*Sankhyā: The Indian Journal of Statistics*
Special issue on Ergodic Theory and Harmonic Analysis
2000, Volume 62, Series A, Pt. 3, pp. 360–366

# ON AUTOMORPHISM GROUPS ACTING ERGODICALLY ON CONNECTED LOCALLY COMPACT GROUPS*

*By* S.G. DANI
*Tata Institute of Fundamental Research, Mumbai*

*SUMMARY.* We show that a connected Lie group admitting an ergodic group of Lie automorphisms is nilpotent. Some extensions of this and examples are discussed.

## 1. Introduction

In his 1956 book on ergodic theory Halmos asked whether an automorphism of a locally compact but non-compact group can be an ergodic measure-preserving transformation (see Halmos, 1956, page 29). The question was addressed by several authors and the answer is known to be in the negative (see Dateyama and Kasuga, 1985 for details; see also Dani, 1982 for the case of connected groups; it may be mentioned that in these papers ergodicity of affine automorphisms is also considered). In this note we consider a similar question with regard to ergodicity of actions of groups of automorphisms (rather than a single automorphism) of locally compact groups. It may be seen that there do exist noncompact groups with ergodic actions by groups of automorphisms; e.g. the group $GL(n, \mathbb{R})$ acts ergodically (in fact transitively except for one point) on $\mathbb{R}^n$, $n \geq 1$, as a group of automorphisms, and there also exist countable subgroups of $GL(n, \mathbb{R})$ such as $SL(n, \mathbb{Z})$ acting 'properly' ergodically on $\mathbb{R}^n$ (see Zimmer, 1984, § 2.2, for example). We show however that a connected finite-dimensional locally compact group satisfying the condition is nilpotent (see Corollary 1.2 below for a precise statement).

For a locally compact group $G$ we denote by Aut $(G)$ the group of all bicontinuous automorphisms of $G$ (by an 'automorphism' we shall always mean a bicontinuous automorphism, with no further mention). We shall in fact prove the following result, which may be compared with the main theorem in Dani, 1982 in the case of a single affine automorphism; on the other hand the result for single automorphism can be readily deduced from the theorem, in the case of connected locally compact groups (see Corollary 2.3).

---

*AMS* (1991) *subject classification.* 22D40, 37A99.
*Key words and phrases.* Ergodic automorphism groups.
*Dedicated to Professor M.G. Nadkarni.

Theorem 1.1. *Let $G$ be a connected finite-dimensional locally compact group. Suppose that the action of $Aut(G)$ on $G$ has a dense orbit. Then $G$ is nilpotent.*

The theorem gives a necessary condition for ergodicity of the actions of automorphism groups. We recall that a ($\sigma$ finite Borel) measure $\mu$ on $G$ is said to be *quasi-invariant* under the action of a subgroup $\Gamma$ of $\mathrm{Aut}(G)$ if for $\gamma \in \Gamma$, $\mu(\gamma^{-1}(E)) = 0$ for a Borel subset $E$ of $G$ if and only if $\mu(E) = 0$; when this holds the $\Gamma$-action is said to be *nonsingular* with respect to $\mu$. The $\Gamma$-action is said to be *ergodic* with respect to a quasi-invariant measure $\mu$, if for every $\Gamma$-invariant Borel subset $E$, either $\mu(E) = 0$ or $\mu(G - E) = 0$. We shall say that a measure $\mu$ on $G$ has *full support* if $\mu(\Omega) > 0$ for all nonempty open subsets of $G$. From the theorem we deduce the following.

Corollary 1.2. *Let $G$ be a connected finite-dimensional locally compact group. Suppose that there exist a subgroup $\Gamma$ of $Aut(G)$ and a measure $\mu$ on $G$ such that $\mu$ has full support and the action of $\Gamma$ on $G$ is nonsingular and ergodic with respect to $\mu$. Then $G$ is a nilpotent group.*

The Corollary applies in particular to the Haar measures on $G$ (whether left or right); that is, if the action of $\mathrm{Aut}(G)$ on $G$ is ergodic with respect to a Haar measure then $G$ is nilpotent. The theorem applies also to measures with full support which may be quasi-invariant only under the action of a proper subgroup $\Gamma$, if the action is ergodic.

It turns out that even all nilpotent Lie groups may not admit groups of automorphisms acting ergodically, or with a dense orbit. We give an example in section 3 in this respect. While there are also many examples of nonabelian nilpotent Lie groups with ergodic actions by the automorphism groups, precisely which groups satisfy the condition is not clear. In section 4 we discuss some complements, including necessity of the finite-dimensionality condition in the above results.

## 2. **Proofs**

We will prove the theorem first for Lie groups and then deduce the general case. Let $G$ be a connected Lie group and let $\mathcal{G}$ be the Lie algebra of $G$. We realise $\mathrm{Aut}(G)$ as a subgroup of $GL(\mathcal{G})$ by identifying each automorphism $\gamma$ with the derivative $\mathrm{d}\gamma$ on $\mathcal{G}$.

Theorem 2.1. *Let $G$ be a connected Lie group such that the action of $Aut(G)$ on $G$ has a dense orbit. Then $G$ is nilpotent.*

Proof. Let $\mathcal{G}$ denote the Lie algebra of $G$ and let $\rho : G \to GL(\mathcal{G})$ be the adjoint representation of $G$. Let $n$ be the dimension of $G$ (and hence also the vector space dimension of $\mathcal{G}$). For $i = 1, \ldots, n$ let $V_i = \wedge^i \mathcal{G}$, the $i$th exterior power of $\mathcal{G}$ as a vector space and $\rho_i : G \to GL(V_i)$ be the $i$th exterior power of $\rho$. Let $c_i : G \to I\!R$, $i = 1, \ldots, n$, be the functions defined by $c_i(g) = \mathrm{Tr}\,\rho_i(g)$, where 'Tr' stands for the trace of the linear transformation. Let $c_0(g) = 1$ for all $g \in G$. Then it is easy to see that for each $g \in G$, $\Sigma_{i=0}^n c_i(g)t^{n-i}$ is the characteristic polynomial of $\rho(g)$. Now let $\mathcal{O}$ be a dense orbit of $\mathrm{Aut}(G)$ on $G$. Consider any $g_1, g_2 \in \mathcal{O}$ and let $\sigma_1$

and $\sigma_2$ be the inner automorphisms of $G$ corresponding to $g_1$ and $g_2$ respectively. There exists $\tau \in \mathrm{Aut}\,(G)$ such that $\tau(g_1) = g_2$. Then for any $g \in G$ we have $\tau \circ \sigma_1(g) = \tau(g_1 g g_1^{-1}) = g_2 \tau(g) g_2^{-1} = \sigma_2 \circ \tau(g)$ and hence $\tau \circ \sigma_1 = \sigma_2 \circ \tau$. Therefore $\mathrm{d}\tau \circ \rho(g_1) = \rho(g_2) \circ \mathrm{d}\tau$. This implies that $\rho_i(g_1)$ and $\rho_i(g_2)$ are conjugate (by $\wedge^i(\mathrm{d}\tau)$) and hence $c_i(g_1) = c_i(g_2)$ for all $i = 1, \ldots, n$. Thus the functions $c_i$ are constant on $\mathcal{O}$. But they are continuous functions on $G$ and since $\mathcal{O}$ is dense in $G$ this implies that $c_i$'s are constant on the whole of $G$. Hence the characteristic polynomial of any $\rho(g)$, $g \in G$, coincides with that of the identity matrix, which means that $\rho(g)$ is unipotent for all $g \in G$. Thus $\rho(G)$ consists of unipotent matrices and hence it is a nilpotent Lie group. Since the kernel of $\rho$ is the center of $G$ it follows that $G$ is a nilpotent Lie group. This proves the theorem.

Lemma 2.2. *Let $G$ be a connected finite-dimensional locally compact group. Let $\mathcal{F}$ be the class of all compact totally disconnected normal subgroups $F$ of $G$ such that $G/F$ is a Lie group. Let $Q$ be the smallest closed subgroup containing all $F$ in $\mathcal{F}$. Then $Q$ is a compact normal subgroup contained in the center of $G$, and $G/Q$ is a Lie group.*

Proof. As a connected locally compact group, $G$ admits maximal compact subgroups and any two of them are conjugate to each other (cf. Montgomery and Zippin, 1955, Theorem 4.13), and hence $G$ has a unique maximal compact normal subgroup. This implies that $Q$ as in the hypothesis is a compact subgroup. Also, since $G$ is connected every neighbourhood $\Omega$ of the identity contains a compact normal subgroup $F$ such that $G/F$ is a Lie group (cf. Montgomery and Zippin, 1955, Theorem 4.6) and, since $G$ is finite-dimensional, when $\Omega$ is sufficiently small the subgroup $F$ is totally disconnected. This shows that $\mathcal{F}$ is nonempty, and in turn that $G/Q$ is a Lie group. We now show that $Q$ is contained in the center of $G$. Let $\Omega$ be any neighbourhood of the identity in $G$ and let $F \in \mathcal{F}$ be contained in $\Omega$. We note that for any $F' \in \mathcal{F}$, $F'F/F$ is a compact totally disconnected subgroup of $G/F$ and since the latter is a Lie group it follows that $F'F/F$ is finite. Since $G/F$ is a connected Lie group and $F'F/F$ is a normal subgroup, it now follows that $F'F/F$ is contained in the center of $G/F$, for all $F' \in \mathcal{F}$. Hence $Q/F$ is contained in the center of $G/F$. This shows that every commutator of the form $gqg^{-1}q^{-1}$, where $g \in G$ and $q \in Q$, is contained in $F$, and hence in $\Omega$. Since this holds for every neighbourhood of the identity it follows that $Q$ is contained in the center of $G$.

Proof of theorem 1.1. Let $Q$ be the subgroup of $G$ as in the conclusion of Lemma 2.2. Then $Q$ is invariant under all bicontinuous automorphisms of $G$ and in particular the $H$-action on $G$ factors to an $H$-action on $G/Q$ by automorphisms. Moreover since the $H$-action on $G$ has a dense orbit the $H$-action on $G/Q$ has a dense orbit. Since $G/Q$ is a Lie group, by Theorem 2.1 it follows that $G/Q$ is a nilpotent Lie group. Since $Q$ is contained in the center of $G$ this shows that $G$ is nilpotent, thus proving the theorem.

Proof of corollary 1.2. If $G$ is second countable then the condition as in the hypothesis implies that the $\Gamma$-action has a dense orbit (see Lemma on page 26 of Halmos, 1956 for an idea of the proof) and hence by Theorem 1.1 $G$ is nilpotent.

This applies in particular when $G$ is a Lie group. The general case follows from this and Lemma 2.2.

Before concluding the section we observe that from Theorem 1.1 the following can be deduced, directly, for the action of a single automorphism.

COROLLARY 2.3. *Let $G$ be a connected locally compact group and suppose that there exists a bicontinuous automorphism with a dense orbit on $G$. Then $G$ is a compact Abelian group.*

PROOF. First suppose that $G$ is a Lie group. Then by Theorem 1.1 $G$ is a nilpotent Lie group. Hence $G$ has a unique compact subgroup $C$ contained in the center, such that $G/C$ is a simply connected nilpotent Lie group. Then $C$ is invariant under all automorphisms and in particular it follows that $G/C$ has an automorphism with a dense orbit. Suppose, if possible, that $G/C$ is nontrivial. Then $[G,G]C$ is a proper closed subgroup of $G/C$ invariant under all automorphisms of $G/C$ and therefore $G/[G,G]C$ has an automorphism with a dense orbit. However $G/[G,G]C$ is a vector group, and any (continuous) automorphism of a vector group is a linear transformation (with respect to the vector space structure) and hence admits no dense orbit. This is a contradiction, showing that $G/C$ is trivial. Thus $G$ is compact, and since it is also nilpotent, it follows that $G$ is a compact abelian group, in the case at hand. The general case follows from this special case, together with Lemma 2.2, by an argument as in the proof of Theorem 1.1.

## 3. **Examples**

As noted in the introduction there are noncompact connected locally compact groups with ergodic actions by groups of automorphisms, with respect to the Haar measure (though for a single automorphism to be ergodic the group has to be compact). In the case of $G = \mathbb{R}^n$, $n \geq 2$, the automorphism group $GL(n, \mathbb{R})$ as well as various subgroups such as $SL(n, \mathbb{Z})$ and more generally any lattice in $SL(n, \mathbb{R})$ act ergodically on $\mathbb{R}^n$, with respect to the Lebesgue measure (cf. Zimmer, 1984, § 2.2). There are also various nilpotent groups $N$ whose automorphism groups have an open orbit on $N$, with a complement of zero Haar measure; e.g. if $V$ is a finite-dimensional real vector space then $V \oplus \wedge^2 V$, where $\wedge^2 V$ denotes the second exterior power of $V$, has the structure of a nilpotent Lie algebra of length 2 (it is the free two-step nilpotent Lie algebra associated to $V$) and the corresponding simply connected Lie group $N$ can be readily seen to satisfy this condition. Similarly for the group of all upper triangular $n \times n$ unipotent matrices it can be verified that the automorphism group has an open orbit of full Haar measure. In these cases there also exist subgroups of the automorphism groups whose action, while ergodic, is nontransitive. We shall not go into details of these observations. It however seems instructive to note the following:

PROPOSITION 3.1. *There exists a 3-step nilpotent Lie group $G$ such that the action of $Aut(G)$ on $G$ has no open orbit.*

PROOF. Let $V = \mathbb{R}^3$, with $\{e_1, e_2, e_3\}$ the standard basis. Let $W$ be the subspace of $\wedge^2 V$ (the second exterior power of $V$) spanned by the elements $f_1 = e_2 \wedge e_3$ and $f_2 = e_3 \wedge e_1$. Now let $X$ be the subspace of $V \otimes W$ spanned by the set $\{e_1 \otimes f_1 + e_2 \otimes f_2,\ e_2 \otimes f_1 - e_3 \otimes f_2\}$ and let $Y = (V \otimes W)/X$. Let $\mathcal{G} = V \oplus W \oplus Y$, as a vector space. We define a Lie algebra structure on $\mathcal{G}$ by prescribing the following relations:

$$[e_1, e_2] = 0,\ [e_2, e_3] = f_1,\ [e_3, e_1] = f_2,$$

$$[e_i, f_j] = (e_i \otimes f_j) \bmod X, \text{ for all } i = 1, 2, 3 \text{ and } j = 1, 2,$$

$$[\alpha, \beta] = 0 \text{ if } \alpha \in \mathcal{G} \text{ and } \beta \in Y, \text{ or } \alpha, \beta \in W.$$

It is straightforward to verify that these relations can be extended (uniquely) to a Lie algebra structure on $\mathcal{G}$. We see also that $[\mathcal{G}, \mathcal{G}] = W + Y$ and $[\mathcal{G}, [\mathcal{G}, \mathcal{G}]] = Y$. Consider the group $\mathrm{Aut}\,(\mathcal{G})$ of all Lie automorphisms of $\mathcal{G}$, acting on $\mathcal{G}$. We claim that the action has no open orbit. To see this it is enough to see that the factor action on $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$ has no open orbit. Let $H$ be the subgroup of $GL(V)$ consisting of all elements which are factors of elements of $\mathrm{Aut}\,(\mathcal{G})$, when $V$ is identified canonically (via the projection) with $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$. It is enough to show that the (natural) $H$-action on $V$ has no open orbit on $V$. It turns out that $H$ is a 2-dimensional subgroup, and this can be seen as follows.

We realise $GL(V)$ as $GL(3, I\!R)$ via the basis $\{e_1, e_2, e_3\}$. Let $A$ denote the subgroup consisting of the diagonal matrices in $H$. Let $d = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3) \in A$ be the factor of $\delta \in \mathrm{Aut}\,(\mathcal{G})$ on $GL(V)$. Since $[e_2, f_1] = [e_3, f_2]$ in $\mathcal{G}$ we have, $\delta([e_2, f_1]) = \delta([e_3, f_2])$, which yields $\lambda_2^2\lambda_3[e_2, f_1] = \lambda_3^2\lambda_1[e_3, f_2] = \lambda_3^2\lambda_1[e_2, f_1]$ and since $[e_2, f_1]$ is a nonzero element, we get that $\lambda_2^2 = \lambda_1\lambda_3$. On the other hand it can be seen that all diagonal matrices satisfying this condition indeed belong to $H$. We note in particular that $A$ contains elements with distinct positive entries on the diagonal. Let $\mathcal{H}$ be the Lie subalgebra corresponding to $H$, viewed as a subalgebra of the Lie algebra of $3 \times 3$ matrices, the latter being the Lie algebra of $GL(3, I\!R)$. For $k, l \in \{1, 2, 3\}$, $k \neq l$, let $E_{kl}$ denote the matrix $(x_{ij})$ with $x_{ij} = 1$ if $i = k$ and $j = l$, and 0 otherwise. Considering the decomposition of $\mathcal{H}$ with respect to the adjoint action of $A$ and using the preceding observation we see that $\mathcal{H}$ is spanned by $\{E_{kl} \mid E_{kl} \in \mathcal{H}\}$ together with the Lie subalgebra of $A$. We now show that in fact no $E_{kl}$ is contained in $\mathcal{H}$. To show this it is enough to show that $I + E_{kl}$ is not contained in $H$ for any $k, l$, $k \neq l$, where $I$ is the identity matrix, since $\{I + tE_{kl}\}$ is the one-parameter subgroup of $H$ tangential to $E_{kl}$. Since $[e_1, e_2] = 0$ but $[e_2, e_3]$ and $[e_3, e_1]$ are nonzero it follows that $I + E_{31}$ and $I + E_{32}$ do not belong to $H$. On the other hand the fact that $[e_3, f_2] = [e_2, f_1]$ can be seen to imply that $I + E_{12}$, $I + E_{13}$, $I + E_{21}$ and $I + E_{23}$ do not belong to $H$ (one can verify that an automorphism of $\mathcal{G}$ factoring to $I + E_{kl}$ on $V$, where $(k, l)$ is one of the pairs $(1, 2)$, $(1, 3)$, $(2, 1)$ and $(2, 3)$, would have to fix one of $[e_2, f_1]$ and $[e_3, f_2]$ but not the other, which gives a contradiction; we omit the details, which are straightforward). Thus $H = A$, which is a 2-dimensional subgroup and hence has no open orbit on $V$. Thus $\mathrm{Aut}\,(\mathcal{G})$ has no open orbit on $\mathcal{G}$.

Now let $G$ be the simply connected Lie group with Lie algebra $\mathcal{G}$. Then $G$ is a nilpotent Lie group and the exponential map $\exp : \mathcal{G} \to G$ is a diffeomorphism of $\mathcal{G}$ onto $G$. Further, under the exponential map the action of $\mathrm{Aut}\,(\mathcal{G})$ on $\mathcal{G}$ corresponds to the action of $\mathrm{Aut}\,(G)$ on $G$. It follows therefore that the action of $\mathrm{Aut}\,(G)$ on $G$ has no open orbit. This proves the proposition.

Proposition 3.1 throws open the question as to which nilpotent Lie groups have automorphism groups acting with an open orbit. In particular it may be of interest to ask the following.

*Question.* Does there exist a 2-step nilpotent Lie group $G$ such that the $\mathrm{Aut}\,(G)$-action on $G$ has no open orbit?

## 4. **Complements**

While here we have restricted to connected locally compact groups, the analogous question may be asked for a general (not necessarily connected) locally compact group. For discrete groups of course the action of the automorphism group cannot be ergodic, since the identity element is fixed and has positive mass. On the other hand there are totally disconnected locally compact groups with automorphism groups acting ergodically; e.g. vector spaces (of finite dimension) over $p$-adic fields, for any prime $p$.

Finally we note the following:

Remark 4.1. The conclusion as in Theorem 1.1 or Corollary 1.2 cannot be expected to hold without the condition of finite-dimensionality. For example, let $\{K_i\}_{i\in\mathbb{Z}}$ be copies of a compact connected simple Lie group $K$ and $G = \Pi_{-\infty}^{\infty} K_i$ be the cartesian product group. Then $G$ is a compact connected group admitting ergodic automorphisms (e.g. the bilateral shift defines such an automorphism) but it is not nilpotent. Conversely it can be seen that if $G$ is a connected locally compact group such that $\mathrm{Aut}\,(G)$ has a dense orbit (or acts ergodically with respect to the Haar measure) then there exists a closed normal subgroup $H$ of $G$ such that $G/H$ is a product of infinitely many copies of a compact simple Lie group; we shall not go into the details of this. Thus, the condition of finite-dimensionality in Theorem 1.1 and Corollary 1.2 can be weakened to $G$ not admitting such a quotient.

*Acknowledgement.* The author is thankful to the referee for useful comments.

## References

Dani, S.G., (1982). Dense orbits of affine automorphisms and compactness of groups, *J. London Math. Soc.* **25**, 241–245.

Dateyama, M. and Kasuga, T., (1985). Ergodic affine maps of locally compact groups, *J. Math. Soc. Japan* **37**, 363–372.

Halmos, P.R., (1956). *Lectures on Ergodic Theory*, The Mathematical Society of Japan.

Hochschild, G., (1965). *The Structure of Lie Groups*, Holden-Day, San Francisco-London-Amsterdam.

Montgomery, D. and Zippin, L., (1955). *Topological Transformation Groups*, Wiley (Interscience), New York.

Zimmer, R.J., (1984). *Ergodic Theory and Semisimple Groups*, Birkhauser.

S.G. Dani
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba
Mumbai 400 005, India
E-mail: dani@math.tifr.res.in