense — it is sometimes called the Poulsen simplex. For more details about this see Glasner and Weiss (1997).

Our main focus in this note will be on a dynamical property introduced by W. Gottschalk called **surjunctivity**. A system (X, G) will be called **surjunctive** if any continuous map $\varphi : X \to X$ that commutes with the G-action, i.e. $\varphi(gx) = g\varphi(x)$ for all $g \in G, x \in X$ that is **injective** is automatically **surjective**. He called a group G **surjunctive** if the standard shift $(\{1, 2, \cdots a\}^G, \sigma_g)$ is surjunctive for any a, and set the problem of determining what is this class of groups (Gottschalk (1973)). We still do not know if there is any group that is not surjunctive, although we shall discuss in the last section some evidence pointing to the existence of such groups. In the next section we shall review the basic properties of surjunctive groups and explain why amenable groups and residually finite groups are surjunctive.

In §2 we shall go on to introduce a class of groups that I call **sofic** that give a simultaneous generalization of these two classes of groups. This class was defined by M. Gromov (1999) who showed that they were surjunctive. This work of M. Gromov has a much wider scope than the dynamics of group shifts that are being discussed here. He defines a notion for graphs, **initially subamenable**, and then a group is sofic in our sense if its Cayley graph is initially subamenable. He proves a very general surjunctivity theorem for a projective system of complex algebraic varieties over such graphs (see Gromov, (1999), §7.G) which implies that the groups are surjunctive. I will give another proof of this in §3.

It is not likely that all groups are sofic — but I don't know of any definite example of a non sofic group. A concrete case that I haven't been able to resolve is the universal Burnside group on a finite set of generators. Finally in §4 I will discuss a natural generalization of the shift — the shifts of finite type — and discuss their surjunctivity properties.

1. **Surjunctive Groups**

Let us repeat the basic definition. A countable group G is said to be **surjunctive** if for any finite set A and any continuous mapping $\varphi : A^G \to A^G$ that commutes with the right shift injectivity implies surjectivity. Mappings φ that commute with the shift are called endomorphisms. Denote by e the identity element of G, and look at $\varphi(\omega)(e), \omega \in A^G$. This is a continuous mapping into a finite set and thus the inverse images of $a \in A$ are clopen sets in A^G. The clopen sets in A^G are the sets defined by a finite number of coordinates. It follows that there is some finite subset $F \subset G$ and mapping $\varphi_0 : A^F \to A$ such that for all ω

$$(\varphi\omega)(e) = \varphi_0(\omega|F)$$

where $\omega|F \in A^F$ denotes the restriction of ω to F. Using the fact that φ commutes with the shift gives in general that

$$(\varphi\omega)(g) = \varphi_0(\sigma_g(\omega)|F) = \varphi_0(\omega|Fg),$$

and thus the value of $\varphi(\omega)$ at g is determined from $\omega(F_g)$ by the same function φ_0.

Conversely, for any finite set F, and any $\varphi_0 : A^F \to A$, a continuous map commuting with the shift, can be constructed by using the same formula. It is not so easy to determine the global properties of φ from the finite mapping φ_0, even in the case when G is $\mathbb{Z}$ or $\mathbb{Z}^d$. Here are some easy facts about the class of surjunctive groups.

LEMMA 1.1. *If G is surjunctive and $H < G$ is any subgroup then H is also surjunctive.*

PROOF. Let ψ be an endomorphism of A^H, defined by a finite mapping $\psi_0 : A^{F_0} \to A$ where $F_0 \subset H$ is a finite set. Since $H \subset G, F \subset G$ and we can use ψ_0 to define an endomorphism φ of G. Now let us verify that if ψ is injective so is φ. Indeed suppose that for some $\omega_1 \neq w_2$ we would have $\varphi(\omega_1) = \varphi(\omega_2)$. Look at the partition of G into H-cosets Hg. For some such coset, say Hg_0 we would have $\omega_1|Hg_0 \neq \omega_2|Hg_0$. The fact that $F_0 \subset H$ means that the values of $\varphi(\omega)$ on Hg_0 are determined by the values of ω on Hg_0 and we could thus find two distinct elements of A^H mapping to the same point. Since G is surjunctive we conclude that φ is onto, and this immediately gives that ψ itself is onto. □

The next lemma shows that we can really restrict our attention to finitely generated groups.

LEMMA 1.2. *If $G = \bigcup_1^\infty H_n$ with $H_1 < H_2 < \cdots$ an increasing sequence of subgroups, each of which is surjunctive then G is surjunctive.*

PROOF. Given φ, an endomorphism, let φ be defined as before by $\varphi_0 : A^F \subset A$ for some finite set F. Since $\bigcup H_n = G$, for some n, $F \subset H_n$. Thus we can use φ_0 to define an endomorphism of $A^{H_{n_0}}$, let us denote it by ψ. The injectivity of φ implies that ψ is injective. To see this suppose ζ_1, ζ_2 distinct points in A^{H_n} with $\psi(\zeta_1) = \psi(\zeta_2)$. On each H_n coset $H_n b$, we fix a coset representative, say b, and then define $\omega_i \in A^G$ by setting, for all $h \in H_n$

$$\omega_i(hb) = \zeta_i(h).$$

Clearly $\omega_1 \neq \omega_2$ but $\varphi(\omega_1) = \varphi(\omega_2)$ contradicting the injectivity of φ.

Now we use the fact that H_n is surjunctive to conclude that ψ is onto. To go back to φ, we consider once more the partition of G into cosets of H_n, and notice that $\omega \in A^G$ restricted to the various cosets are completely independent copies of A^{H_n}. Thus the fact that φ is onto on each coset combines to give the overall surjectivity. □

I cannot show that the product of surjunctive groups is surjunctive — nor do I know what happens under homomorphisms. As we shall see shortly, the free group on any number of generators is surjunctive so that a positive answer to the last question would immediately show that all groups are surjunctive.

Here is a proof of the fact that $\mathbb{Z}$ is a surjunctive group:

In $A^{\mathbb{Z}}$ consider the periodic points of period p. Their number is clearly $|A|^p$ in particular it is finite, and if ψ is any endomorphism of the shift ψ maps periodic points of period p to periodic points of period p. Since an injective mapping of a finite set is necessarily surjective, we see that $\psi(A^{\mathbb{Z}})$ contains all of the periodic points. But the periodic points are dense, and ψ, being continuous, maps $A^{\mathbb{Z}}$ onto a closed subset of $A^{\mathbb{Z}}$ whence we conclude that ψ is surjective.

This proof can be generalized to any residually finite group as follows. Suppose that $H < G$ is of finite index. Then the set of $\omega \in A^G$ that satisfy $\sigma_h \omega = \omega$ for all $h \in H$ form a finite set — which we can call the H-periodic points. If φ commutes with the shift it maps any H-periodic point to an H-periodic point and thus the image $\varphi(A^G)$ contains all H-periodic points. The group G is residually finite if and only if H-periodic points, for H of finite index, are dense in A^G. Thus we have established the following theorem due to Lawton (see Gottschalk, 1973):

THEOREM 1.3. *If G is a residually finite group then G is surjunctive.*

Here is another proof of the fact that $A^{\mathbb{Z}}$ is surjunctive. First let us recall that a **subshift** $X \subset A^{\mathbb{Z}}$ is a closed subset of $A^{\mathbb{Z}}$ that is invariant under the shift σ. Such a subshift is determined by $X_n \subset A^{[0,n-1]}$ which consists of the restrictions of elements of X to $[0, n-1]$, as n ranges over the positive integers.

DEFINITION 1.4. *The entropy of X, denoted by $h(X)$, is defined to be:*

$$h(X) = \inf_{n \geq 1} \frac{1}{n} \log |X_n|.$$

The easily verified fact that $|X_{n+m}| \leq |X_n|\,|X_m|$ shows that the infimum is also the limit as $n \to \infty$ of $\frac{1}{n} \log |X_n|$.

PROPOSITION 1.5. *For any subshift $X \subset A^{\mathbb{Z}}$ we have*

$$h(X) \leq \log |A|$$

with equality if and only if $X = A^{\mathbb{Z}}$.

PROOF. The inequality is obvious. Suppose then that $X \neq A^{\mathbb{Z}}$. Since X is closed that means that for some finite n, say n_0, $X_{n_0} \neq A^{n_0}$. We obtain

$$h(X) \leq \frac{1}{n_0} \log(|A|^{n_0} - 1) < \log |A|$$

as required. □

Suppose then that φ commutes with the shift and is injective, and suppose that $X = \varphi(A^{\mathbb{Z}})$. Since φ is injective the inverse function $\psi : X \to A^{\mathbb{Z}}$ is well defined and continuous and it also commutes with the shift. Thus ψ is given by a finite map ψ_0, say with domain $[-N_0, +N_0]$. From this we easily deduce that

$$|X_{n+2N_0}| \geq |A|^n$$

for all n, since the coordinates of a point ω in the interval $[0, n-1]$ are given via ψ_0 by the coordinates in X that lie in the interval $[-N_0, n+N_0-1]$.

We conclude that $h(X) = \log |A|$ which by Proposition 1.5 shows that $X = A^{\mathbb{Z}}$, i.e. φ is onto.

This argument can be generalized to any amenable group, where the entropy of a subshift $X \subset A^G$ can be defined as for $\mathbb{Z}$ by replacing the intervals $[0, n-1]$ by a sequence of Følner sets. We do not go into detail here since in the following sections we will prove in detail a more general fact. But we do record

Theorem 1.6. *If G is amenable then it is surjunctive.*

The results so far suffice to show that a very large class of groups, including all linear groups, are surjunctive. This fact follows because any finitely generated linear group is residually finite.

2. Sofic Groups

Throughout this section we shall deal only with finitely generated groups. The key notion will be that of the Cayley graph $\Gamma(G, B)$ of a group G generated by a finite symmetric set B. The canonical edge labeling will be important for us so that we will use a directed version of the Cayley graph. The vertices of $\Gamma(G, B)$ are the elements of G, and there is a directed edge labeled by $b \in B$ going from g_1 to g_2 if and only if $g_2 = bg_1$. The natural graph metric in Γ gives a right invariant metric on G. The r-ball in $\Gamma(G, B)$ centered at e will be denoted by $N_r(B)$, or simply by N_r if the B is understood. Abstract graphs will be denoted by (V, E) with $V =$ vertices and $E =$ edges.

Definition 2.1. *A finitely generated group G will be called* **sofic** *if for some finite symmetric set of generators B, and any $\epsilon > 0$, and $r \in \mathbb{N}$, there is a finite directed graph (V, E) edge labeled by B, which has a finite subset of its vertices $V_0 \subset V$ satisfying:*

(1) for each $v \in V_0$, the r-neighborhood of v in (V, E) is graph isomorphic (as a labeled graph) to N_r.

(2) $|V_0| \geq (1-\epsilon)|V|$.

Suppose that G is residually finite. Then there is a homomorphism θ of G onto a finite group G_0, that is one to one on N_r. In this case, the Cayley graph of $(G_0, \theta(B))$ is a finite graph which will satisfy the above requirement with $\epsilon = 0$. Thus any residually finite group is sofic. Before showing that amenable groups are sofic, it is worth pointing out that in the definition we could have required the property for **any** symmetric generating set B, since if one set will verify the condition so will any other.

To see this notice that if B' is another generating set then for some $r_0, N_{r_0}(B) \supset B'$ and thus for any r, $N_{rr_0}(B) \supset N_r(B')$. If (V_0, E_0) is edge labeled by B, and v_1, v_2 are such that for both of them their r-neighborhoods are isomorphic to $N_{r_0}(B)$ then we can connect v_1 and v_2 by an edge labeled by $b' \in B'$ in a well defined way if and only if there is a path from v_1 to v_2 corresponding to b'. Doing this, if ϵ' is chosen sufficiently small (given ϵ) then if (V_0, E_0) satisfied our requirements for $(N_{rr_0}(B), \epsilon')$, the new graph will satisfy the requirements with $(N_r(B'), \epsilon)$.

Now let us turn to amenable groups. Følner's characterization of amenable groups G is that for any finite set K, and any $\epsilon > 0$ there is a finite set $F \subset G$ that satisfies

$$|\{f \in F : fK \subset F\}| \geq (1-\epsilon) \cdot |F|.$$

Taking for K the set $N_r(B)$ one easily sees that the subgraph of $\Gamma(G, B)$ defined by the vertices in F give a finite graph that satisfies the definition. We have so far shown.

Theorem 2.2. *All residually finite groups and all amenable groups are sofic.*

There is a more direct way to combine residually finite and amenable groups, namely:

Definition 2.3. *A group G will be said to be residually amenable* (RA) *if homomorphisms from G onto amenable groups separate points in G.*

A combination of the two arguments above shows that any RA group is sofic. It appears to be unlikely that all sofic groups are RA, but I don't know of any example. For examples of sofic groups that are neither amenable nor residually finite it suffices to see that RA are closed under direct products, while there are examples of non residually finite amenable groups. For a simple example take the natural embedding of A_n, the even permutations on n elements, into A_{n+1}, and let $G_1 = \bigcup_1^\infty A_n$. As an increasing union of finite groups it is amenable, but if H is any normal subgroup then $H \cap A_n$ is normal in A_n and hence equals either $\{e\}$ or all of A_n whence G_1 is certainly not residually finite. Thus $F_2 \times G_1 = G$ will be an example of an RA group that is neither amenable nor residually finite. We put on the record the following:

Proposition 2.4. *The direct product of sofic groups is sofic.*

The proof of this proposition is an easy exercise in the notion of the product of two graphs, and we will content ourselves with a sketch. If $G_1 = \langle B_1 \rangle, G_2 = \langle B_2 \rangle$ are sofic we take for a set of generators for $G = G_1 \times G_2, B = (B_1 \times \{e_2\}) \cup (\{e_1\} \times B_2)$. For two abstract graphs $\Gamma_1 = (V_1, E_1), \Gamma_2 = (V_2, E_2)$, form $\Gamma = \Gamma_1 \times \Gamma_2$ by putting $V = V_1 \times V_2$ and setting E to be those pairs $((u_1, u_2), (v_1, v_2))$ such that either $u_2 = v_2$ and $(u_1, v_1) \in E_1$ or $u_1 = v_1$ and $(u_2, v_2) \in E_2$. With this definition it is not hard to see that the product of graphs for G_1, G_2 will give a graph with the required properties for $G_1 \times G_2$.

In contrast to this proposition I don't know what happens when you take the free product of sofic groups. An even more basic question is: If in a short exact sequence $1 \to K \to G \to L \to 1$ both K and L are sofic is G sofic? It seems that the answer is probably affirmative if L is finite, but the general case is wide open.

3. Sofic Groups are Surjunctive

With a view to establishing that sofic groups are surjunctive let $G = \langle B \rangle$ be a fixed sofic group with B a finite symmetric set of generators. Suppose that $\Gamma_0 = (V_0, E_0)$ is a finite directed graph, edge labeled by B, that has a set of vertices $V_1 \subset V_0$, which is large, at which the $2r+1$ ball is isometric to $N_{2r+1}(B)$. We would like to find a set of points $V_2 \subset V_1$, such that the r-balls centered at $v \in V_2$ are disjoint, but such that $|V_2|$ is a reasonable fraction of $|V_0|$. To this end we let V_2 be any maximal subset of V_1 such that indeed for u in V_2 the r-balls centered at u are disjoint. If any $v \in V_1 \setminus V_2$ is at a distance at least $2r+1$ from all $u \in V_2$, then clearly the r-ball centered at v would be disjoint from all r-balls centered at points of V_2, and V_2 would not be maximal. It follows that

$$|V_1| \leq |N_{2r+1}| \cdot |V_2|.$$

If $|V_1|$ contains at least half the points of V_0 we would have $|V_2| \geq \frac{1}{2|N_{2r+1}|} \cdot |V_0|$. We formulate this as follows:

Lemma 3.1. *If $\Gamma_0 = (V_0, E_0)$ is a finite directed graph, edge labeled by B, and for at least half the vertices of Γ_0 their $2r+1$-neighborhoods are isometric to $N_{2r+1}(B)$, then there is a set V_2 of these vertices with*

$$|V_2|/|V_0| \geq 1/2|N_{2r+1}(B)|$$

and such that the minimal distance between any two vertices in V_2 is $2r+1$.

It doesn't matter that the lower bound is very small, the main thing is that it is independent of the size of Γ_0.

Suppose that φ is an endomorphism of A^G which is injective. We suppose that φ is given by a finite mapping φ_0 defined on $A^{N_{r_0}(B)}$ for some finite r_0, as discussed in §1. Furthermore, we may suppose that r_0 is large enough, so that the inverse mapping ψ from $X = \varphi(A^G)$, is defined by a finite map ψ_0 that is defined on $X_{r_0} \subset A^{N_{r_0}(B)}$, where X_{r_0} consists of the restrictions to $N_{r_0}(B)$ of elements of X. The fact that φ and ψ are inverses has a finite expression in the fact that if ψ_0 is used to define a mapping from X_{2r_0} to $A^{N_{r_0}}$ then upon composition with φ_0 we get the identity function at e.

Conversely, if φ_0 is used to define a mapping from $A^{N_{2r_0}}$ to $A^{N_{r_0}}$ then the image will lie in X_{r_0} and composing with ψ_0 we will get the identity mapping at e.

Having determined r_0, we take $\epsilon > 0$ to be very small, its precise size will be given later, and let $\Gamma_0 = (V_0, E_0)$ be a finite graph whose edges are labeled by B and with a set $V_1 \subset V_0$ that satisfies:

(i) $|V_1| \geq (1-\epsilon)|V_0|$

(ii) for each $v \in V_1$, the $5r_0$-neighborhood is isometric to N_{5r_0}.

We proceed now to do the following. Form A^{V_0}, and apply φ whenever we can, namely, for any $y \in A^{V_0}$ $v \in V_0$ with its r_0-neighborhood isometric to N_{r_0} define $(\varphi y)(v)$ by applying φ_0 to the restriction of y to the r_0-neighborhood of v. If $\overline{V}_0 \subset V_0$ is that subset of V_0, we get a mapping from A^{V_0} to $A^{\overline{V}_0}$. Let's denote the range of this mapping by Z_0. Now for vertices v in $\overline{V}_0$ all of whose r_0-neighboring vertices also belong to $\overline{V}_0$, we can define $(\psi z)(v)$ for z in Z_0 by applying ψ_0 to the restriction of z to this neighborhood. Clearly this includes all of V_1, so that we get a mapping from Z_0 **onto** A^{V_1}. The surjectivity follows from the discussion above. We obtain in this way an inequality:

$$|A|^{|V_1|} \leq |Z_0|.$$

After this machinery has been set in place we turn to the question of surjectivity of φ. If φ is not surjective then we can assume that in addition to our earlier requirements for r_0 we also demand now that r_0 be chosen large enough so that X_{r_0} is a proper subset of $A^{N_{r_0}}$. Applying the lemma we get a set of vertices $V_2 \subset V_1$ such that for $u_1 \neq v \in V_2$ the distance between u and v is at least $2r_0 + 1$, and $|V_2|/|V_0| \geq \frac{1}{2|N_{2r_0+1}|}$. Clearly the r_0-neighborhoods of $v \in V_2$ are disjoint, and lie in $\overline{V}_0$. Furthermore, it is also clear that the restriction of any $z \in Z_0$ to such neighborhoods lies in X_{r_0}. We can thus give an upper bound for the size of Z_0 as follows:

$$|Z_0| \leq (|A|^{|N_{r_0}|} - 1)^{|V_2|} \cdot |A|^{(|\overline{V}_0| - |V_2| \cdot |N_{r_0}|)}.$$

Combining these inequalities yields

$$|A|^{(1-\epsilon)\cdot|V_0|} \leq (|A|^{|N_{r_0}|} - 1)^{|V_2|} |A|^{(|\overline{V}_0| - |V_2| \cdot |N_{r_0}|)}.$$

Recalling the estimate for $|V_2|$ we see that if ϵ is chosen sufficiently small so that

$$(1 - \frac{1}{|A|^{N_{r_0}}})^{\frac{1}{2|N_{2r_0+1}|}} \lneq |A|^{-\epsilon}$$

we get a contradiction. This is the choice that we make for ϵ and this concludes the proof of:

THEOREM 3.2. *If G is a sofic group and A is any finite set then any injective endomorphism of A^G is surjective.*

As we have already remarked any increasing union of surjunctive groups is surjunctive. From this it follows that if we extend the definition of sofic group by saying that a group G is **sofic** if for any finite symmetric set $B \subset G$, $H = \langle B \rangle$ is a sofic group then the theorem above remains valid.

4. Surjunctivity of Systems of Finite Type

A very useful generalization of the full shift $(\{1, 2, \cdots a\}^G, \sigma_g)$ is a class of closed shift invariant subspaces of Ω that are defined by a finite set of conditions. More precisely, one fixes a finite set $F \subset G$ and a subset $W \subset \{1, 2, \cdots a\}^F$ and defines

$$\Omega(W) = \{\omega \in \Omega : \text{ all } g \in G, \sigma_g \omega|_F \in W\}.$$

In case $G = \mathbb{Z}$ these are the shifts of finite type which have been extensively studied for the last half century. There is no guarantee that $\Omega(W)$ is non empty and indeed there is not even an algorithm to decide this question for groups like $\mathbb{Z}^d, d \geq 2$.

As far as surjunctivity goes the situation is this. Already for $\mathbb{Z}$ there is a simple example of a shift of finite type that is not surjunctive.

Consider the following set of pairs $\{00, 01, 11, 12, 22\} \subset \{0, 1, 2\}^2$. Any point ω in the shift defined by this set has at most one block of 1's, which, if finite is bordered by an infinite string of zeros to the left and 2's to the right. It is easy to define a continuous injective map of this system to itself which detects a 12 is it occurs at all and changes that 2 to a 1, i.e. it elongates the block of 1's. Clearly this is not surjective since the string 012 will not be found in the image.

It is not hard to impose a reasonable condition which eliminates this phenomenon. Recall that a subshift X of Ω is said to be **mixing** if for any two non empty open sets U, V if g is large enough (outside some finite set) $\sigma_g U \cap V \neq \emptyset$. We then have

THEOREM 4.1. *If $X \subset \{1, 2, \cdots a\}^{\mathbb{Z}}$ is a mixing shift of finite type then it is surjunctive.*

This theorem is a consequence of the following result which holds for any amenable group G. To formulate it concisely let's recall that a system (X, G) is **intrinsically ergodic** if there is a unique **invariant** measure μ that maximizes the entropy of (X, μ, G).

THEOREM 4.2. *If (X, G) is intrinsically ergodic, and the maximizing measure μ has global support then (X, G) is surjunctive.*

Theorem 4.1 will follow by the well known fact that a mixing shift of finite type is intrinsically ergodic (see Weiss, (1970)). Theorem 4.2 follows almost immediately from the definitions. If $\varphi : X \to X$ is injective then it defines a conjugacy between (X, G) and $(\varphi(X), G)$ and if $\varphi(X) \neq X$ $\varphi \circ \mu$ could not equal μ, since μ has global support. But since entropy is a conjugacy invariant $\varphi \circ \mu$ would have the same entropy as μ contradicting intrinsic ergodicity.

Unfortunately, this argument ceases to hold already in $\mathbb{Z}^2$ where it is known that mixing shifts of finite type need not be intrinsically ergodic. In fact one can exhibit examples of mixing shifts of finite type in $\mathbb{Z}^2$ that are not surjunctive. Here is a rough description of such a system. There will be four symbols $\{0, 1, 2, *\}$, and a typical point will have a background of $*$'s and an infinite disjoint collection of "one dimensional" paths labeled by $0, 1, 2$ according to the one dimensional system that was described above.

To begin a more formal description we use two symbols $\{s, *\}$ and make the following requirements: the s-symbols shall form infinite paths of width one from $(-\infty, -\infty)$ to $(+\infty, +\infty)$ in $\mathbb{Z}^2$) with no more than three consecutive steps to the right or in the vertical direction. Furthermore such paths should be separated in both the vertical and horizontal direction by at least five $*$'s.

It is clear that this can be accomplished by a shift of finite type in $\mathbb{Z}^2$, and since there is no upper limit on the number of $*$'s separating the s-paths it is also clear that this shift is mixing. Now on each one of these s-paths we describe the one dimensional shift of finite type that was not surjunctive, by replacing all s's by $\{0, 1, 2\}$ (from $-\infty$ to $+\infty$). Once again it is not hard to verify that the mixing condition is still valid. The same mapping along these paths is injective but not surjective and thus we get an example of a mixing system of finite type that is not surjunctive.

This example gives some indication that there may exist non surjunctive groups since some constructions of this type may be convertible into constructions of groups via the techniques of combinatorial group theory.

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