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LIMIT SETS OF GROUPS OF LINEAR TRANSFORMATIONS*

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SUMMARY. For a subgroup Γ of the linear group of $\mathbb{R}^d$, we describe the Γ-orbit closures of points of $\mathbb{R}^d$ in terms of the limit set of Γ in the projective space $\mathbb{P}^{d-1}$, under proximality and irreducibility conditions. In particular, we show the minimality of the action of Γ on the corresponding asymptotic set in the linear space when Γ is a Schottky group.

1. **Introduction**

Let Γ be a group of linear transformations acting on a vector space $V = \mathbb{R}^d$, $d \geq 2$. Several authors have considered the situation where Γ is a discrete subgroup contained in a Lie subgroup of $GL(d, \mathbb{R})$ whose action satisfies certain transitivity conditions, and studied closure of the Γ-orbit in relation to that of G. The problem was first treated by Greenberg (1963), where it was shown that if V is a finite-dimensional vector space over $\mathbb{D}$, where $\mathbb{D}$ is $\mathbb{R}$, $\mathbb{C}$ or the division ring or quaternions over $\mathbb{R}$, G a subgroup of $GL(d, \mathbb{D})$ acting 'strongly transitively' (a condition satisfied by $SL(d, \mathbb{R})$, $SL(d, \mathbb{C})$, $Sp(2d, \mathbb{R})$), and Γ is a uniform lattice (i.e. a discrete subgroup of G such that the quotient G/Γ is compact), then the Γ-orbits of all nonzero vectors of V are dense in V. Various generalisations of this were obtained later by J.S. Dani (1975), Dani and Dani (1973), Dani and Raghavan (1980).

Greenberg's results are obtained by methods of elementary linear algebra. They can be deduced from the later work on the orbits of horospherical subgroups, due to W.A. Veech and other authors, and subsequent general results of M. Ratner on orbits of unipotent flows (see Dani, for example, for details). If G is a semisimple Lie group and Γ is a uniform lattice in G, then, for any horospherical subgroup acting ergodically on G/Γ, all the orbits of the action are dense in G/Γ.

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*Dedicated to Professor M.G. Nadkarni.

By a 'duality' argument this implies that all Γ-orbits on the space of euclidean p-frames are dense in the space of p-frames (by a p-frame we mean a p-tuple $(v_1, \ldots, v_p)$, where $v_1, \ldots, v_p$ are linearly independent vectors in $\mathbb{R}^d$).

Dani and Raghavan, (1980) gave certain sufficient conditions for density of the orbit of a p-frame $v = (v_1, \ldots, v_p)$, where $1 \leq p \leq d-1$, under the action of a lattice Γ in a Lie subgroup G of $GL(d, \mathbb{R})$, which is not necessarily cocompact in G. They showed in particular that, if $\Gamma = SL(d, \mathbb{Z})$, then Γv is dense if and only if v is 'irrational' in the sense that the p-dimensional subspace spanned by the coordinates of v does not contain any nonzero rational vector. In particular the $SL(d, \mathbb{Z})$-orbit of a vector in $\mathbb{R}^d$ is dense in $\mathbb{R}^d$ if and only if the corresponding direction is irrational.

In this paper we study actions of more general discrete subgroups Γ of $SL(d, \mathbb{R})$ (see § 2 for the conditions involved), firstly on the projective space $\mathbb{P}^{d-1}$ and the sphere S^{d-1} (realised as quotients of $\mathbb{R}^d - \{0\}$ in a natural way), and then on $\mathbb{R}^d$, and generalise the results of Greenberg. The subgroups considered will be "non-elementary" but possibly "small", like for example the Schottky groups, or discrete subgroups obtained by deformation in $SL(d, \mathbb{R})$ of a lattice in $SO(d-1, 1)$.

In a subsequent work (Conze and Guivarc'h) (1999) the methods of this paper are extended to actions on homogeneous Grassmannian manifolds and, by duality arguments, information on actions of subgroups on $SL(d, \mathbb{R})/\Gamma$, when Γ is discrete, is obtained.

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2. Notation, Hypotheses, Asymptotic Sets

In what follows Γ will denote a group of $d \times d$ matrices of determinant 1. We consider the actions of Γ on the vector space $V = \mathbb{R}^d$, the sphere S^{d-1} and projective space $\mathbb{P}^{d-1}$. We denote by $\pi : \mathbb{R}^d - \{0\} \to \mathbb{P}^{d-1}$ and $\eta : S^{d-1} \to \mathbb{P}^{d-1}$ the canonical quotient maps.

An important notion in this study is that of a dominant eigenvector.

Definition 2.1. A matrix $\gamma \in GL(d, \mathbb{R})$ is said to be *proximal* if it has an eigenvalue λ_0 such that $|\lambda_0| > |\lambda|$ for all other eigenvalues λ of γ (whether real or complex). For such a γ an eigenvector $v_0 \in V$ corresponding to the eigenvalue λ_0 is called a dominant eigenvector of γ.

A vector v_0 of V is called a dominant eigenvector (or simply a 'dominant vector') for Γ, if there exists $\gamma \in \Gamma$ such that v_0 is a dominant eigenvector of γ.

Let v_0 be a dominant eigenvector for a proximal matrix $\gamma \in \Gamma$ and let λ_0 be the corresponding eigenvalue. Using the Jordan canonical form for γ one can decompose V as $V = \mathbb{R}v_0 + W$, where W is the subspace

$$W = \{v \in V \mid \lambda_0^{-n} \gamma^n v \to 0, \text{ as } n \to \infty\}.$$

Thus any vector x can be expressed in the form $x = \phi(x) v_0 + w(x)$, with $w(x) \in W$, ϕ being a linear form on V with kernel W.

If v_0 is a dominant vector for Γ, so is σv_0 for any $\sigma \in \Gamma$, since if v_0 is a dominant eigenvector of γ, then σv_0 is a dominant eigenvector of $\sigma\gamma\sigma^{-1}$.

Our main results are valid for subgroups Γ of $GL(d, \mathbb{R})$ satisfying the following conditions (we note that in the formulation of the conditions Γ is not assumed to be discrete):

(H_1) The Γ-action is strongly irreducible; i.e. there does not exist any proper nonzero subspace of V invariant under the action of a subgroup of finite index in Γ.

(H_2) Γ contains a matrix γ which is proximal.

REMARK 2.2. In general, the action of a group Γ on a metric space (E, δ) is said to be proximal if, for all $x, y \in E$, there exists a sequence $\{\gamma_n\}$ in Γ such that $\delta(\gamma_n x, \gamma_n y) \to 0$ as $n \to \infty$. It is known that the conditions (H_1) and (H_2) imply the action of Γ on $\mathbb{P}^{d-1}$ to be proximal on $\mathbb{P}^{d-1}$ (see Furstenberg, 1972). (The metric δ on $\mathbb{P}^{d-1}$ may be taken as defined below, in § 5, or equivalent to it). Conversely if the Γ-action is irreducible and proximal then conditions (H_1) and (H_2) are satisfied (cf. Guivarc'h, 1990).

REMARK 2.3. It can be seen using elementary linear algebra that conditions (H_1) and (H_2) hold for the contragradient action of Γ on the dual vector space V^* (defined by $\gamma(f)(v) = f(\gamma^{-1}v)$ for all $f \in V^*$, $v \in V$ and $\gamma \in \Gamma$) whenever they hold for the Γ-action on V.

Asymptotic sets for Γ-*actions*

Let Γ be a subgroup of $GL(d, \mathbb{R})$ and consider its action on $\mathbb{P}^{d-1}$. We denote by $L_\Gamma(\mathbb{P}^{d-1})$ the closure of the subset of $\mathbb{P}^{d-1}$ consisting of all $\pi(v_0)$ with v_0 a dominant vector for Γ. It is a Γ-invariant subset of $\mathbb{P}^{d-1}$. We recall that a nonempty closed invariant subset F of a space X with an action of a group Γ is said to be minimal if it contains no nonempty proper subset which is closed and Γ-invariant.

PROPOSITION 2.4. *Suppose that the* Γ-*action satisfies conditions* (H_1) *and* (H_2). *Then* $L_\Gamma(\mathbb{P}^{d-1})$ *is minimal, and it is the only minimal set for the action.*

PROOF. Clearly it is enough to show that for any dominant vector v_0, $\pi(v_0)$ is contained in every nonempty closed invariant subset F. Let v_0 be a dominant eigenvector of a matrix $\gamma_0 \in \Gamma$. Using the Jordan decomposition for γ_0 we docompose V as $V = \mathbb{R}v_0 + W$, where W is the subspace $\{v \in V \mid \lambda_0^{-n}\gamma_0^n v \to 0 \text{ as } n \to \infty\}$. Let $x \in V$ be given. Then by the irreducibility condition there exists a $\gamma \in \Gamma$ such that $\gamma x \notin W$. Let $\gamma x = av_0 + w$, where $a \in \mathbb{R} - \{0\}$ and $w \in W$, be the decomposition of γx. Then

$$\lambda_0^{-n}\gamma_0^n\gamma x = av_0 + \lambda_0^{-n}\gamma_0^n w \to av_0 \text{ as } n \to \infty.$$

Therefore in $\mathbb{P}^{d-1}$, $\pi(\gamma_0^n\gamma x) \to \pi(v_0)$ as $n \to \infty$. This shows that $\pi(v_0)$ is contained in every Γ-invariant nonempty closed subset.

REMARK 2.5. One can show that $L_\Gamma(\mathbb{P}^{d-1})$ is the set of those $v \in \mathbb{P}^{d-1}$ for which there exists a sequence $\{\gamma_n\}$ such that $\gamma_n\mu \to \delta_v$, where μ is a probability measure on $\mathbb{P}^{d-1}$ not supported on any proper linear subspace (image of a vector subspace) of $\mathbb{P}^{d-1}$.

We next consider the Γ-action on S^{d-1}. We denote by σ the symmetry map of the sphere, defined by $\sigma(x) = -x$, namely the other point with the same image as

x in $\mathbb{P}^{d-1}$, for all $x \in S^{d-1}$. The pre-image of $L_\Gamma(\mathbb{P}^{d-1})$ in S^{d-1} will be denoted by $L_\Gamma(S^{d-1})$.

PROPOSITION 2.6. *Suppose the Γ-action is irreducible and proximal on $\mathbb{P}^{d-1}$. Then either $L_\Gamma(S^{d-1})$ is the unique nonempty closed Γ-invariant subset of S^{d-1}, or it can be expressed as $L_\Gamma(S^{d-1}) = L_\Gamma^+(S^{d-1}) \cup L_\Gamma^-(S^{d-1})$, where $L_\Gamma^+(S^{d-1})$and $L_\Gamma^-(S^{d-1})$ (simply denoted by $L_\Gamma^+$and L_Γ^-) are the only nonempty minimal closed Γ-invariant subsets. Further, in the latter case we have $\sigma(L_\Gamma^+) = L_\Gamma^-$, $L_\Gamma^+ \cap L_\Gamma^- = \emptyset$ and the restrictions to L_Γ^+ and L_Γ^- of the quotient map η of S^{d-1} onto $\mathbb{P}^{d-1}$ are Γ-equivariant isomorphisms onto $L_\Gamma(\mathbb{P}^{d-1})$.*

PROOF. For any nonempty closed invariant subset F, we have $F \cup \sigma(F) = \eta^{-1}(\eta(F))$ and hence by Proposition 2.4 it contains $\eta^{-1}(L_\Gamma(\mathbb{P}^{d-1})) = L_\Gamma(S^{d-1})$. If $L_\Gamma(S^{d-1})$ is minimal this shows that it is the only minimal set. Now suppose that $L_\Gamma(S^{d-1})$ is not minimal. Let F be a proper nonempty closed invariant subset of $L_\Gamma(S^{d-1})$. Then by the above observation $F \cup \sigma(F) = L_\Gamma(S^{d-1})$. Clearly $F \cap \sigma(F)$ is the pre-image of a proper closed invariant subset of $L_\Gamma(\mathbb{P}^{d-1})$ and hence it must be empty. Hence the restriction of η to F is injective. Since $\eta(F) = L_\Gamma(\mathbb{P}^{d-1})$ and η is Γ-equivariant, it follows that the restriction map is a Γ-equivariant isomorphism of F onto $L_\Gamma(\mathbb{P}^{d-1})$. In particular F is a minimal invariant subset. The desired conclusion now follows if we put $L_\Gamma^+ = F$ and $L_\Gamma^- = \sigma(F)$.

We say that Γ is of *type 1* or *type 2* according to whether the action of Γ on S^{d-1} has one or two minimal sets, respectively.

REMARK 2.7. a) Both cases in Proposition 2.6 can in fact occur. For $\Gamma = SO(d-1,1)$ the first alternative holds if d is even and the second one holds if d is odd. The cone defined by the equation $x_1^2 + \ldots + x_{d-1}^2 = x_d^2$ meets S^{d-1} in two connected components which are in the same Γ-orbit in the first case, but in different orbits in the second case.

b) If $L_\Gamma(\mathbb{P}^{d-1}) = \mathbb{P}^{d-1}$ then Γ is of type 1, since otherwise $S^{d-1} = L_\Gamma(S^{d-1})$ would be a disjoint union of two closed subsets L_Γ^+ and L_Γ^-, which is impossible as S^{d-1} is connected.

c) If $-Id \in \Gamma$ then Γ is of type 1.

d) Proposition 2.4 remains valid under a weaker hypothesis. Namely, condition (H_1) can be repaced by the following condition (H_0) :

(H_0) : There does not exist any proper non zero subspace of V invariant under the action of Γ.

2.1 *Discussion on Conditions* (H_1) *and* (H_2).

LEMMA 2.8. *The Γ-action satisfies the strong irreducibility condition (H_1), if and only if there does not exist any nonzero vector v such that Γv is contained in a finite union of proper (vector) subspaces of V.*

PROOF. Suppose that (H_1) is satisfied. Let $v \neq 0$ and consider the collection $\mathcal{W}$ of all sets which are finite unions of subspaces, containing Γv. Then the intersection, say W, of all elements of $\mathcal{W}$ is again a finite union of subspaces. Clearly W is Γ-invariant. Now if $W = \cup_1^p V_j$, with V_j, $j = 1, \ldots, p$, the maximal subspaces contained in W, then Γ permutes the subspaces $V_1, \ldots, V_p$. Let Γ' be the subgroup of Γ consisting of elements leaving each V_j invariant. Then Γ' is normal, Γ/Γ' is finite and all V_j's are Γ'-invariant. Therefore by condition (H_1) $V_j = V$ for all j and hence $W = V$ which means that Γv is not contained in a finite union of proper subspaces.

Conversely suppose that the condition as in the hypothesis is satisfied. If Γ' is a subgroup of finite index in Γ leaving invariant a proper subspace V_1 then $\cup_{\gamma\in\Gamma}\gamma V_1$ is a finite union of subspaces which is Γ-invariant. Thus, for $v \in V_1$, Γv is contained in a finite union of subspaces, contradicting the assumption. Hence (H_1) is satisfied.

Under the dual form as in the Remark 2.3, we have the following:

LEMMA 2.9. *The Γ-action satisfies the strong irreducibility condition (H_1), if and only if, for any linear form ϕ which is not identically 0 and any r-tuple of vectors $(x_1, \ldots, x_r)$, there exists a $\gamma \in \Gamma$ such that $\phi(\gamma x_j) \neq 0$ for all $j = 1, \ldots, r$.*

For a subgroup Γ of $GL(d, \mathbb{R})$ we denote by $Z^c(\Gamma)$ the Zariski closure of Γ in $GL(d, \mathbb{R})$. The following result relates conditions (H_1) and (H_2) being satisfied for Γ to their being satisfied for $Z^c(\Gamma)$; see Guivarc'h and Raugi (1989) and Prasad (1994) for details.

PROPOSITION 2.10. *Γ satisfies conditions (H_1) and (H_2), if and only if the conditions are satisfied by $Z^c(\Gamma)$.*

Let G be a connected semisimple Lie subgroup of $GL(d, \mathbb{R})$, having no non-trivial compact factors, and Γ be a lattice in G (that is G/Γ has finite G-invariant measure). Then by Borel's density theorem Γ is Zariski-dense in G and hence by Proposition 2.10 conditions (H_1) and (H_2) hold for Γ provided they hold for G. Since conditions (H_1) and (H_2) hold for $SL(d, \mathbb{R})$, $SO(d-p, p)$ $(p \geq 1)$ and for the symplectic groups $Sp(2m, \mathbb{R})$ (for $d = 2m$), by Proposition 2.10 they hold for Zariski-dense subgroups of these groups, and for lattices in these groups, in particular. For a more general G as above, conditions (H_1) and (H_2) are satisfied if the G-action on $\mathbb{R}^d$ is irreducible and the highest weight corresponding to the representation on $\mathbb{R}^d$ has multiplicity 1 (see Guivarc'h, Ji and Taylor, 1998, Chap. 4).

3. Closures of Orbits in $\mathbb{R}^d$

We now consider density properties of the Γ-action on the vector space $V = \mathbb{R}^d$. We describe two methods: an elementary one inspired by (Greenberg, 1963) and another based on probabilistic techniques from (Guivarc'h and Raugi, 1989) and (Guivarc'h, 1990).

To begin with we assume that the Γ-action is irreducible and satisfies the proximality condition (H_2). For convenience we shall denote $L_\Gamma(\mathbb{P}^{d-1})$ by L_Γ.

PROPOSITION 3.1. *Let v_0 be a dominant vector for $\gamma_0 \in \Gamma$, with eigenvalue λ_0, and let $v \in V - \{0\}$ be such that $\pi(v) \in L_\Gamma$. Then there exists a scalar β such that $1 \leq |\beta| \leq |\lambda_0|$ and $\beta v \in \overline{\Gamma v_0}$.*

PROOF. By the minimality of L_Γ there exists a sequence $\{\gamma_n\}$ in Γ such that $\gamma_n \pi(v_0) \to \pi(v)$ and hence there exists a sequence of scalars $\{\alpha_n\}$ such that

$\alpha_n \gamma_n v_0 \to v$. Also, there exists a sequence $\{p_n\}$ of integers such that $|\lambda_0^{p_n} \alpha_n^{-1}| \in [1, |\lambda_0|]$. Passing to a subsequence one may suppose that $\lambda_0^{p_n} \alpha_n^{-1}$ converges, to say β, with $|\beta| \in [1, |\lambda_0|]$, so we have $\gamma_n \gamma_0^{p_n} v_0 = \gamma_n \lambda_0^{p_n} v_0 = (\lambda_0^{p_n} \alpha_n^{-1})(\alpha_n \gamma_n v_0) \to \beta v$, which proves the proposition.

The following result gives a conclusion in the other direction.

Proposition 3.2. *Let v be a nonzero vector in V such that $0 \in \overline{\Gamma v}$. Then for any dominant vector v_0 there exists a scalar α such that $\alpha v_0 \in \overline{\Gamma v}$.*

Proof. Let v_0 be a dominant eigenvector of $\gamma_0 \in \Gamma$, with eigenvalue λ_0. Let $(v_0, v_1, \ldots, v_{d-1})$ be a Jordan basis for γ_0. Then there exist linear forms $\phi_0, \phi_1, \ldots, \phi_{d-1}$ on V, such that for all $x \in V$ we have

$$x = \phi_0(x) v_0 + \Sigma_{i=1}^{d-1} \phi_i(x) v_i.$$

Since $0 \in \overline{\Gamma v}$, there exists a sequence $\{\gamma_n\}$ in Γ such that $\gamma_n v \to 0$. Passing to a subsequence one may assume that there exists a sequence of scalars $\{\alpha_n\}$ such that $\{\alpha_n \gamma_n v\}$ converges to a nonzero vector, say u. By an argument as before, using irreducibility we conclude that there exists a $\gamma' \in \Gamma$ such that $\phi_0(\gamma' u) \neq 0$. Replacing the sequence $\{\gamma_n\}$ by $\{\gamma' \gamma_n\}$ and changing notation we may in fact assume that $\phi_0(u) \neq 0$.

With respect to the basis as above, for any sequence $\{p_n\}$ of integers we have

$$\gamma_0^{p_n} \gamma_n v = \lambda_0^{p_n} \phi_0(\gamma_n v) v_0 + \Sigma_{i=1}^{d-1} \phi_i(\gamma_n v) \gamma_0^{p_n} v_i.$$

The sequence $\{p_n\}$ may be chosen such that $\lambda_0^{p_n} \phi_0(\gamma_n v)$, $n \geq 1$, are all of absolute value between 1 and $|\lambda_0|$ and, further, by passing to a subsequence we may in fact assume that the sequence converges to a limit, say α, with $|\alpha| \in [1, |\lambda_0|]$.

Now consider the other terms on the right hand side. We have

$$|\phi_i(\gamma_n v)| \|\gamma_0^{p_n} v_i\| = |\lambda_0^{p_n} \phi_0(\gamma_n v)| \frac{|\phi_i(\gamma_n v)|}{|\phi_0(\gamma_n v)|} \frac{\|\gamma_0^{p_n} v_i\|}{|\lambda_0^{p_n}|}.$$

The first factor in the product on the right hand side remains bounded; the second converges to a finite limit $|\phi_i(u)/\phi_0(u)|$ and the third factor tends to 0, by the properties of dominant eigenvectors. This implies that

$$\gamma^{p_n} \gamma_n v \to \alpha v_0,$$

thus proving the proposition.

Combining Propositions 3.1 and 3.2 we get:

Corollary 3.3. *For any nonzero vector v such that $0 \in \overline{\Gamma v}$ and any nonzero w such that $\pi(w) \in L_\Gamma$, there exists a scalar θ such that $\theta w \in \overline{\Gamma v}$.*

Proof. Let v and w be as in the hypothesis and let v_0 be a dominant vector. The orbit-closure $\overline{\Gamma v}$ contains αv_0 for a nonzero scalar α, and hence $\alpha \overline{\Gamma v_0}$. As $\pi(w) \in L_\Gamma$ there exists β such that $\beta w \in \overline{\Gamma v_0}$. Hence $\alpha \beta w \in \alpha \overline{\Gamma v_0} \subset \overline{\Gamma v}$.

The Corollary signifies that under the condition that the origin be contained in the orbit closure, the orbit meets all lines corresponding to elements of L_Γ.

Henceforth we assume that Γ satisfies also the strong irreducibility condition (H_1). We denote by $L_\Gamma(\mathbb{R}^d)$ the set of vectors $v \neq 0$ such that $\pi(v) \in L_\Gamma(\mathbb{P}^{d-1})$. In the type 2 case the pre-image of $L_\Gamma^+(S^{d-1})$ in $\mathbb{R}^d - \{0\}$ will be denoted by $L_\Gamma^+(\mathbb{R}^d)$.

THEOREM 3.4. *Let $L = L_\Gamma(\mathbb{R}^d)$ or $L_\Gamma^+(\mathbb{R}^d)$ according to whether Γ is of type 1 or type 2, respectively. Then there exists a dense G_δ subset E of L such that the Γ-orbit of any $v \in E$ is dense in L.*

PROOF. By an argument due to Hedlund it is enough to show that if O_1 and O_2 are two open sets intersecting L then there exists a $\gamma \in \Gamma$ such that γO_1 intersects O_2; indeed if this holds and $\{U_i\}$ is a countable base of L then $\cap_i \cup_{\gamma\in\Gamma} \gamma^{-1}U_i$ is a dense G_δ set of points of L whose orbits are dense in L. We may also suppose that the open sets O_1 and O_2 are convex and do not contain 0. Let Φ_1 and Φ_2 be the open cones in V generated by O_1 and O_2 respectively.

By hypothesis Γ contains a proximal matrix, say γ. Let x_1 be the dominant eigenvector and λ_1 be the corresponding eigenvalue. Since γ has determinant 1 it must also have an eigenvalue, say λ_2, (real or complex) such that $|\lambda_2| < 1$; it follows in particular that there exists $x_2 \in V - \{0\}$ such that $\gamma^n x_2 \to 0$.

In Φ_1 there exists by Proposition 2.4 a vector v_1 which is dominant for a $\gamma_1 \in \Gamma$, with eigenvalue say μ_1. Then there exists a linear form ϕ such that $\mu_1^{-n}\gamma_1^n x \to \phi(x)v_1$ for all $x \in V$. By condition (H_1) and Lemma 2.9 there exists a $\gamma' \in \Gamma$ such that simultaneously we have $\phi(\gamma' x_1) \neq 0$ and $\phi(\gamma' x_2) \neq 0$. Replacing γ be $\gamma'\gamma{\gamma'}^{-1}$, x_1 by $\gamma' x_1$ and x_2 by $\gamma' x_2$, we may suppose that $\gamma x_1 = \lambda_1 x_1$, $\gamma^n x_2 \to 0$ and $\phi(x_1)$ and $\phi(x_2)$ are nonzero. Since $v_1 \in \Phi_1$ and $\mu_1^{-n}\gamma_1^n x \to \phi(x)v_1$ for all $x \in V$, this implies that $\gamma_1^m x_1$ and $\gamma_1^m x_2$ are contained in Φ_1 for all large m. Hence there exist $\alpha_1, \alpha_2 > 0$ and an integer m such that $\alpha_1\gamma_1^m x_1, \alpha_2\gamma_1^m x_2 \in O_1$. By convexity of O_1 therefore the segment joining $\alpha_1\gamma_1^m x_1$ and $\alpha_2\gamma_1^m x_2$ is contained in O_1.

The cone Φ_2 contains a dominant vector and hence by Proposition 2.4 there exists $\gamma_2 \in \Gamma$ such that $\gamma_2 x_1 \in \Phi_2$.

Let $x_1' = \alpha_1 x_1$ and $x_2' = \alpha_2 x_2$. The segment joining the points $\gamma_2\gamma^n x_1' = \lambda_1^n \gamma_2 x_1'$ and $\gamma_2\gamma^n x_2'$ converges as $n \to \infty$, to a ray from the origin, passing through $\gamma_2 x_1$. This ray, say R, is the limit of the images under $\gamma_2\gamma^n\gamma_1^{-m}$ of the segment S joining $\gamma_1^m x_1'$ and $\gamma_1^m x_2'$; as $\gamma_1^m x_1'$ and $\gamma_1^m x_2'$ are contained in O_1 and O_1 is convex, $S \subset O_1$.

As $\gamma_2 x_1 \in \Phi_2$ the ray R meets one of O_2 or $\sigma(O_2)$ in a segment, where σ is the symmetry map of $\mathbb{R}^d$. If it meets O_2 then O_2 contains images of points from the segment $S \subset O_1$ and hence we are through.

To complete the proof we consider the cases of type 1 and type 2 separately. Suppose first that Γ is of type 1. Then Γ acts minimally on $L_\Gamma(S^{d-1})$ and hence there exists $\theta \in \Gamma$ such that $\theta(R)$ meets O_2 and then the above argument applies, with $\theta(R)$ in place of R. If Γ is of type 2, then R is contained in $L_\Gamma^+(\mathbb{R}^d)$ and does not intersect $\sigma(O_2)$; it therefore has to meet $\sigma(O_1)$; so we are through by the earlier argument. This proves the theorem.

THEOREM 3.5. *Let v be a vector in $\mathbb{R}^d - \{0\}$ such that $0 \in \overline{\Gamma v}$. Then according to whether Γ is of type 1 or type 2, $\overline{\Gamma v}$ contains either $L_\Gamma(\mathbb{R}^d)$ or one of the closed subsets $L_\Gamma^+(\mathbb{R}^d)$ or $L_\Gamma^-(\mathbb{R}^d)$, respectively.*

PROOF. Suppose first that Γ is of type 1. Let $w \in L_\Gamma(\mathbb{R}^d)$ be a point with dense orbit (see Theorem 3.4). As $0 \in \overline{\Gamma v}$ there exists, by Corollary 3.3, a scalar θ with $\theta w \in \overline{\Gamma v}$. The orbit of θw is dense in $L_\Gamma(\mathbb{R}^d)$ and is contained in $\overline{\Gamma v}$.

If Γ is of type 2 we take $w \in L_\Gamma^+(\mathbb{R}^d)$ with dense orbit in $L_\Gamma^+(\mathbb{R}^d)$. By Corollary 3.3 there exists $\theta \in \mathbb{R}$ such that $\theta w \in \overline{\Gamma v}$. Depending on whether $\theta > 0$ or $\theta < 0$, the vector θw has dense orbit in $L_\Gamma^+(\mathbb{R}^d)$ or $L_\Gamma^-(\mathbb{R}^d)$ contained in $\overline{\Gamma v}$.

A probabilistic method. We have seen the role played in the study of Γ-orbits by the set of vectors v for which there exists a sequence $\{\gamma_n\}$ in Γ such that $\gamma_n v \to 0$. In fact, in the examples such a condition is obtained in the following stronger form:

$$\lim_n \|\gamma_n^{-1}\| = \infty \text{ and } \sup_n \|\gamma_n^{-1}\| \|\gamma_n v\| < \infty.$$

DEFINITION 3.6. *Let L_Γ^h (resp. L_Γ^c) be the set of points of L_Γ of the form $\pi(v)$, with v such that there exists a sequence $\{\gamma_n\}$ in Γ with $\gamma_n^{-1} v \to 0$ (resp. $\|\gamma_n^{-1}\| \to \infty$ and $\{\|\gamma_n^{-1}\|\|\gamma_n v\| \mid n = 1, 2, \ldots\}$ is bounded).*

Then one has $L_\Gamma^c \subset L_\Gamma^h$, and it can be shown that L_Γ^c is a 'thick' subset of L_Γ^h. By Proposition 2.4 it contains at least all the dominant vectors of Γ.

We are going to study the set L_Γ^c in detail. For this purpose we introduce on Γ a probability measure μ whose support generates Γ. We denote by $\check{\mu}$ the measure which is 'symmetric' of μ, defined by $\check{\mu}(E) = \mu(E^{-1})$ for all Borel subsets E. It is known that there exists on $\mathbb{P}^{d-1}$ a unique probability measure ν' which is stationary with respect to $\check{\mu}$, i.e. such that $\check{\mu} * \nu' = \nu'$. The uniqueness of ν' is a consequence of the conditions (H_1) and (H_2) by the results of (Furstenberg, 1972). The minimality of the action of Γ on $L_\Gamma = L_\Gamma(\mathbb{P}^{d-1})$ implies that the support of ν' is the whole of L_Γ (see Guivarc'h, 1990).

The method described below depends on a construction of a thick set of limit points by a procedure similar to coding of irrationals by continued fraction expansions and study of the corresponding matrix products. This may be viewed as a study of a certain transformations on $\mathbb{P}^{d-1} \times \mathbb{R}^*$; if ν is the μ-stationary measure on $\mathbb{P}^{d-1}$, then $\nu \otimes l$, where l is the Lebesgue measure on $\mathbb{R}^*$, is a natural invariant measure on $\mathbb{P}^{d-1} \times \mathbb{R}^*$. The ergodic properties of such transformations (with infinite invariant measure) are studied in (Conze, 1976), in relation to certain cohomological equations. Here we will be concerned only with certain topological aspects.

THEOREM 3.7. *Suppose that Γ satisfies conditions (H_1) and (H_2). Let μ be a probability measure on Γ whose support generates Γ. Suppose also that*

$$\int \log \|\gamma\| \mathrm{d}\mu(\gamma) < \infty.$$

Let ν' be the unique $\check{\mu}$-stationary measure on $\mathbb{P}^{d-1}$. Then $\nu'(L_\Gamma^c) = 1$ and the Hausdorff dimension of L_Γ^c is strictly positive. In particular, if Γ is of type 1, the orbit Γx is dense in $L_\Gamma(\mathbb{R}^d)$ for ν'-almost all x, and the set of points of S^{d-1} with orbits dense in $L_\Gamma(\mathbb{R}^d)$ is of strictly positive dimension.

PROOF. Consider a sequence of independent identically distributed matrix valued random variables $\{\gamma_k\}_{k\in\mathbb{Z}}$ with distribution μ. The product $S_n(\omega) = \gamma_n(\omega)\cdots\gamma_1(\omega)$, $n \geq 1$, of the matrices is a random sequence of elements of Γ.

We denote by θ the shift on the product space $\Omega = \Gamma^{\mathbb{Z}}$, on which the γ_k's are defined. We note that $\gamma_k = \gamma_1 \circ \theta^{k-1}$ for all k. Let $e_1, \ldots, e_d$ be the canonical basis of $\mathbb{R}^d$.

Since the conditions (H_1) and (H_2) are satisfied, it follows from (Guivarc'h and Raugi, 1986) that the first and the last Lyapunov exponents, say ρ_1 and ρ_d, of the products $S_n(\omega)$ are simple. Recall that ρ_1 and ρ_d are given by

$$\rho_1 = \lim_n \frac{1}{n}\int \log\|S_n(\omega)\|\mathrm{d}P(\omega) \text{ and } \rho_d = -\lim_n \frac{1}{n}\int \log\|S_n^{-1}(\omega)\|\mathrm{d}P(\omega),$$

where P is the product measure on Ω. Since det $S_n = 1$ we have $\rho_1 > 0$ and $\rho_d < 0$.

By the multiplicative ergodic theorem of Oseledets (1968), there exists a measurable function $\varphi(\omega)$, from Ω to $GL(d, \mathbb{R})$, such that P-almost surely,

$$\gamma_1(\omega)\varphi(\omega) = \varphi(\theta\omega)\Lambda(\omega),$$

where $\Lambda(\omega) = \mathrm{diag}\,[\lambda_1(\omega), \Lambda'(\omega), \lambda_d(\omega)]$ and λ_1 and λ_d are (measurable) scalar functions while Λ' is a function with values in $(d-2)\times(d-2)$ matrices. Now

$$S_n(\omega) = \varphi(\theta^n\omega)\Delta_n(\omega)\varphi^{-1}(\omega),$$

with $\Delta_n(\omega) = \mathrm{diag}\,[\lambda_{n,1}, \Delta'_n, \lambda_{n,d}]$, $\lambda_{n,1} = \Pi_{j=0}^{n-1}\lambda_1(\theta^j\omega)$, $\lambda_{n,d} = \Pi_{j=0}^{n-1}\lambda_d(\theta^j\omega)$, and $\Delta'_n = \Pi_{j=0}^{n-1}\Lambda'(\theta^j\omega)$.

One knows that $\lim_n \|\varphi(\theta^n\omega)\|^{1/n} = 1$,

$$\rho_1 = \lim_n \frac{1}{n}\log|\lambda_{n,1}(\omega)| > 0 \text{ and } \rho_d = \lim_n \frac{1}{n}\log|\lambda_{n,d}(\omega)| < 0.$$

The point of $\mathbb{P}^{d-1}$ defined by $z(\omega) = \varphi(\omega)e_d$ corresponds to the contracting direction of $S_n(\omega)$; it satisfies the functional equation (in terms of action on the projective space)

$$z(\omega) = \gamma_1^{-1}(\omega)\cdot z(\theta\omega).$$

We shall show that $z(\omega) \in L_\Gamma^c$ almost surely.

By the multiplicative ergodic theorem one knows, as the last exponent of $S_n(\omega)$ is simple, that the equation has a unique solution, depending on the forward coordinates (those with positive index) of ω. This solution is $z(\omega) = \varphi(\omega)\cdot e_d$. Then $\gamma_1^{-1}(\omega)$ and $z(\theta\omega)$ are independent and the distribution η of $z(\omega) \in \mathbb{P}^{d-1}$ satisfies the condition $\eta = \check{\mu} * \eta$.

By the observations preceding the statement of the theorem the support of $\eta = \nu'$ equals L_Γ. One has therefore $z(\omega) \in L_\Gamma$ P-almost surely. As

$$S_n(\omega)[\varphi(\omega)\cdot e_d] = \lambda_{n,d}\varphi(\theta^n\omega)e_d,$$

we have $S_n(\omega)v(\omega) = \lambda_{n,d}\dfrac{\varphi(\theta^n\omega)e_d}{\|\varphi(\omega)e_d\|}$, with $v(\omega) = \dfrac{\varphi(\omega)e_d}{\|\varphi(\omega)e_d\|}$.

On the other hand

$$S_n^{-1}(\omega) = \varphi(\omega)\Delta_n^{-1}(\omega)\varphi^{-1}(\theta^n\omega),$$

and hence

$$\|S_n^{-1}(\omega)\| \leq \|\varphi(\omega)\|\|\varphi^{-1}(\theta^n\omega)\| \sup\left[|\lambda_{n,1}^{-1}(\omega)|, \|{\Delta'}_n^{-1}(\omega)\|, |\lambda_{n,d}^{-1}(\omega)|\right].$$

Since ρ_d is of multiplicity 1 we have

$$\lim_n \frac{1}{n}[\log|\lambda_{n,d}(\omega)| + \log\|{\Delta'}_n^{-1}(\omega)\|] < 0.$$

Hence, almost surely, for n sufficiently large

$$\sup[|\lambda_{n,1}^{-1}(\omega)|, \|{\Delta'}_n^{-1}(\omega)\|, |\lambda_{n,d}^{-1}(\omega)|] = \frac{1}{|\lambda_{n,d}(\omega)|}$$

and hence

$$\|S_n^{-1}(\omega)\|\|S_n(\omega)v(\omega)\| \leq \frac{\varphi(\theta^n\omega)e_d}{\|\varphi(\omega)e_d\|}\|\varphi(\omega)\|\|\varphi^{-1}(\theta^n\omega)\|.$$

By the Poincaré recurrence theorem, for almost all ω there exists a sequence $\{n_k(\omega)\}$ such that the right hand side of the inequality is bounded by a constant $C > 0$ for all $n = n_k(\omega)$. Hence $\|S_{n_k}^{-1}(\omega)\|\|S_{n_k}(\omega)v(\omega)\| \leq C$ and $z(\omega) = \pi[v(\omega)] \in L_\Gamma^c$. Since $z(\omega)$ is distributed according to ν', it now follows that $\nu'(L_\Gamma^c) = 1$.

By Guivarc'h (1990) one has

$$\int \delta^{-\epsilon}(x,y)\mathrm{d}\nu'(y) < C'$$

for some $\epsilon > 0$ and $C' < \infty$, for all μ satisfying the condition $\int \|g\|^\alpha \mathrm{d}\mu(g) < \infty$, with $\alpha > 0$. Thus ν' has finite energy for the ϵ-Riesz potential $\delta^{-\epsilon}(x,y)$ on $\mathbb{P}^{d-1}$. The lemma of Frostman (cf. Kahane and Salem, 1963, p. 35) therefore implies that any Borel subset supporting ν' has Hausdorff dimension at least ϵ. This holds therefore, in particular, for L_Γ^c.

The last assertion follows from Theorem 3.4 and the fact that $L_\Gamma^c \subset L_\Gamma^h$.

Remark 3.8. Suppose that $G = SO(d-1,1) \subset SL(d,\mathbb{R})$ and Γ is a Zariski-dense subgroup of G. Then L_Γ^c is the set of 'conical' limit points and L_Γ^h is the set of horospherical limit points (cf. Sullivan, 1979). If Γ is geometrically finite then $L_\Gamma = L_\Gamma^c \cup L_\Gamma^p$, where L_Γ^p is the countable set of bounded parabolic points. For $d = 2$ there are examples due to Pommerenke, of Fuchsian groups (not finitely generated) for which $\dim L_\Gamma^c < \dim L_\Gamma$ (see Starkov, 1995).

4. **Subgroups of Finite Covolume**

We can apply the results of the preceding section to subgroups of a Lie group G such that G/Γ has finite volume (finite invariant measure). We obtain results analogous to those of Greenberg (1963), under weaker conditions. As in (Greenberg, 1963) we begin with a lemma on stability of eigenvalues of matrices.

LEMMA 4.1. *Let A be a proximal matrix. Let λ_1 be the eigenvalue of A with highest absolute value and v_1 be the eigenvector corresponding to λ_1. Then for all $\epsilon > 0$ there exists a neighbourhood U of the identity in $GL(d, \mathbb{C})$ such that, for any $B \in U$ and any integer $n > 0$, the matrix BA^n is proximal and its eigenvalue μ_1 of highest absolute value and an eigenvector w_1 corresponding to μ_1 satisfy the following conditions:*

$$|\lambda_1^n - \mu_1| < \epsilon \text{ and } \|v_1 - w_1\| < \epsilon.$$

PROOF. It can be given by an argument analogous to the one in (Greenberg, 1963); we omit the details.

LEMMA 4.2. *If G/Γ has finite G-invariant measure, then, for any $g \in G$ and any nonempty open subset U of G, there exist an integer $n > 0$ and elements $\gamma \in \Gamma$ and $h_1, h_2 \in U$ such that $g^n = h_1 \gamma h_2$.*

PROOF. Since the sets $g^k U\Gamma/\Gamma$, $k = 1, 2, \ldots$ are all of same positive measure in G/Γ they can not be pairwise disjoint. This implies the lemma.

LEMMA 4.3. *Let G be a subgroup of $GL(d, \mathbb{R})$ and suppose that the G-action on $\mathbb{R}^d$ satisfies $(H_1)$and (H_2). Let Γ be a subgroup of G such that $Z^c(\Gamma) = G$. Let $N(\Gamma) = \{g \in G \mid g\Gamma g^{-1} = \Gamma\}$, the normaliser of Γ in G. Suppose that the quotient $G/N(\Gamma)$ has finite invariant measure. Then $L_G = L_\Gamma$.*

PROOF. By Lemmas 4.1 and 4.2 if $g \in G$ has a dominant eigenvector v with eigenvalue λ, then there exists an integer n and a matrix γ_0 in $N(\Gamma)$ in a neighbourhood of g^n, with a dominant eigenvector near to v. This shows that the directions of the dominant vectors for G, which form a dense subset of L_G are in fact contained in $L_{N(\Gamma)}$. Since $Z^c(\Gamma) = G$, Γ satisfies conditions (H_1) and (H_2), hence $L_\Gamma \neq \emptyset$. But $L_{N(\Gamma)} = L_\Gamma$, since L_Γ is a closed non trivial subset of $L_{N(\Gamma)}$ invariant under the action of $N(\Gamma)$, while by Proposition 2.4 the action of $N(\Gamma)$ on $L_{N(\Gamma)}$ is minimal. This proves the lemma.

Using Theorem 3.5 we can now deduce the following result from (Dani, 1975).

COROLLARY 4.4 *Let Γ be a lattice in $SL(d, \mathbb{R})$. Then for a nonzero v in $\mathbb{R}^d$, Γv is dense in $L_G(\mathbb{R}^d)$ if and only if $0 \in \overline{\Gamma v}$.*

PROOF. By Lemma 4.3 $L_\Gamma(\mathbb{P}^{d-1}) = L_G(\mathbb{P}^{d-1}) = \mathbb{P}^{d-1}$. Furthermore, by Remark 2.7 b), Γ is of type 1. In view of Borel's density theorem Γ satisfies condition (H_1). Lemmas 4.1 and 4.2 show that Γ satisfies condition (H_2). So, if $0 \in \overline{\Gamma v}$, by Theorem 3.5 we have $\overline{\Gamma v} = \mathbb{R}^d$. The converse is obvious.

For the lattice $\Gamma = SL(d, \mathbb{Z})$ it turns out that $0 \in \overline{\Gamma v}$ if and only if v is not a multiple of a vector with rational coordinates; a proof of this may be found in Dani (1975); using multidimensional continued fractions and the Jacobi-Perron algorithm (see Broise and Guivarc'h, 1999), one can in fact show that for such a v, $\|\gamma_n\| \, \|\gamma_n^{-1} v\|$ is bounded for the sequence $\{\gamma_n\}$ associated to the algorithm (for which we also have $\|\gamma_n\| \to \infty$).

The following example illustrates another instance where Theorem 3.5 can be applied together with Lemma 4.3.

EXAMPLE 4.5. Let Γ_2 be the congruence subgroup of $SL(2,\mathbb{Z})$ modulo 2; i.e. the subgroup consisting of integral matrices γ such that all entries of the matrix $\gamma - Id$ are even. It is of finite index in $SL(2,\mathbb{Z})$ and is a free group on 2 generators. Therefore it has plenty of normal subgroups. If H is such a normal subgroup $\neq \{e\}$ then $Z^c(H)$ is normal in $Z^c(\Gamma_2) = SL(2,\mathbb{R}) = G$; hence $Z^c(H) = G$. Then Lemma 4.3 implies $L_H = L_{\Gamma_2} = \mathbb{P}^1$.

The following theorem extends results of Greenberg in (Greenberg, 1963).

THEOREM 4.6. *Let Γ be a subgroup of $SL(d,\mathbb{R})$ satisfying the conditions (H_1) and (H_2) and let $G = Z^c(\Gamma)$, the Zariski closure of Γ. Then G is a semisimple Lie group with no nontrivial compact factors and the connected component G_0 of the identity in G acts irreducibly on $\mathbb{R}^d$. The G-action has a unique compact orbit Q on $\mathbb{P}^{d-1}$. The orbit Q is an algebraic subvariety of $\mathbb{P}^{d-1}$ and it is a factor of the Furstenberg boundary of G. Also, $Q = L_G$.*

Let $\widehat{Q}$ be the cone in $\mathbb{R}^d$ over Q. Then $\widehat{Q}$ is the smallest algebraic Γ-invariant cone of $\mathbb{R}^d$ and it equals the Zariski closure of $L_\Gamma(\mathbb{R}^d)$. Moreover, if $\widehat{Q} - \{0\}$ is not connected, then it decomposes into two connected orbits of G_0 (we denote these by $\widehat{Q}^+$ and $\widehat{Q}^-$).

If $L_G = L_\Gamma$ and $v \in \widehat{Q} - \{0\}$ is such that $0 \in \overline{\Gamma v}$ then according to whether Γ is of type 1 or 2, we have either $\overline{\Gamma v} = \widehat{Q} - \{0\}$, or $\overline{\Gamma v} = \widehat{Q}^+$ or $\widehat{Q}^-$, respectively; this holds in particular if $G/N(\Gamma)$ is of finite volume.

If G/Γ is compact then $L_\Gamma^c = L_\Gamma = L_G$; in this case, according to whether G is of type 1 or 2 we have either $\overline{\Gamma v} = \widehat{Q} - \{0\}$ or $\overline{\Gamma v} = \widehat{Q}^+$ or $\widehat{Q}^-$.

PROOF. Since G acts irreducibly on $\mathbb{R}^d$, G is a direct product of a semisimple group with an abelian group A. By irreducibility of G, the subalgebra of $\operatorname{End} V$ generated by A is a field, isomorphic to $\mathbb{R}$ or $\mathbb{C}$. If the second case holds then A acts nontrivially by isometries on $\mathbb{P}^{d-1}$, and the proximality condition can not hold for Γ. Therefore A is isomorphic to $\mathbb{R}$, and its action on $\mathbb{R}^d$ is by scalars. Since G is contained in $SL(d,\mathbb{R})$ it follows that $A = \{\pm Id\}$. Hence G is a semisimple Lie group. A similar argument shows that G has no nontrivial compact factors.

We next show that the identity component G_0 of G acts irreducibly on $\mathbb{R}^d$. Suppose this is not the case. Then $\mathbb{R}^d$ decomposes as direct sum of G-invariant proper subspaces permuted by G into each other. Then the union of the component subspaces is a Γ-invariant set. This contradicts condition (H_1). Hence G_0 acts irreducibly on $\mathbb{R}^d$.

We denote by U a maximal unipotent subgroup of G and by T the maximal connected triangular subgroup of G containing U (Onischik and Vinberg, 1990, p. 276). Then we have the Iwasawa decomposition $G = L \cdot T$, where L is a maximal compact subgroup of G. Let $p \in \mathbb{P}^{d-1}$ be a point fixed by the T-action. Since $G = LT$ we have $Gp = Lp$ and hence Gp is a compact orbit of G. Since the G-action is irreducible $Gp \neq \{p\}$. Moreover, by the proximality of the Γ-action on $\mathbb{P}^{d-1}$, any two compact G-invariant subsets have to intersect and hence Gp is the only compact G-orbit; we shall denote it by Q.

Since for any point fixed under the T-action the G-orbit is compact, any such point must be contained in Q. As Q contains a unique fixed point of T it follows that T has a unique fixed point on $\mathbb{P}^{d-1}$. Moreover, it has to be the direction of a highest weight vector of the irreducible representation of the semisimple Lie group G_0 (see Guivarc'h, Ji and Taylor, 1998, Ch. 4). Let P be the minimal parabolic subgroup of G containing T, namely the normaliser of T in G. Since p is the unique fixed point of the T-action, P fixes p and $Gp = Q$ is a factor of the flag variety G/P of G. As G and G_0 satisfy conditions (H_1) and (H_2) and act minimally (in fact transitively), the uniqueness assertion as in Proposition 2.4 yields that $L_G = Q = L_{G_0}$.

Let $\underline{G}$ be the Zariski closure of G in $SL(d, \mathbb{C})$ and let $\underline{S}^p$ and S^p be the stabilisers of p in $\underline{G}$ and G respectively. Since S^p is the set of real points of the algebraic group $\underline{S}^p$, which is defined over $\mathbb{R}$, and $P \subset S^p$ it follows that $\underline{S}^p$ contains a Borel subgroup of $\underline{G}$. Hence G/S^p is the set of real points of a complete algebraic variety $\underline{G}/\underline{S}^p$ (Borel, 1969, p. 75). Starting with the rational map $g \mapsto g \cdot p$ of $\underline{G}$ into the complex projective space $\mathbb{P}^{d-1}(\mathbb{C})$ one obtains, by passing to the quotient, an $\mathbb{R}$-defined isomorphism, of $\underline{G}/\underline{S}^p$ on its image $\mathcal{V} \subset \mathbb{P}^{d-1}(\mathbb{C})$ which is a complete algebraic variety, and hence closed in $\mathbb{P}^{d-1}(\mathbb{C})$, in the algebraic sense. In particular $\mathcal{V}$ is a set of common zeros in $\mathbb{P}^{d-1}(\mathbb{C})$ of a family of homogeneous polynomials with real coefficients, since the image Gp of G/S^p is dense in $\mathcal{V}$; moreover by the isomorphism of $\underline{G}/\underline{S}^p$ onto $\mathcal{V}$ as above, Gp is the set of real points of $\mathcal{V}$. Hence $Q = Gp$ is a the set of points of $\mathbb{P}^{d-1}$ which are common zeros of a family of polynomials with real coefficients and $\widehat{Q}$ is an algebraic cone of $\mathbb{R}^d$, which is Γ-invariant. If $\mathcal{C}$ is such a cone, then it is also G-invariant and its projection in $\mathbb{P}^{d-1}$ is a compact G-invariant subset. From what we have seen, it contains $Gp = Q$ and hence $\widehat{Q} \subset \mathcal{C}$. The condition $Z^c(\Gamma) = G$ then implies that $Z^c(L_\Gamma(\mathbb{R}^d)) = \widehat{Q}$.

We now consider the inverse image $\tilde{Q}$ of Q in the unit sphere S^{d-1}. As Q is connected, $\tilde{Q}$ is either connected or decomposes into two symmetric connected components $\tilde{Q}_1$ and $\tilde{Q}_2 = -\tilde{Q}_1$. In the first case $\widehat{Q} - \{0\} = \mathbb{R}^+\tilde{Q}$ is connected. In the second case $\mathbb{R}^+\tilde{Q}_1$ and $\mathbb{R}^+\tilde{Q}_2$ are connected and disjoint and $\widehat{Q} - \{0\} = \mathbb{R}^+\tilde{Q}_1 \cup \mathbb{R}^+\tilde{Q}_2$; hence $\widehat{Q}^+ = \mathbb{R}^+\tilde{Q}^+ = -\widehat{Q}^-$ and $\widehat{Q}^+$ and $\widehat{Q}^-$ are the connected components of $\widehat{Q} - \{0\}$. Theorem 3.5 therefore implies that either $\widehat{Q}^+$ and $\widehat{Q}^-$ are the orbits of G_0 in $\widehat{Q} - \{0\}$ or G_0 is transitive on $\widehat{Q} - \{0\}$.

The assertion about the case $L_\Gamma = L_G$ follows from Theorem 3.5. If $G/N(\Gamma)$ has finite volume then by Lemma 4.3 $L_\Gamma = L_G$ and the assertion applies to this case.

Now suppose that G/Γ is compact and let $v \in \widehat{Q} - \{0\}$. Let $a \in G$ be a diagonalisable element such that $\|a^{-n}v\| \le C/\|a^n\|$ for all $n \ge 1$, where C is a suitable constant. By the compactness of G/Γ one can find a sequence $\{\gamma_n\}$ in Γ such that $\gamma_n a^{-n}$ is bounded in G. Hence with suitable new constants we have

$$\|\gamma_n^{-1}v\| \le \frac{C'}{\|a^n\|} \le \frac{C''}{\|\gamma_n\|}.$$

This shows that $L_\Gamma = L_\Gamma^c$. Hence it follows from the preceding part that $\overline{\Gamma v}$ equals either $\widehat{Q} - \{0\}$, or one of $\widehat{Q}^+$ or $\widehat{Q}^-$ depending on whether Γ is of type 1 or type 2.

5. Groups of Schottky Type

The aim of this section is to illustrate our results of the previous section for a special class of groups; namely groups of Schottky type. This is a class of discrete groups whose limit sets can be described explicitly by a coding. They are generated by transformations satisfying a 'ping pong' property. This class of groups appeared in (Tits, 1972) (see also Margulis, 1991, p. 351). In a context similar to the present one, these groups were involved in Benoist (1997a) and (Guivarc'h, 1990).

We first give a general construction of groups of Schottky type.

Definition 5.1. Let (X, δ) be a complete metric space, p a point in X, and Σ a finite set of homeomorphisms of X which is symmetric (namely $a^{-1} \in \Sigma$ for all $a \in \Sigma$). Let $\{A_a\}_{a\in\Sigma}$ be a family of compact subsets of X such that $p \notin \cup_{a\in\Sigma} A_a$ and $a(p) \in A_a$ for all $a \in \Sigma$. We say that the system $\{(a, A_a) \mid a \in \Sigma\}$ has property (S) if the following conditions are satisfied:

1) for $a, b \in \Sigma$, $A_a \cap A_b = \emptyset$, unless $a = b$;

2) for $a, b \in \Sigma$, $a(A_b) \subset \operatorname{Int} A_a$, except when $a = b^{-1}$;

3) for all sequences $\{a_n\}$ such that $a_n \neq a_{n+1}^{-1}$ for all $n \geq 1$, the diameter of $a_1 \cdots a_n A_{a_{n+1}}$ tends to 0 as n tends to ∞.

The group Γ of homeomorphisms of X generated by Σ satisfying the conditions is called a group of Schottky type.

A sequence $\{a_n\}$ (or a word $a_1 \cdots a_k$ in Γ) is said to be *admissible* if $a_n \neq a_{n+1}^{-1}$ for all $n \geq 1$. We denote by Ω the set of all admissible sequences. For all $\omega = \{a_n\} \in \Omega$ and $n \geq 1$ we define compact subsets $A_{a_1\cdots a_n}$ by

$$A_{a_1\cdots a_n} = a_1 \cdots a_{n-1} A_{a_n}.$$

We denote by Homeo(X) the group of homeomorphisms of X, equipped with the topology of uniform convergence on compact subsets.

Let Γ be a group of Schottky type associated to a system $\{(a, A_a) \mid a \in \Sigma\}$ as above.

Proposition 5.2. *Γ is a free group over Σ and it is a discrete subgroup of Homeo(X). For all admissible sequences $\omega = \{a_n\}$, the sequence $\{a_1 \cdots a_n(p)\}$ converges to a point $\alpha(\omega) \in X$ and the map α is a homeomorphism of Ω onto $\alpha(\Omega)$.*

Proof. This follows from classical arguments; we include some details for the convenience of the reader.

Suppose if possible that Γ is either not free or not discrete. In either case there exists admissible words $a_1 \cdots a_n$ arbitrarily close to the identity; in particular $a_1 \cdots a_n(p)$ are arbitrarily close to p, contrary to the inequality

$$\delta(p, a_1 \cdots a_n(p)) \geq \delta(p, A_{a_1}) \geq \inf_{a\in\Sigma} \delta(p, A_a) > 0.$$

Hence Γ is free and discrete.

For all admissible sequences $\omega = \{a_n\}$, by condition (2) $\{A_{a_1\cdots a_n}\}_{n\geq 1}$ is a decreasing sequence. Also using conditions (1) and (2) it easy to see that if

$a_1 \cdots a_n \neq b_1 \cdots b_n$, where $a_1, \ldots, a_n, b_1, \ldots b_n \in \Sigma$, then the sets $A_{a_1 \cdots a_n}$ and $A_{b_1 \cdots b_n}$ are disjoint. For an admissible word $\gamma = a_1 \cdots a_n$ we denote by $|\gamma|$ the length n of γ and by A_γ the set $A_{a_1 \cdots a_n}$. If $\omega = \{a_n\} \in \Omega$ then $\{A_{a_1 \cdots a_n}\}$ is a nested sequence of sets with diameters tending to 0. Therefore for each $\omega \in \Omega$ there exists a unique point contained in $\cap_n A_{a_1 \cdots a_n}$ and we define that as $\alpha(\omega)$.

If $\omega = \{a_n\}$ and $\omega' = \{b_n\}$ are two admissible sequences such that $a_i = b_i$ for all $i = 1, \ldots, k$ and $a_{k+1} \neq b_{k+1}$, then $\alpha(\omega) \in A_{a_1 \cdots a_k a_{k+1}}$ while $\alpha(\omega') \in A_{b_1 \cdots b_k b_{k+1}}$. As these sets are disjoint, this shows that α is injective. It is straightforward to see that α is continuous, and hence it follows that it is a homeomorphism onto its image $\alpha(\Omega) = \cap F_n$, where $F_n = \cup_{|\gamma|=n} A_\gamma$ for all n.

Definition 5.3. A closed subset C of $I\!P^{d-1}$ is said to be convex if it is contained in the complement of a projective hyperplane H and it is convex as a subset of the affine space $I\!P^{d-1} - H$.

If C is a closed convex subset with nonempty interior we denote by d_C the distance of Hilbert associated to C, defined for $x, y \in \operatorname{Int} C$, by

$$d_C(x, y) = |\log(xyuv)|,$$

where u and v are the endpoints of the intersection of the line joining x, y with C and $(xyuv)$ is the cross ratio of the four points.

The definition of the distance d_C is independent of the choice of the hyperplane H. For properties of the distance d_C the reader is referred to (Benoist, 1997b). In particular we recall that d_C is not bounded, but the space $(\operatorname{Int} C, d_C)$ is complete.

Definition 5.4. If C and D are two convex subsets of $I\!P^{d-1}$ and g is a projective transformation such that $g(C) \subset D$, we put

$$\rho_g^{C,D} = \sup_{x,y \in \operatorname{Int} C} \frac{d_D(g \cdot x, g \cdot y)}{d_C(x, y)}.$$

Remark 5.5. Let C and D be closed convex subsets such that $g(C) \subset \operatorname{Int} D$. Then the diameter β of $g(C)$ for the metric d_D is finite and, by a theorem of G. Birkhoff (1957), we have

$$\rho_g^{C,D} \leq \operatorname{th}(\beta/4) < 1.$$

In the particular case with $C = D$, with $g(C) \subset \operatorname{Int} C$, the Perron-Frobenius theorem implies that g is proximal, with the dominant vector contained in C.

Lemma 5.6. *Let C, D and $\rho < 1$ be given and consider the set H_ρ of $h \in G$ such that $\rho_h^{C,D} \leq \rho$. Then there exists a constant K_ρ such that for all $h \in H_\rho$ and $x \in I\!R^d$ such that $\|x\| = 1$ and $\pi(x) \in C$, we have $\|hx\| \geq K_\rho \|h\|$.*

Proof. Indeed, if it is not the case then there would exist $h_n \in H_\rho$, $x_n \in I\!R^d$ with $\|x_n\| = 1$ and $\pi(x_n) \in C$ such that $h_n x_n / \|h_n\| \to 0$. However, by definition H_ρ is equicontinuous on C and one may therefore suppose that $h_n / \|h_n\| \to \tau \in \operatorname{End} V$, with $\|\tau\| = 1$, where τ is defined as a continuous map of C into D. One may also suppose that x_n converges to x, with $\|x\| = 1$ and $\tau x = 0$. Hence the kernel $\operatorname{Ker} \tau$ of

τ intersects C. One sees easily that the sequence $h_n/\|h_n\|$ cannot be equicontinuous in a neighbourhood of a point of $\operatorname{Ker}\tau$, which contradicts the preceding observation (see Broise and Guivarc'h, 1999). This proves the lemma.

DEFINITION 5.7. We denote by δ the distance on $\mathbb{P}^{d-1}$ defined, for $x = \pi(v)$ and $y = \pi(w)$ with v and w unit vectors in $V = \mathbb{R}^d$, by

$$\delta(x, y) = \|v \wedge w\|.$$

The distance corresponds to the sine of the angle between the vectors in $\mathbb{R}^d$.

REMARK 5.8. We denote by $\sigma(g, \epsilon)$ the infinitesimal multiplication coefficient of the lengths in the direction $\epsilon \in T^1(\mathbb{P}^{d-1})$, when one applies the projective transformation g. An explicit formula, with canonical norms on $\mathbb{R}^d$ and $\wedge^2 \mathbb{R}^d$ is given by

$$\sigma(g, \epsilon) = \frac{\|gx \wedge gy\|}{\|gx\|^2} : \frac{\|x \wedge y\|}{\|x\|^2},$$

for ϵ based at x, in the direction of $x \wedge y$.

Hypothesis. Returning to the notation as before, we now consider the following situation. Let $X = \mathbb{P}^{d-1}$ equipped with the distance δ as above, Σ be a set of projective transformations and let A_a, $a \in \Sigma$ be convex subsets. We suppose also that conditions (1) and (2) in Definition 5.1 are satisfied. Under these conditions we say that the system satisfies condition (S^+).

We put $\rho_{ab} = \rho_a^{A_b, A_a}$, for $a \neq b^{-1}$ and $\rho = \sup_{ab \neq e} \rho_{ab}$. We note that $\rho < 1$. Let $\overline{\rho}$ be the lower bound of the Lipschitz coefficients $\sigma(a, \epsilon)$ as a runs over Σ and ϵ is based at a point of A_b, with $b \neq a^{-1}$.

PROPOSITION 5.9. *Under the preceding conditions there exists a constant $C > 0$ such that, for all admissible sequences $\{a_n\}$, we have*

$$\overline{\rho}^n \delta(x, y) \leq \delta(a_1 \cdots a_n \cdot x, a_1 \cdots a_n \cdot y) \leq C \rho^n \delta(x, y),$$

for all $x, y \in A_{a_{n+1}}$. In particular Condition (3) as in Definition 5.1 holds.

PROOF. It is clear that

$$d_{A_{a_1}}(a_1 \cdots a_n \cdot x, a_1 \cdots a_n \cdot y) \leq \rho^n d_{A_{a_{n+1}}}(x, y).$$

There exists a constant $K > 0$ such that, for $x, y \in A_{a_k}$, we have

$$K^{-1} d_{A_{a_k}}(x, y) \leq \delta(x, y) \leq K d_{A_{a_k}}(x, y).$$

We deduce that, for $x, y \in A_{a_{n+1}}$,

$$\delta(a_1 \cdots a_n \cdot x, a_1 \cdots a_n \cdot y) \leq K d_{A_{a_1}}(a_1 \cdots a_n \cdot x, a_1 \cdots a_n \cdot y)$$

$$\leq K \rho^n d_{A_{a_{n+1}}}(x, y) \leq K^2 \rho^n \delta(x, y).$$

We then get the desired upper bound with $C = K^2$. The lower bound follows from the definition of $\overline{\rho}$ and the fact that the sequence $\{a_k\}$ is admissible.

The following proposition enables construction of systems satisfying condition (S^+), starting with sequences of hyperbolic matrices.

For any proximal projective transformation a we denote by a^+ its dominant eigenvector, by λ_a the corresponding eigenvalue, and by H_a^- the subspace

$$H_a^- = \{w \mid \lambda_a^{-n} a^n w \to 0 \text{ as } n \to \infty\}.$$

PROPOSITION 5.10. *Consider a system* $\widehat{\Sigma} = \{(a, A_a) \mid a \in \Sigma\}$, *where* Σ *is a set of projective transformations and* A_a *are compact convex sets. If* $\widehat{\Sigma}$ *satisfies condition* (S^+) *then all elements of* Σ *are proximal, with* $a^+ \in A_a$, *and* $H_b^- \cap A_a = \emptyset$ *for* $b \neq a^{-1}$. *Conversely, given a system* $\widehat{\Sigma} = \{a, A_a \mid a \in \Sigma\}$, *where* A_a *are disjoint compact convex sets and* Σ *is a set of proximal projective transformations with* $a^+ \in A_a$ *and* $H_b^- \cap A_a = \emptyset$ *if* $b \neq a^{-1}$, *then for all sufficiently large* n *the system* $\widehat{\Sigma}_n = \{a^n, A_a \mid a \in \Sigma\}$ *satisfies condition* (S^+).

PROOF. Replacing a by a^{-1} if necessary, we may suppose that a preserves the convex cones of $\mathbb{R}^d$ associated to A_a. As $a(A_a) \subset \operatorname{Int} A_a$, the Perron-Frobenius theorem implies that a is proximal, $a^+ \in A_a$, $a^+ \notin H_a^-$ and $H_a^- \cap A_a = \emptyset$.

If $x \in A_b$, with $b \neq a^{-1}$, the condition $a(A_b) \subset \operatorname{Int} A_a$ implies that $a^n \cdot x \to a^+$ and hence $x \notin H_a^-$, so $H_a^- \cap A_b = \emptyset$.

Conversely, if Σ consists of proximal elements and $H_a^- \cap A_b = \emptyset$ for $b \neq a^{-1}$, the sequence of closed sets $\{a^k(A_b)\}$ converges to a^+ and hence for k sufficiently large, $a^k(A_b) \subset \operatorname{Int} A_a$. This shows that condition (S^+) is satisfied for all large k.

Limit sets and minimality of $L_\Gamma(\mathbb{R}^d)$. In what follows we denote by d the distance on Ω such that the distance of two sequences which coincide precisely up to n coordinates equals 2^{-n}. On $L_\Gamma = L_\Gamma(\mathbb{P}^{d-1})$ we put the distance δ as on $\mathbb{P}^{d-1}$, defined earlier.

THEOREM 5.11. *Suppose that the system* $\widehat{\Sigma} = \{(a, A_a) \mid a \in \Sigma\}$ *satisfies Condition* (S^+). *Let* Γ *be the group generated by* Σ *and let* ρ *and* $\overline{\rho}$ *be the coefficients as in Proposition 5.9. Then the map* α *of* Ω *into* $\mathbb{P}^{d-1}$ *is a biHölderian homeomorphism of* (Ω, d) *onto* (L_Γ, δ): *there exists* $C > 0$ *such that*

$$C^{-1} d^\lambda(\omega, \omega') \leq \delta(\alpha(\omega), \alpha(\omega')) \leq C d^\mu(\omega, \omega'), \ \forall \omega, \omega' \in \Omega,$$

with $\lambda = -\log \overline{\rho}/\log 2$ *and* $\mu = -\log \rho/\log 2$. *If* $a_1 \cdots a_n$ *is admissible and* $x_{n+1} \in A_{a_{n+1}}$ *we have*

$$\|a_1 \cdots a_n \cdot x_{n+1}\| \geq (C\rho^n)^{d^{-1}-1}.$$

Moreover there exists a constant K_ρ *such that under the same conditions*

$$\|a_1 \cdots a_n \cdot x_{n+1}\| \geq K_\rho \|a_1 \cdots a_n\|.$$

PROOF. Let $\alpha(\omega') = \lim_n b_1 \cdots b_n \cdot p$ with $a_i = b_i$ for $1 \leq i \leq k$ and $a_{k+1} \neq b_{k+1}$. We have $\alpha(\omega) = a_1 \cdots a_k \cdot x_{k+1}$ and $\alpha(\omega') = a_1 \cdots a_k \cdot y_{k+1}$, with $x_{k+1} \in A_{a_{k+1}}$, $y_{k+1} \in A_{b_{k+1}}$, and by Proposition 5.9 there exists a constant C such that

$$\delta(\alpha(\omega), \alpha(\omega')) \leq C\rho^k \delta(x_{k+1}, y_{k+1}) \leq C\rho^k,$$

and

$$\delta(\alpha(\omega), \alpha(\omega')) \geq C\overline{\rho}^k \delta(x_{k+1}, y_{k+1}) \geq C^{-1}\overline{\rho}^k.$$

Since $d(\omega, \omega') = 2^{-k}$ we get

$$C^{-1} d^{\lambda}(\omega, \omega') \leq \delta(\alpha(\omega), \alpha(\omega')) \leq C d^{\mu}(\omega, \omega'),$$

with $\lambda = -\log \overline{\rho}/\log 2$ and $\mu = -\log \rho/\log 2$.

By Proposition 5.9, if $a_1 \cdots a_{n+1}$ is admissible, the Lipschitz coefficient of $\gamma = a_1 \cdots a_n$ on $A_{a_{n+1}}$ is bounded by $C\rho^n$. The Jacobian of γ is therefore bounded by $C^{d-1}\rho^{n(d-1)}$. However, since this Jacobian is $\|\gamma x\|^{-d}$, it follows that for $x_{n+1} \in A_{a_{n+1}}$,

$$\|a_1 \cdots a_n x_{n+1}\| \geq (C\rho^n)^{(d^{-1}-1)}.$$

Finally, since the Lipschitz coefficient of $a_1 \cdots a_n$ on $A_{a_{n+1}}$ are bounded by $C\rho^n$, by Lemma 5.6 there exists $K_\rho > 0$ such that

$$\|a_1 \cdots a_n \cdot x_{n+1}\| \geq K_\rho \|a_1 \cdots a_n\|.$$

REMARK 5.12. We note that the Hausdorff dimension of L_Γ, which is positive (see Theorem 3.7), is bounded by $\log(|\Sigma| - 1)/\log \rho^{-1}$; this follows from the fact that the closed sets $F_n = \cup_{|\gamma|=n} A_\gamma$ are covered by $|\Sigma|(|\Sigma| - 1)^{n-1}$ balls of radius $C\rho^n$.

THEOREM 5.13. *Let $\widehat{\Sigma} = \{(a, A_a) \mid a \in \Sigma\}$ be a system satisfying condition (S^+), let Γ be the subgroup of Homeo$(\mathbb{P}^{d-1})$ generated by Σ and suppose that Γ satisfies Condition (H_1). Then Γ satisfies (H_2) and $L_\Gamma = L_\Gamma^c$. If Γ is of type 1 (resp. type 2) then the orbit of any point of $L_\Gamma(\mathbb{R}^d)$ (resp. $L_\Gamma^+(\mathbb{R}^d)$) is dense in $L_\Gamma(\mathbb{R}^d)$ (resp. $L_\Gamma^+(\mathbb{R}^d)$). Hence $L_\Gamma(\mathbb{R}^d)$ is either minimal or a union of two minimal subsets $L_\Gamma^+(\mathbb{R}^d)$ and $L_\Gamma^-(\mathbb{R}^d)$.*

PROOF. By Theorem 3.5 it suffices to see that for all $v \in L_\Gamma(\mathbb{R}^d)$ there exists a sequence $\{\gamma_n\}$ in Γ such that $\|\gamma_n\| \, \|\gamma_n^{-1} v\|$ is bounded and $\|\gamma_n\| \to \infty$. Let $v \in L_\Gamma(\mathbb{R}^d)$ be given. Then by Theorem 5.11 there exists an admissible sequence $\{a_n\}$ such that $a_1 \cdots a_n \cdot p \to \pi(v)$. For all $n \geq 1$ we put

$$\gamma_n = a_1 \cdots a_n \text{ and } x_{n+1} = \gamma_n^{-1} v / \|\gamma_n^{-1} v\|.$$

Then $x_{n+1} \in A_{a_{n+1}}$ and hence Theorem 5.11 implies that

$$\|\gamma_n x_{n+1}\| \geq K \|\gamma_n\|$$

for all n, for a suitable constant K. Thus $\|v\|/\|\gamma_n^{-1} v\| \geq K\|\gamma_n\|$ or, in other words, $\|\gamma_n\| \|\gamma_n^{-1} v\|$ is bounded. Theorem 5.11 also shows that $\|a_1 \cdots a_n\| \to \infty$. This proves the theorem.

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