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COUNTING INTEGRAL MATRICES WITH A GIVEN CHARACTERISTIC POLYNOMIAL*

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SUMMARY. We give a simpler proof of an earlier result giving an asymptotic estimate for the number of integral matrices in large balls whose characteristic polynomial is a given monic integral irreducible polynomial. The proof uses a result on equidistributions of multi-dimensional polynomial trajectories on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ which is a generalization of Ratner's theorem on equidistributions of unipotent trajectories.

We also compute the exact constants appearing in the above mentioned asymptotic estimates.

1. **Introduction**

Let P be a monic polynomial of degree n ($n \geq 2$) with integral coefficients which is irreducible over $\mathbb{Q}$. Let

$$V_P = \{X \in \mathrm{M}_n(\mathbb{R}) : \det(\lambda I - X) = P(\lambda)\}.$$

Since P has n distinct roots, V_P is the set of real $n \times n$-matrices X such that the roots of P are the eigenvalues of X. Let $V_P(\mathbb{Z})$ denote that set of matrices in V_P with integral entries. Let B_T denote the ball in $M_n(\mathbb{R})$ centred at 0 and of radius T with respect to the Euclidean norm: $\|(x_{ij})\| = (\sum_{i,j} x_{ij}^2)^{1/2}$. We are interested in estimating, for large T, the number of integer matrices in B_T with characteristic polynomial P.

Theorem 1.1 (Eskin, Mozes and Shah (1996)). *There exists a constant $C_P > 0$ such that*

$$\lim_{T\to\infty} \frac{\#(V_P(\mathbb{Z}) \cap B_T)}{T^{n(n-1)/2}} = C_P.$$

A formula for C_P, in the general case, is given in Theorem 5.1. Under an additional hypothesis, the formula for C_P is simpler and it can be given as follows (Cf. Eskin, Mozes and Shah (1996)):

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*To Professor M.G. Nadkarni, on the occasion of his sixtieth birthday

Theorem 1.2. *Let α be a root of P and $K = \mathbb{Q}(\alpha)$. Suppose that $\mathbb{Z}[\alpha]$ is the integral closure of $\mathbb{Z}$ in K. Then*

$$C_P = \frac{2^{r_1}(2\pi)^{r_2}hR}{wD^{1/2}} \cdot \frac{\pi^{m/2}/\Gamma(1+(m/2))}{\prod_{s=2}^{n}\pi^{-s/2}\Gamma(s/2)\zeta(s)},$$

where h = ideal class number of K, R = regulator of K, w = order of the group of roots of unity in K, D = discriminant of K, r_1 (resp. r_2) = number of real (resp. complex) places of K, and $m = n(n-1)/2$.

Remark 1.1. The three components of the above formula for C_P are volumes of certain standard entities in geometry of numbers (with respect to the canonical volume forms on the respective spaces):

$$\begin{aligned} \mathrm{Vol}(J^0(K)/K^\times) &= \frac{2^{r_1}(2\pi)^{r_2}hR}{wD^{1/2}}, \\ \mathrm{Vol}(B^m) &= \pi^{m/2}/\Gamma(1+(m/2)), \\ \mathrm{Vol}(\mathcal{SM}_n) &= \prod_{s=2}^{n}\pi^{-s/2}\Gamma(s/2)\zeta(s). \end{aligned}$$

Here $J^0(K)/K^\times$ = the group of principal ideals of K modulo $K^\times$ (see Koch (1997), Chap. 1, §5.4), B^m = the unit ball in $\mathbb{R}^m$, and $\mathcal{SM}_n$ = the determinant one surface in the Minkowski fundamental domain M_n in the space of $n \times n$ real positive symmetric matrices with respect to the action of $\mathrm{GL}_n(\mathbb{Z})$ (see Terras (1988, Sect. 4.4.4)).

Remark 1.2. The hypothesis of Theorem 1.2 is satisfied if α is a root of unity (see Koch (1997), Theorem 1.61).

The conclusion of Theorem 1.2 was obtained in Eskin, Mozes and Shah (1996) under a further hypothesis that all roots of P are real.

In Eskin, Mozes and Shah (1996), the proof of Theorem 1.1 is based on the following: (1) the existence of limits of large translates of certain algebraic measures as proved in Eskin, Mozes and Shah (1997); (2) showing that such limiting distributions are actually algebraic measures, using Ratner's description of ergodic invariant measures of unipotent flows Ratner (1991a); and (3) the verification that a certain condition, called the *non-focusing condition*, holds in the case of Theorem 1.1 (See Ratner, 1995).

A main purpose of this article is to provide a simple and direct proof of this theorem using the following result on equidistributions of 'polynomial like' trajectories on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$:

Theorem 1.3. *Let Γ be a lattice in $\mathrm{SL}_n(\mathbb{R})$, μ the $\mathrm{SL}_n(\mathbb{R})$-invariant probability measure on $\mathrm{SL}_n(\mathbb{R})/\Gamma$, and $x \in \mathrm{SL}_n(\mathbb{R})/\Gamma$. Let*

$$\Theta = (\Theta_{ij})_{i,j=1}^{n} : \mathbb{R}^m \to \mathrm{SL}_n(\mathbb{R})$$

be a map such that each Θ_{ij} is a real valued polynomial in m variables, and $\Theta(0) = I$, the identity matrix. Suppose that $\Theta(\mathbb{R}^m)$ is not contained in any proper closed sub-

group L of $\mathrm{SL}_n(\mathbb{R})$ *such that the orbit Lx is closed. Then for any* $f \in \mathrm{C_c}(\mathrm{SL}_n(\mathbb{R})/\Gamma)$,

$$\lim_{T\to\infty} \frac{1}{\mathrm{Vol}(B(T))} \int_{B(T)} f(\Theta(\boldsymbol{s})x)\, d\boldsymbol{s} = \int f\, d\mu,$$

where $B(T)$ denotes the ball of radius T in $\mathbb{R}^m$ centered at 0.

Take $0 \le r \le m$. Put $B^+(T) = B(T) \cap (\mathbb{R}_+)^r \times \mathbb{R}^{m-r}$. Then

$$\lim_{T\to\infty} \frac{1}{\mathrm{Vol}(B_T^+)} \int_{B_T^+} f(\tilde{\Theta}(\boldsymbol{s})x)\, d\boldsymbol{s} = \int f\, d\mu, \qquad \forall\ f \in \mathrm{C_c}(\mathrm{SL}_n(\mathbb{R})/\Gamma),$$

where $\tilde{\Theta}(\boldsymbol{s}) := \Theta(s_1^{1/2}, \ldots, s_r^{1/2}, s_{r+1}, \ldots, s_m)$, $\forall\ \boldsymbol{s} \in (\mathbb{R}_+)^r \times \mathbb{R}^{m-r}$.

The first part of the theorem is a particular case of Corollary 1.1 of Shah (1994), whose proof can be readily modified to prove the second part. This result is a generalization of Ratner's theorem on equidistribution of orbits of one-dimensional unipotent flows Ratner (1991b). The main ingredient in its proof is, just as in Ratner (1991b), the classification of ergodic invariant measures for unipotent flows.

As in Eskin, Mozes and Shah (1996), the first step in the proof of Theorem 1.1 is its reformulation to a question in ergodic theory of subgroup actions on homogeneous spaces of Lie groups; we follow the approach of Duke, Rudnick and Sarnak (1993). The second step is to reduce this question to one about equidistribution of polynomial trajectories, so that Theorem 1.3 can be applied.

Another purpose of this article is to obtain an expression for C_P in terms of algebraic number theoretic constants associated with P; this is carried out in Section 5.

2. **Reduction to a Question in Ergodic Theory**

We write

$${}^gX := gXg^{-1}, \qquad \forall\ g \in \mathrm{GL}_n(\mathbb{R}), \forall\ X \in \mathrm{M}_n(\mathbb{R}).$$

Put

$$\Gamma = \mathrm{GL}_n(\mathbb{Z}) = \{X \in \mathrm{M}_n(\mathbb{Z}) : \det(X) = \pm 1\}.$$

If $X \in V_P(\mathbb{Z})$ and $\gamma \in \Gamma$, then ${}^\gamma X \in V_P(\mathbb{Z})$; and we denote the Γ-orbit through X by

$${}^\Gamma X := \{\, {}^\gamma X : \gamma \in \Gamma\}.$$

2.1 *Finitely many Γ-orbits in $V_P(\mathbb{Z})$.* Using a correspondence between Γ-orbits and ideal classes due to Latimer and MacDuffee (1933) and the finiteness of class numbers of orders, one has the following (see Proposition 5.3).

Proposition 2.1 (Latimer and MacDuffee). *There are only finitely many distinct Γ-orbits in $V_P(\mathbb{Z})$.*

REMARK 2.1. The above proposition is a particular case of a general 'finiteness theorem' due to Borel and Harish-Chandra (1962).

By Proposition 2.1, to prove Theorem 1.1 it is enough to prove the following.

THEOREM 2.2. *Let $X \in V_P(\mathbb{Z})$. Then there exists $c_X > 0$ such that*

$$\lim_{T\to\infty} \frac{\#({}^{\Gamma}X \cap B_T)}{T^{n(n-1)/2}} = c_X.$$

2.2 *Considering a fixed* Γ*-orbit.* Put $G = \{g \in \mathrm{GL}_n(\mathbb{R}) : \det g = \pm 1\}$. Since the conjugation action of $\mathrm{GL}_n(\mathbb{R})$ on V_P is transitive, the same holds for the action of G on V_P. Note that $\Gamma = \mathrm{GL}_n(\mathbb{Z})$ is a lattice in G. Fix any $X_0 \in V_P(\mathbb{Z})$. Put

$$H = \{g \in G : {}^{g}X_0 = X_0\}.$$

Since all the eigenvalues of X_0 are distinct, H is an abelian group consisting of elements diagonalizable over $\mathbb{C}$. In fact, H is a real algebraic torus defined over $\mathbb{Q}$. Using Dirichlet's unit theorem we will show the following (Theorem 5.4):

PROPOSITION 2.3. *$H/H \cap \Gamma$ is compact.*

Define

$$R_T = \{g \in G : {}^{g}X_0 \in B_T\}/H \subset G/H,$$

and χ_T denote its characteristic function. Then

$$\#({}^{\Gamma}X_0 \cap B_T) = \#(\Gamma[H] \cap R_T) = \sum_{\dot\gamma \in \Gamma/\Gamma\cap H} \chi_T(\gamma[H]). \tag{1}$$

2.3 *Choosing Haar measures on G and H.* We choose Haar measures $\tilde\mu$ (resp. $\tilde\nu$) on G (resp. H). Let μ (resp. ν) denote the left invariant measure on G/Γ (resp. $H/H\cap\Gamma$) corresponding to the measure $\tilde\mu$ (resp. $\tilde\nu$). Let η be the corresponding left G-invariant measure on G/H (see Raghunathan (1972), Lemma 1.4); that is, $\forall\ f \in \mathrm{C_c}(G)$,

$$\int_G f\, d\tilde\mu = \int_{gH\in G/H} \left(\int_H f(gh)\, d\tilde\nu(h)\right)\, d\eta(gH). \tag{2}$$

In Section 3.8 we show that there exists a constant $c_\eta > 0$ (see (45)) depending on X_0 such that

$$\lim_{T\to\infty} \eta(R_T)/T^{n(n-1)/2} = c_\eta. \tag{3}$$

2.4 *Introducing an auxiliary counting function.* For all $T > 0$ and $g \in G$, let

$$F_T(g\Gamma) := \#(g\Gamma[H] \cap R_T) = \sum_{\dot\gamma\in\Gamma/(\Gamma\cap H)} \chi_T(g\gamma H). \tag{4}$$

Note that F_T is bounded, measurable, and vanishes outside a compact set in G/Γ. By (1) and (3), in order to prove Theorem 2.2, it is enough to prove the following:

THEOREM 2.4.
$$\lim_{T\to\infty}\frac{F_T(e\Gamma)}{\eta(R_T)}=\frac{\nu(H/H\cap\Gamma)}{\mu(G/\Gamma)}.$$

Although the precise constant, given by a volume ratio, on the right hand side of the above equation is not needed for proving Theorem 2.2, it will be used in the computation of C_P.

2.5 *Weak convergence is enough.* From the computations in Sections 3.5 and 3.6, one can deduce the following: Given any $\kappa > 1$ there exists a neighbourhood Ω of e in G such that
$$R_{\kappa^{-1}T}\subset \Omega R_T\subset R_{\kappa T}. \tag{5}$$

Now by (3),
$$\lim_{\kappa\to 1}\lim_{T\to\infty}\eta(R_{\kappa T})/\eta(R_T)=1 \tag{6}$$

By (5) and (6), in order to prove Theorem 2.4, it is enough to prove the following weak convergence (see Eskin and McMullen, 1993):

THEOREM 2.5. *For any* $f\in \mathrm{C_c}(G/\Gamma)$,
$$\lim_{T\to\infty}\frac{\langle f,F_T\rangle}{\eta(R_T)}=\frac{\nu(H/H\cap\Gamma)}{\mu(G/\Gamma)}\cdot\langle f,1\rangle.$$

Using Fubini's theorem (Raghunathan (1972), Lemma 1.6), we have the following (Duke, Rudnick and Sarnak (1993), Eskin and McMullen (1993)):

PROPOSITION 2.6. *For any* $f\in \mathrm{C_c}(G/\Gamma)$,
$$\begin{aligned}\langle f,F_T\rangle &= \int_{G/\Gamma} f(g\Gamma)\left(\textstyle\sum_{\dot\gamma\in\Gamma/(H\cap\Gamma)}\chi_T(g\gamma H)\right)\,d\mu(\dot g)\\ &= \int_{G/H\cap\Gamma} f(g\Gamma)\chi_T(gH)\,d\bar\mu(\dot g)\\ &= \int_{G/H}\chi_T(gH)\left(\int_{H/H\cap\Gamma} f(gh\Gamma)\,d\nu(\dot h)\right)\,d\eta(gH)\\ &= \int_{R_T}\left(\int_{H/H\cap\Gamma} f(gh\Gamma)\,d\nu(\dot h)\right)\,d\eta(gH),\end{aligned} \tag{7}$$

where $\bar\mu$ is the left G-invariant measure on $G/(H\cap\Gamma)$ corresponding to $\tilde\mu$, and $\dot x$ denotes the coset $x(H\cap\Gamma)$.

In Eskin, Mozes and Shah (1996) further analysis of the limit was carried out by showing that, as $T_i\to\infty$, for 'almost all' sequences $g_iH\to\infty$ in G/H, where $g_iH\in R_{T_i}$,
$$\int_{H/H\cap\Gamma} f(g_ih\Gamma)\,d\nu(\dot h)\to\frac{\nu(H/H\cap\Gamma)}{\mu(G/\Gamma)}\langle f,1\rangle \qquad \text{as } i\to\infty.$$

In view of (7), this implies Theorem 2.5.

In this article, our approach is to change the order of integration in the final expression in (7), and then apply Theorem 1.3 to find the limit. For this purpose, we need an explicit description of R_T, and of the measure η. We will show that

R_T is a 'polynomial like' image of a ball, and η is the push forward of a Lebesgue measure under this map.

3. Integration on R_T

NOTATION 3.1. Let r_1 be the number of real roots of P and r_2 be the number of pairs of complex conjugate roots of P. Since P is irreducible, all roots of P are distinct, and $n = r_1 + 2r_2$. Fix a root α of P. Let $\sigma_1, \dots, \sigma_{r_1}$ be the distinct real embeddings of $\mathbb{Q}(\alpha)$. Let $\sigma_{r_1+1}, \dots, \sigma_{r_1+2r_2}$ be the distinct non-real complex embeddings of $\mathbb{Q}(\alpha)$, such that

$$\sigma_{r_1+r_2+i} = \overline{\sigma_{r_1+i}}, \qquad 1 \le i \le r_2. \tag{8}$$

Put

$$d_i = \begin{cases} \sigma_i(\alpha) & \text{if } 1 \le i \le r_1 \\ \begin{pmatrix} a_{i-r_1} & -b_{i-r_1} \\ b_{i-r_1} & a_{i-r_1} \end{pmatrix} & \text{if } r_1 < i \le r_1 + r_2, \end{cases} \tag{9}$$

where $a_i + b_i\sqrt{-1} := \sigma_{r_1+i}(\alpha)$, $i = 1, \dots, r_2$.

3.1 *Diagonalization of X and H.* Let

$$\begin{aligned} X_1 &= \operatorname{diag}(d_1, \dots, d_{r_1+r_2}) \\ H_1 &= \{g \in G : {}^gX_1 = X_1\} \\ R_T^1 &= \{g \in G : {}^gX_1 \in B_T\}/H_1. \end{aligned}$$

Since the eigenvalues of X_1 are same as the roots of P, $X_1 \in V_P$. Let $g_0 \in G$ be such that ${}^{g_0}X_0 = X_1$.

Define $\psi : G \to G$ as $\psi(g) = g_0 g {g_0}^{-1}$, $\forall\, g \in G$. Then $H_1 = \psi(H)$ and $\psi_*(\tilde{\mu}) = \tilde{\mu}$. We choose a Haar measure $\tilde{\nu}_1$ on H_1 defined by

$$\tilde{\nu}_1 := \psi_*(\tilde{\nu}). \tag{10}$$

Define $\bar{\phi} : G/H \to G/H_1$ as $\bar{\phi}(gH) = g{g_0}^{-1}H_1$, $\forall\, g \in G$. Let $\eta_1 := \bar{\phi}_*(\eta)$. Then by (2), $\forall\, f \in \mathrm{C_c}(G)$,

$$\int_G f\, d\tilde{\mu} = \int_{G/H_1} \left(\int_{H_1} f(gh_1)\, \tilde{\nu}_1(h_1) \right) d\eta_1(gH_1). \tag{11}$$

Also

$$R_T^1 = \bar{\phi}(R_T) \qquad \text{and} \qquad \eta_1(R_T^1) = \eta(R_T). \tag{12}$$

Put $\Gamma_1 = \psi(\Gamma)$. Define $\bar{\psi} : G/\Gamma \to G/\Gamma_1$ as $\bar{\psi}(g\Gamma) = \psi(g)\Gamma_1$, $\forall\, g \in G$. Let $\mu_1 := \bar{\psi}_*(\mu)$ and $\nu_1 := \bar{\psi}_*(\nu)$. Then μ_1 is the G-invariant measure on G/Γ_1 associated to $\tilde{\mu}$. Also ν_1 is the H_1-invariant measure on

$$H_1/(H_1 \cap \Gamma_1) \cong H_1\Gamma_1/\Gamma_1 = \bar{\psi}(H\Gamma/\Gamma)$$

associated to $\tilde{\nu}_1$, and

$$\nu_1(H_1/H_1 \cap \Gamma_1) = \nu(H/H \cap \Gamma). \tag{13}$$

Now we can rewrite Proposition 2.6 as follows:

PROPOSITION 3.1. *$\forall\, f \in \mathrm{C_c}(G/\Gamma)$, and $f_1 := f \circ \bar{\psi}^{-1} \in \mathrm{C_c}(G/\Gamma_1)$,*

$$\begin{aligned} \langle f, F_T \rangle &= \int_{R_T} \left(\int_{H/H\cap\Gamma} f(gh\Gamma)\, d\nu(\dot{h}) \right) d\eta(gH) \\ &= \int_{R_T^1} \left(\int_{H_1/H_1\cap\Gamma_1} f_1(gh\Gamma_1)\, d\nu_1(\dot{h}) \right) d\eta_1(gH_1). \end{aligned}$$

Due to this proposition, instead of integrating on R_T, it suffices to integrate on R_T^1. Therefore we describe the measure η_1 on G/H_1. For this purpose we want to express G as $G = YH_1$, where Y is a product of certain subgroups and subsemigroups of G (see (23)). Later, in Section 3.3 we will decompose the Haar measure of G into products of appropriate Haar measures on these subgroups. This will allow us to describe η_1 as a product of the chosen Haar measures on the subgroups and subsemigroups, whose product is Y (Proposition 3.2).

3.2 *Product decompositions of G.* In view of the above, first we will describe various subgroups of G, and then obtain different product decompositions of G into those subgroups and their subsemigroups.

Let $\mathrm{O}(n)$ denote the group of orthogonal matrices in $\mathrm{GL}_n(\mathbb{R})$. Let

$$N = \{\boldsymbol{n} := (n_{ij})_{i,j=1}^n : n_{ij} \in \mathbb{R},\, n_{ij} = 0 \text{ if } i > j,\, n_{ii} = 1\} \tag{14}$$

$$A = \{\boldsymbol{a} := \mathrm{diag}(a_1, \ldots, a_n) : a_i > 0, \prod_{i=1}^n a_i = 1\}. \tag{15}$$

By Iwasawa decomposition, the map

$$(k, n, a) \mapsto kna \,:\, \mathrm{O}(n) \times N \times A \to G$$

is a diffeomorphism.

For $i, j = 1, \ldots, r_1 + r_2$, let

$$M_{ij} = \begin{cases} \mathbb{R} & \text{if } i \le r_1,\, j \le r_1 \\ \mathrm{M}_{1\times 2}(\mathbb{R}) & \text{if } i \le r_1,\, j > r_1 \\ \mathrm{M}_{2\times 1}(\mathbb{R}) & \text{if } i > r_1,\, j \le r_1 \\ \mathrm{M}_2(\mathbb{R}) & \text{if } i > r_1,\, j > r_1. \end{cases} \tag{16}$$

We will express $g \in \mathrm{M}_n(\mathbb{R})$ as $g = (g_{ij})_{i,j=1}^{r_1+r_2}$, where $g_{ij} \in M_{ij}$.

Put

$$\begin{aligned}
\mathcal{U} &= \left(\prod_{1\le i<j\le r_1+r_2} M_{ij}\right) \cong \mathbb{R}^{\frac{1}{2}n(n-1)-r_2},\\
u(\boldsymbol{x}) &= (u_{ij});\ \boldsymbol{x}=(x_{ij})\in\mathcal{U}, M_{ij}\ni u_{ij} = \begin{cases} 0 & \text{if } i>j\\ 1 & \text{if } i=j\\ x_{ij} & \text{if } i<j,\end{cases}\\
h(t) &= \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix},\ \forall\, t\in\mathbb{R}.
\end{aligned}$$

Define

$$\begin{aligned}
L_1 &= \{\mathrm{diag}(1,\dots,1,g_1,\dots,g_{r_2})\in G : g_i\in \mathrm{SL}_2(\mathbb{R})\}\\
K_1 &= \{\mathrm{diag}(1,\dots,1,k_1,\dots,k_{r_2})\in G : k_i\in \mathrm{SO}(2)\}\\
N_1 &= \{h(\boldsymbol{t}) = \mathrm{diag}(1,\dots,1,h(t_1),\dots,h(t_{r_2}) :\\
&\qquad\qquad \boldsymbol{t}=(t_i)\in\mathbb{R}^{r_2}\}
\end{aligned}$$

$$\begin{aligned}
A_1 &= \{\boldsymbol{a}_1 = \mathrm{diag}(1,\dots,1,b_1,\dots,b_{r_2}) :\\
&\qquad\qquad b_i = \mathrm{diag}(\beta_i, {\beta_i}^{-1}),\ \beta_i>0\}
\end{aligned}\tag{17}$$

$$\begin{aligned}
U &= \{u(\boldsymbol{x}) : \boldsymbol{x}=(x_{ij})\in\mathcal{U}\}\\
C &= \{\boldsymbol{c} = \mathrm{diag}(c_1,\dots,c_{r_1},c_{r_1+1}^{1/2}I_2,\dots,c_{r_1+r_2}^{1/2}I_2)\in G :\\
&\qquad\qquad c_i>0,\ \textstyle\prod_{i=1}^{r_1+r_2} c_i = 1\}
\end{aligned}\tag{18}$$

$$\Sigma = \{\mathrm{diag}(\epsilon_1,\dots,\epsilon_{r_1},I_2,\dots,I_2)\in G : \epsilon_i=\pm 1\},\tag{19}$$

where I_2 denotes the identity matrix in $\mathrm{M}_2(\mathbb{R})$.

We have the following product decompositions:

$$\begin{aligned}
N &= N_1\cdot U, & A &= A_1\cdot C,\\
H_1 &= \Sigma\cdot K_1\cdot C, & L &= K_1\cdot N_1\cdot A_1.
\end{aligned}\tag{20}$$

In each of the above decompositions, the product map, from the direct product of the subgroups on the right hand side to the group on the left hand side, is a diffeomorphism. We also note that

$$\Sigma\cdot C\subset Z_G(L),\quad N_G(U) = \Sigma\cdot C\cdot L\cdot U.\tag{21}$$

Therefore

$$\begin{aligned}
G &= \mathrm{O}(n)NA = \mathrm{O}(n)K_1\cdot N_1U\cdot A_1C\\
&= \mathrm{O}(n)\cdot K_1N_1A_1\cdot UC\\
&= \mathrm{O}(n)\cdot L\cdot U\cdot C.
\end{aligned}\tag{22}$$

One has that $\mathrm{SL}_2(\mathbb{R}) = \mathrm{SO}(2)\cdot h(\mathbb{R}_+)\cdot \mathrm{SO}(2)$ (see Proposition A.3). Since $L\cong(\mathrm{SL}_2(\mathbb{R}))^{r_2}$, we have that

$$L = K_1N_1^+K_1,$$

where $N_1^+ = \{h(\boldsymbol{t}) : \boldsymbol{t} \in (\mathbb{R}_+)^{r_2}\}$. Now, in view of (20)–(22), we have

$$\begin{aligned} G &= \mathrm{O}(n) \cdot K_1 N_1^+ K_1 \cdot U \cdot C \\ &= \mathrm{O}(n) \cdot N_1^+ U \cdot K_1 C \\ &= \mathrm{O}(n)\Sigma \cdot N_1^+ U \cdot K_1 C \\ &= \mathrm{O}(n) \cdot N_1^+ U \cdot \Sigma K_1 C \\ &= \mathrm{O}(n) \cdot N_1^+ U \cdot H_1. \end{aligned} \tag{23}$$

3.3 *Choice of Haar measures on subgroups of G.* Our next aim is to choose the Haar measures on each of the subgroups defined in the previous section, so that the analogues of the decompositions (20), (22) and (23) also hold in terms of products of the Haar measures on these subgroups.

Choice of Haar measure $\tilde{\mu}$ on G. We choose a Haar integral dk on $\mathrm{O}(n)$ such that $\mathrm{Vol}(\mathrm{SO}(n)) = 1$; in particular,

$$\mathrm{Vol}(\mathrm{O}(n)) = \int_{\mathrm{O}(n)} 1 \, dk = 2. \tag{24}$$

We choose the Haar integral $d\boldsymbol{n}$ on N (see (14)) such that

$$d\boldsymbol{n} = \prod_{i<j} dn_{ij}.$$

We choose the Haar integral $d\boldsymbol{a}$ on A such that $\forall\ f \in \mathrm{C_c}(A)$,

$$\int_A f(\boldsymbol{a}) \, d\boldsymbol{a} = \int_{(\mathbb{R}_{>0})^{n-1}} f(\boldsymbol{a}) \, \frac{da_1}{a_1} \cdots \frac{da_{n-1}}{a_{n-1}}; \qquad \text{(see (15))}$$

alternative notation: $d\boldsymbol{a} = \prod_{i=1}^{n-1} da_i/a_i$.

We choose a Haar measure $\tilde{\mu}$ on G such that,

$$\int_G f \, d\tilde{\mu} = \int_{\mathrm{O}(n)\times N \times A} f(k\boldsymbol{n}\boldsymbol{a}) \, dk \, d\boldsymbol{n} \, d\boldsymbol{a}, \ \forall\ f \in \mathrm{C_c}(G). \tag{25}$$

Decomposition of integrals on A and N. We choose a Haar integral $d\boldsymbol{c}$ on C such that (see (18))

$$d\boldsymbol{c} = (dc_1/c_1) \cdots (dc_{r_1+r_2-1}/c_{r_1+r_2-1}).$$

Choose the Haar integral $d\boldsymbol{a}_1 := \prod_{i=1}^{r_2} d\beta_i/\beta_i$ on A_1 (see (17)). Then $d\boldsymbol{a} = d\boldsymbol{a}_1 \, d\boldsymbol{c}$, where $\boldsymbol{a} = \boldsymbol{a}_1\boldsymbol{c}$, $(\boldsymbol{a}_1, \boldsymbol{c}) \in A_1 \times C$ (see (20)).

Let $d\boldsymbol{t}$ denote the standard Lebesgue measure on $\mathbb{R}^{r_2}$. Let $d\boldsymbol{x}$ denote the standard Lebesgue measure on $\mathcal{U}$. Then $d\boldsymbol{n} = d\boldsymbol{t}\, d\boldsymbol{x}$, where $\boldsymbol{n} = h(\boldsymbol{t})u(\boldsymbol{x})$, $(\boldsymbol{t}, \boldsymbol{x}) \in \mathbb{R}^{r_2} \times \mathcal{U}$.

Choice of Haar integral dl on L. Let dl be a Haar integral on L such that,

$$\int_L f(l) \, dl = \int_{K_1 \times \mathbb{R}^{r_2} \times A_1} f(kh(\boldsymbol{t})\boldsymbol{a}_1) \, d\theta(k) \, d\boldsymbol{t} \, d\boldsymbol{a}_1, \ \forall\ f \in \mathrm{C_c}(L), \tag{26}$$

where θ denotes the Haar measure on K_1 such that

$$\theta(K_1) = 1. \tag{27}$$

Decomposition of Haar integral $d\tilde{\mu}$. From the above choices of Haar integrals on various subgroups of G, their interrelations, (21) and (22) we have

$$\int_G f(g)\, d\tilde{\mu}(g) = \int_{\mathrm{O}(n)\times L\times \mathcal{U}\times C} f(kl\boldsymbol{x}\boldsymbol{c})\, dk\, dl\, d\boldsymbol{x}\, d\boldsymbol{c},\ \forall\ f \in \mathrm{C_c}(G). \tag{28}$$

Choice of Haar measure $\tilde{\nu}$ on H. We also choose a Haar measure $\tilde{\nu}$ on H such that for the Haar measure $\tilde{\nu}_1 := \psi_*(\tilde{\nu})$ on H_1 (see (10)), we have (see (20))

$$\int_{H_1} f\, d\tilde{\nu}_1 = \sum_{\sigma\in\Sigma} \int_{K_1\times C} f(\sigma k\boldsymbol{c})\, d\theta(k)\, d\boldsymbol{c}, \qquad \forall\ f \in \mathrm{C_c}(H_1). \tag{29}$$

3.4 *Description of integral η_1 on G/H_1.* In order to describe η_1, we will express the integral $d\tilde{\mu}$ as a product of an integral on certain subset of G and the integral $d\tilde{\nu}_1$ using the expressions (28) and (29).

A new description of the integral dl. First we will express the Haar integral on L in terms of the product decomposition $L = K_1N_1^+K_1$.

By Proposition A.3 (stated and proved in Appendix A), the following holds: $\forall\ f \in \mathrm{C_c}(\mathrm{SL}_2(\mathbb{R}))$,

$$\begin{aligned} &\textstyle\int_{\mathrm{SO}(2)\times\mathbb{R}\times\mathbb{R}_{>0}} f(kh(t)\,\mathrm{diag}(\beta,\beta^{-1}))\, d\vartheta(k)\, dt\, (d\beta/\beta) \\ &\quad = (\pi/2) \textstyle\int_{\mathrm{SO}(2)\times\mathbb{R}_+\times\mathrm{SO}(2)} f(k_1h(t^{1/2})k_2)\, d\vartheta(k_1)\, dt\, d\vartheta(k_1), \end{aligned} \tag{30}$$

where ϑ is a probability Haar measure on SO(2).

Since $L \cong \mathrm{SL}_2(\mathbb{R})^{r_2}$, by (26) and (30), $\forall\ f \in \mathrm{C_c}(L)$,

$$\int_L f(l)\, dl = (\pi/2)^{r_2} \int_{K_1\times(\mathbb{R}_+)^{r_2}\times K_1} f(kh(\boldsymbol{t}^{1/2})k')\, d\theta(k)\, d\boldsymbol{t}\, d\theta(k'), \tag{31}$$

where the notation is

$$\boldsymbol{t}^{1/2} := (t_1^{1/2},\ldots,t_{r_2}^{1/2}),\ \forall\ \boldsymbol{t} = (t_1,\ldots,t_{r_2}) \in (\mathbb{R}_+)^{r_2}. \tag{32}$$

From (23) and (28)–(31), $\forall\ f \in \mathrm{C_c}(G)$,

$$\begin{aligned} &\int_G f(g)\, d\tilde{\mu}(g) \\ = \quad &(\pi/2)^{r_2} \int_{\mathrm{O}(n)\times K_1\times(\mathbb{R}_+)^{r_2}\times K_1\times\mathcal{U}\times C} f(kk_1'h(\boldsymbol{t}^{1/2})k_1u(\boldsymbol{x})\boldsymbol{c}) \\ &\qquad\qquad \times dk\, d\theta(k_1')\, d\boldsymbol{t}\, d\theta(k_1)\, d\boldsymbol{x}\, d\boldsymbol{c} \\ = \quad &(\pi/2)^{r_2}(\#\Sigma)^{-1} \sum_{\sigma\in\Sigma} \int_{\mathrm{O}(n)\times(\mathbb{R}_+)^{r_2}\times\mathcal{U}\times K_1\times C} f(k\sigma h(\boldsymbol{t}^{1/2})u(\boldsymbol{x})k_1\boldsymbol{c}) \\ &\qquad\qquad \times dk\, d\boldsymbol{t}\, d\boldsymbol{x}\, d\theta(k_1)\, d\boldsymbol{c}. \\ = \quad &\pi^{r_2}2^{-r_1-r_2} \int_{\mathrm{O}(n)\times(\mathbb{R}_+)^{r_2}\times\mathcal{U}\times H_1} f(kh(\boldsymbol{t}^{1/2})u(\boldsymbol{x})h_1)\, dk\, d\boldsymbol{t}\, d\boldsymbol{x}\, d\tilde{\nu}_1(h_1). \end{aligned}$$

Now in view of (11), we have the following:

PROPOSITION 3.2. *For any* $\bar{f} \in \mathrm{C_c}(G/H_1)$,

$$\int_{G/H_1} \bar{f}\, d\eta_1 = (2\pi)^{r_2} 2^{-n} \int_{\mathrm{O}(n)\times(\mathbb{R}_+)^{r_2}\times\mathcal{U}} \bar{f}(kh(\boldsymbol{t}^{1/2})u(\boldsymbol{x})H_1)\, dk\, d\boldsymbol{t}\, d\boldsymbol{x}.$$

3.5 *Changing the order of Integration.* The Euclidean norm on $\mathrm{M}_n(\mathbb{R})$ is invariant under the left and the right multiplication by the elements of $\mathrm{O}(n)$. Therefore

$$R_T^1 = \mathrm{O}(n)\Psi(D_T^1)H_1/H_1,$$

where

$$\begin{array}{lcl} \Psi(\boldsymbol{t},\boldsymbol{x}) & = & h(\boldsymbol{t}^{1/2})\boldsymbol{u}(\boldsymbol{x}),\ \forall\ (\boldsymbol{t},\boldsymbol{x}) \in (\mathbb{R}_+)^{r_2}\times\mathcal{U},\ (\text{see } (32)) \\ D_T^1 & = & \{(\boldsymbol{t},\boldsymbol{x}) \in (\mathbb{R}_+)^{r_2}\times\mathcal{U} : \|{}^{\Psi(\boldsymbol{t},\boldsymbol{x})}X_1\| < T\} \end{array} \tag{33}$$

Since $\mathcal{U} \cong \mathbb{R}^{\frac{1}{2}n(n-1)-r_2}$, let ℓ denote the standard Lebesgue measure on $(\mathbb{R}_+)^{r_2}\times\mathcal{U}$. Then by (24) and Proposition 3.2,

$$\eta_1(R_T^1) = (2\pi)^{r_2}2^{-(n-1)}\ell(D_T^1). \tag{34}$$

For the purpose of analysing the limit in Theorem 2.5, we change the order of integration in Proposition 3.1 using Proposition 3.2 and (33)–(34) as follows:

PROPOSITION 3.3. *For all* $f \in \mathrm{C_c}(G/\Gamma_1)$,

$$\begin{aligned} &\tfrac{1}{\eta_1(R_T^1)} \textstyle\int_{R_T^1} \left(\int_{H_1/H_1\cap\Gamma_1} f(gh\Gamma_1)\, d\nu_1(\dot{h})\right) d\eta_1(gH_1) \\ &\qquad = (1/2)\textstyle\int_{\mathrm{O}(n)} dk \cdot \int_{H_1/H_1\cap\Gamma_1} d\nu_1(\dot{h}) \times \\ &\qquad\qquad \times \left(\tfrac{1}{\ell(D_T^1)} \textstyle\int_{(\boldsymbol{t},\boldsymbol{x})\in D_T^1} f(k\Psi(\boldsymbol{t},\boldsymbol{x})h\Gamma_1)\, d\boldsymbol{t}\, d\boldsymbol{x}\right), \end{aligned}$$

where $\dot{h}$ denotes the coset $h(H_1\cap\Gamma_1)$.

3.6 *Description of the set* D_T^1. Our aim for this subsection is to show that D_T^1 is asymptotically the image of a ball of radius T under a 'polynomial like' map (see Propositions 3.4-5).

Coordinates of ${}^{\Psi(\boldsymbol{t},\boldsymbol{x})}X_1$. Take $\boldsymbol{x} = (x_{ij}) \in \mathcal{U}$. If $u(\boldsymbol{x})^{-1} = u(\boldsymbol{y})$, $\boldsymbol{y} = (y_{ij}) \in \mathcal{U}$, then

$$y_{ij} = -x_{ij} + B_{ij}((x_{kl})_{0<l-k<j-i})$$

where $B_{ij} : \prod_{0<l-k<j-i} M_{kl} \to M_{ij}$ is a polynomial map for $i < j-1$, and $B_{ij} \equiv 0$ if $i = j-1$.

If ${}^{u(\boldsymbol{x})}X_1 = u(\boldsymbol{x})X_1u(\boldsymbol{y}) = (\omega_{ij})_{i,j=1}^{r_1+r_2}$, then $w_{ij} = 0$ if $i > j$, and

$$\omega_{ij} = \begin{cases} d_i & \text{if } i = j \ \ (\text{see } (9)) \\ S_{ij}(x_{ij}) + Q_{ij}((x_{kl})_{0<l-k<j-i}) & \text{if } i < j, \end{cases} \tag{35a}$$

where $S_{ij} : M_{ij} \to M_{ij}$ $(i < j)$ is defined as

$$S_{ij}(x) = xd_j - d_i x, \ \forall\ x \in M_{ij}, \tag{35b}$$

and $Q_{ij} : \prod_{0<l-k<j-i} M_{kl} \to M_{ij}$ is a polynomial map for $i < j-1$, and $Q_{ij} \equiv 0$ if $i = j-1$; an explicit formula for Q_{ij} will not be needed later.

Let $\boldsymbol{t} = (t_i) \in (\mathbb{R}_+)^{r_2}$. If we write

$$^{\Psi(\boldsymbol{t},\boldsymbol{x})}X_1 = {}^{h(\boldsymbol{t}^{1/2})}\left({}^{u(\boldsymbol{x})}X_1\right) = (\zeta_{ij})_{i,j=1}^{r_1+r_2}, \tag{35c}$$

then $\zeta_{ij} = 0$ if $i > j$, and

$$\zeta_{ij} = h(t_{i-r_1}^{1/2})\omega_{ij}h(-t_{j-r_1}^{1/2}) \qquad \text{if } i \leq j, \tag{35d}$$

where by convention: $h(t_{i-r_1}^{1/2}) = h(-t_{i-r_1}^{1/2}) = 1$ for $1 \leq i \leq r_1$.

Note that for $i = 1, \ldots, r_2$, (see (9))

$$h(t^{1/2})d_{r_1+i}h(-t^{1/2}) = \begin{pmatrix} a_i - t^{1/2}b_i & -(1+t)b_i \\ b_i & a_i + t^{1/2}b_i \end{pmatrix}. \tag{35e}$$

Therefore by (35c),

$$\| {}^{\Psi(\boldsymbol{t},\boldsymbol{x})}X_1\|^2 = \|X_1\|^2 + \sum_{i=1}^{r_2} b_i^2(t_i^2 + 4t_i) + \sum_{i<j} |\zeta_{ij}|^2, \tag{35f}$$

where the sum $\sum_{i=1}^{r_1+r_2} |\zeta_{ii}|^2$ is evaluated using (35d), (35a), and (35e).

Expressing D_T^1 as an image of a ball. Now, in view of (33), we want to find a function

$$\tilde{\delta} : (\mathbb{R}_+)^{r_2} \times \mathcal{U} \to (\mathbb{R}_+)^{r_2} \times \mathcal{U}$$

such that

$$\tilde{\delta}(B^+_{(T^2 - \|X_1\|^2)^{1/2}}) = D_T^1, \tag{36}$$

where

$$B_T^+ := \{(\boldsymbol{s}, \boldsymbol{z}) \in (\mathbb{R}_+)^{r_2} \times \mathcal{U} : \|\boldsymbol{s}\|^2 + \|\boldsymbol{z}\|^2 < T^2\}.$$

For $(\boldsymbol{s}, \boldsymbol{z}) \in (\mathbb{R}_+)^{r_2} \times \mathcal{U}$, if we write $\tilde{\delta}(\boldsymbol{s}, \boldsymbol{z}) = (\boldsymbol{t}, \boldsymbol{x})$, where $\boldsymbol{t} = (t_i) \in (\mathbb{R}_+)^{r_2}$ and $\boldsymbol{x} = (x_{ij}) \in \mathcal{U}$, then in view of (33), (35f), and (36), we want that the following equations are satisfied:

$$s_i = [b_i^2(t_i^2 + 4t_i)]^{1/2}, \qquad (1 \leq i \leq r_2) \tag{37}$$

$$z_{ij} = \zeta_{ij}, \qquad (1 \leq i < j \leq r_1 + r_2), \tag{38}$$

where ζ_{ij} is a function of x_{ij} and $\{x_{kl} : 0 < l - k < j - i\}$ (see (35a) and (35d)).

By first solving the equations in (37), we get

$$t_i = (b_i^{-2}s_i^2 + 4)^{1/2} - 2.$$

Then we proceed to solve the equations in (38) inductively, in the following order: for any (i,j) they are solved for all $\{x_{kl} : 0 < l-k < j-i\}$ before solving it for the x_{ij}. Therefore it is enough to express x_{ij} in terms of $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{t}$, and $\{x_{kl} : 0 < l-k < j-i\}$. We get

$$\begin{aligned} x_{ij} &= x_{ij}(\boldsymbol{t}, \{x_{kl} : 0 < l-k < j-i\}) \\ &= {S_{ij}}^{-1}\left(h(-t_{i-r_1}^{1/2})z_{ij}h(t_{j-r_1}^{1/2}) - Q_{ij}((x_{kl})_{0<l-k<j-i}),\right). \end{aligned} \tag{39}$$

where S_{ij} and Q_{ij} are as in (35a) and (35b).

'Polynomial like' approximation for $\tilde{\delta}$. We define $\boldsymbol{t}' := (t'_i) \in (\mathbb{R}_+)^{r_2}$,

$$t'_i = |b_i|^{-1}s_i, \qquad 1 \le i \le r_2.$$

Next we define $\boldsymbol{x}' := (x'_{ij}) \in \mathcal{U}$ inductively, using the formula in (39), as follows:

$$x'_{ij} = x_{ij}(\boldsymbol{t}', \{x'_{kl} : 0 < k-l < j-i\}), \qquad (1 \le i < j \le r_1 + r_2).$$

Then we define

$$\delta(\boldsymbol{s},\boldsymbol{z}) = (\boldsymbol{t}', \boldsymbol{x}').$$

It is straightforward to verify that

$$0 \le t'_i - t_i < 2, \qquad 1 \le i \le r_2.$$

Therefore

$$\delta(B^+_{T-2}) \subset \tilde{\delta}(B^+_T) \subset \delta(B^+_T),\ \forall\ T > 0. \tag{40}$$

Also note that if $T > \|X_1\|$, then

$$T - \|X_1\|^2\, T^{-1} < (T^2 - \|X_1\|^2)^{1/2}\ < T.$$

Therefore, since (36) and (40) hold, we get the following:

Proposition 3.4. *For* $T > \|X_1\| + 2$,

$$\delta(B^+_{T-2-\|X_1\|^2 T^{-1}}) \subset D^1_T \subset \delta(B^+_T).$$

Proposition 3.5. *The map* $\Theta : \mathbb{R}^{\frac{1}{2}n(n-1)} \to G$ *defined by*

$$\Theta(\boldsymbol{s},\boldsymbol{z}) := \Psi(\delta((s_1^2,\ldots,s_{r_2}^2),\boldsymbol{z})), \qquad \forall\ (\boldsymbol{s},\boldsymbol{z}) \in \mathbb{R}^{r_2} \times \mathcal{U} = \mathbb{R}^{\frac{1}{2}n(n-1)},$$

is a polynomial map; that is, each coordinate function of Θ *is a polynomial in* $\frac{1}{2}n(n-1)$*-variables.*

3.7 *Jacobian of* δ. Let the notation be as in the definition of δ. The Jacobian of δ at $(\boldsymbol{s},\boldsymbol{z})$ is given by:

$$\mathrm{Jac}(\delta)(\boldsymbol{s},\boldsymbol{z}) = |\partial(\boldsymbol{t}',\boldsymbol{x}')/\partial(\boldsymbol{s},\boldsymbol{z})| = \prod_{i=1}^{r_2} |\partial t'_i/\partial s_i| \cdot \prod_{i<j} |\partial x'_{ij}/\partial z_{ij}| \tag{41}$$

$$= \prod_{i=1}^{r_2} |b_i|^{-1} \cdot \prod_{i<j} |\det(S_{ij})^{-1}|, \tag{42}$$

where (41) holds because $\partial t_i'/\partial z_{kl} = 0$ for all i, k, l, and $\partial x_{kl}'/\partial z_{ij} = 0$ for all $0 < l - k < j - i$, and (42) holds because $\det h(t) = 1$ for all t. In particular, $\mathrm{Jac}(\delta)$ is a constant function.

Computation of $\det(S_{ij})$. By (16)

$$M_{ij} = \mathrm{Hom}(\mathbb{R}^{\nu_i}, \mathbb{R}^{\nu_j}) \cong \mathbb{R}^{\nu_i} \otimes (\mathbb{R}^{\nu_j})^*, \qquad (1 \le i < j \le r_1 + r_2),$$

where $\nu_k = 1$ if $1 \le k \le r_1$, and $\nu_k = 2$ if $r_1 < k \le r_2$. Under this canonical isomorphism, S_{ij} corresponds to

$$(1 \otimes d_j^*) - (d_i \otimes 1), \qquad \text{(see (35b))}$$

whose eigenvalues are distinct, and by (8) they are

$$\sigma_{j'}(\alpha) - \sigma_{i'}(\alpha), \qquad i' \in \hat{i},\ j' \in \hat{j},$$

where $\hat{k} = \{k\}$ if $\nu_k = 1$, and $\hat{k} = \{k, r_2 + k\}$ if $\nu_k = 2$. Therefore by (42)

$$\mathrm{Jac}(\delta) = 2^{r_2} \prod_{1 \le i < j \le n} |\sigma_i(\alpha) - \sigma_j(\alpha)|^{-1} = 2^{r_2}/|D_{\mathbb{Q}(\alpha)/\mathbb{Q}}|^{1/2}, \tag{43}$$

where $D_{\mathbb{Q}(\alpha)/\mathbb{Q}}$ denotes the discriminant of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

3.8 *Volume of* R_T. We note that

$$\ell(B_T^+) = 2^{-r_2}\,\mathrm{Vol}(B^{n(n-1)/2})T^{n(n-1)/2}, \tag{44}$$

where $\mathrm{Vol}(B^m)$ denotes the volume of a unit ball in $\mathbb{R}^m$. Also note that for any $m \in \mathbb{N}$ and $a, b > 0$, if $T > \max\{a, b\}$ then

$$((T + a)^m - (T - b)^m)/T^m < m(a + b)T^{-1}.$$

Therefore by (12), (34), Proposition 3.4, and since $\mathrm{Jac}(\delta)$ is a constant,

$$\begin{aligned} \lim_{T\to\infty} \eta(R_T)/\ell(B_T^+) &= \lim_{T\to\infty} \eta_1(R_T^1)/\ell(B_T^+) \\ &= (2\pi)^{r_2} 2^{-(n-1)} \lim_{T\to\infty} \ell(D_T^1)/\ell(B_T^+) \\ &= (2\pi)^{r_2} 2^{-(n-1)}\,\mathrm{Jac}(\delta). \end{aligned}$$

Now by (43) and (44),

$$c_\eta := \lim_{T\to\infty} \eta(R_T)/T^{n(n-1)/2} = \frac{(2\pi)^{r_2}\,\mathrm{Vol}(B^{n(n-1)/2})}{2^{n-1}|D_{\mathbb{Q}(\alpha)/\mathbb{Q}}|^{1/2}}. \tag{45}$$

4. Equidistribution of Trajectories

In view of Propositions 3.1 and 3.3, by Propositions 3.4-5 and since $\mathrm{Jac}(\delta)$ is a constant we have the following: for any $f_1 \in \mathrm{C_c}(G/\Gamma_1)$, and any $x_1 \in G/\Gamma_1$,

$$\begin{aligned} &\lim_{T\to\infty} \tfrac{1}{\ell(D_T^1)} \textstyle\int_{(\boldsymbol{t},\boldsymbol{x})\in D_T^1} f_1(\Psi(\boldsymbol{t},\boldsymbol{x})x_1)\, d\boldsymbol{t}\, d\boldsymbol{x} \\ &\quad = \lim_{T\to\infty} \tfrac{1}{\ell(B_T^+)} \textstyle\int_{(\boldsymbol{s},\boldsymbol{z})\in B_T^+} f_1(\Theta(\boldsymbol{s}^{1/2},\boldsymbol{z})x_1)\, d\boldsymbol{s}\, d\boldsymbol{z}, \end{aligned} \tag{46}$$

Since $\Theta(\mathbb{R}^{r_2} \times \mathcal{U}) \supset U$, in order to apply Theorem 1.3 we show the following:

Lemma 4.1. *For $x \in G/\Gamma_1$, if H_1x is compact then $\overline{Ux} = G/\Gamma_1$.*

Proof. Choose $\boldsymbol{c} \in C$, such that $c_1 > \ldots > c_{r_1+r_2} > 0$ (see (18)). Then $U = \{u \in G : \boldsymbol{c}^{-m}u\boldsymbol{c}^m \to 1 \text{ as } m \to \infty\}$, which is the expanding horospherical subgroup of G^0 associated to $\boldsymbol{c}$. Therefore by Proposition 1.5 of Dani and Raghavan (1980)

$$\overline{\cup_{n=1}^{\infty} \boldsymbol{c}^m U y} = G^0\Gamma_1/\Gamma_1 = G/\Gamma_1, \qquad \forall\, y \in G/\Gamma_1. \tag{47}$$

By (20)-(21), $C \subset H_1$ and $H_1 \subset N_G(U)$. Let F be a compact subset of H_1 such that $Fx = H_1x$. Then by (47)

$$G/\Gamma_1 = \overline{CUx} \subset \overline{H_1Ux} = \overline{UH_1x} = \overline{UFx} = F\overline{Ux}. \tag{48}$$

By Moore's ergodicity theorem (Moore, 1966), U acts ergodically on G/Γ_1. Hence there exists $x_1 \in G/\Gamma_1$ such that $\overline{Ux_1} = G/\Gamma_1$. By (48) there exist $h \in F$ and $x_2 \in \overline{Ux}$ such that $x_1 = hx_2$. Therefore, since $h \in N_G(U)$,

$$G/\Gamma_1 = \overline{Ux_1} = \overline{Uhx_2} = h\overline{Ux_2} \subset h\overline{Ux}.$$

Hence $\overline{Ux} = G/\Gamma_1$. □

Proposition 4.2. *For all $f_1 \in \mathrm{C_c}(G/\Gamma_1)$, $k \in K$ and $h \in H_1$:*

$$\lim_{T\to\infty} \frac{1}{\ell(B_T^+)} \int_{(\boldsymbol{s},\boldsymbol{z})\in B_T^+} f_1(k\Theta(\boldsymbol{s}^{1/2},\boldsymbol{z})h\Gamma_1)\, d\boldsymbol{s}\, d\boldsymbol{z} = \frac{1}{\mu_1(G/\Gamma_1)} \int_{G/\Gamma_1} f_1\, d\mu_1,$$

where Θ is as in Proposition 3.5.

Proof. Note that $G/\Gamma_1 = G^0/(\Gamma_1 \cap G^0)$ and $G^0 = \mathrm{SL}_n(\mathbb{R})$. We apply Theorem 1.3 for $\Gamma_1 \cap G^0$ in place of Γ, $x = h\Gamma_1$ and the function $f_2 \in \mathrm{C_c}(G/\Gamma_1)$, where $f_2(g\Gamma_1) := f_1(kg\Gamma_1), \forall\, g \in G$. Since $H_1\Gamma_1/\Gamma_1 = \bar{\psi}(H\Gamma/\Gamma)$, by Proposition 2.3, H_1x is compact. Therefore by Lemma 4.1, U_1x is dense in G/Γ_1. Since $\Theta(\mathbb{R}^{r_2} \times \mathcal{U}) \supset U$, the conclusion of Theorem 1.3 holds, and hence the proposition follows. □

Proof of Theorem 1.1. By a series of reductions in Section 3, we showed that it is enough to prove Theorem 2.5. Now it is straightforward to deduce this result from Propositions 3.1 and 3.3, Equation (46), Proposition 4.2, Lebesgue's dominated convergence theorem, Equation (13), and the fact that $\mu_1 = \bar{\psi}_*(\mu)$. □

5. Computation of C_P

The rest of the article is devoted to proving the following:

Theorem 5.1. *Let the notation be as in Theorem 1.1. Fix any root α of P. Then*

$$C_P = \sum_{\mathcal{O}\supset\mathbb{Z}[\alpha]} \kappa(\mathcal{O}) \cdot \frac{\mathrm{Vol}(B^{n(n-1)/2})}{\mathrm{Vol}(\mathcal{SM}_n)},$$

where the sum is over all orders $\mathcal{O}$ of the number field $K = \mathbb{Q}(\alpha)$ containing $\mathbb{Z}[\alpha]$,

$$\kappa(\mathcal{O}) = \frac{2^{r_1}(2\pi)^{r_2} h_{\mathcal{O}} R_{\mathcal{O}}}{w_{\mathcal{O}}|D_{K/\mathbb{Q}}|^{1/2}},$$

here

$$\begin{aligned} r_1 &= \textit{Number of real places of } K, \\ r_2 &= \textit{Number of complex places of } K, \\ h_{\mathcal{O}} &= \textit{Number of modules classes with order } \mathcal{O} \\ R_{\mathcal{O}} &= \textit{Regulator of } \mathcal{O}^\times, \ \textit{(see (56))} \\ w_{\mathcal{O}} &= \textit{Order of the group of roots of unity in } \mathcal{O}^\times, \\ D_{K/Q} &= \textit{Discriminant of } K, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Vol}(B^m) &= \pi^{m/2}/\Gamma(1+m/2) \\ \mathrm{Vol}(\mathcal{SM}_n) &= \prod_{s=2}^{n} \pi^{-s/2}\Gamma(s/2)\zeta(s) \end{aligned}$$

The number theoretic terms involved in the above statement will be explained in the course of the proof of this theorem; see Koch (1997, pp. 10–17) for their definitions. See also Remark 1.1.

For computing C_P, by Proposition 2.1 and Theorem 2.2, we need to count the number of distinct Γ-orbits in $V_P(\mathbb{Z})$, and compute C_{X_0} for each $X_0 \in V_P(\mathbb{Z})$. To compute C_{X_0}, by (1), (3), (4), and Theorem 2.4, we need to compute c_η, $\nu(H/H(\mathbb{Z}))$, and $\mu(G/\Gamma$. Note that (45) already gives a formula for c_η.

5.1 Γ-*Orbits in* $V_P(\mathbb{Z})$. We will describe a result on a correspondence between the classes of matrices and classes of ideals; here two matrices are said to be in the *same equivalence class* if they are in the same Γ-orbit.

Fix any root α of P. Any (nonzero) ideal I of $\mathbb{Z}[\alpha]$ is a free $\mathbb{Z}$-module of rank n. We say that ideals I and J of $\mathbb{Z}[\alpha]$ are *equivalent* if and only if $aI = bJ$ for some nonzero $a, b \in \mathbb{Z}[\alpha]$. Let $[I]$ denote the class of ideals in $\mathbb{Z}[\alpha]$ equivalent to I.

For any $X \in V_P(\mathbb{Z})$, α is an eigenvalue of X, and there exists a nonzero eigenvector $\boldsymbol{\omega} := {}^{\mathbf{t}}(\omega_1, \ldots, \omega_n) \in \mathbb{Q}(\alpha)^n$ such that

$$X\boldsymbol{\omega} = \alpha\boldsymbol{\omega} \tag{49}$$

Replacing $\boldsymbol{\omega}$ by some integral multiple, we may assume that $\omega_i \in \mathbb{Z}[\alpha]$ for $1 \le i \le n$. Put $I_X = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$. Then by (49), $\alpha I_X \subset I_X$. Hence I_X is an ideal of $\mathbb{Z}[\alpha]$. The ideal class $[I_X]$ depends only on X, and not on the choice of the eigenvector $\boldsymbol{\omega}$.

Let $\gamma \in \Gamma = \mathrm{GL}_n(\mathbb{Z})$. Put $Y = {}^{\gamma}X$, and $\boldsymbol{\omega}' = \gamma\boldsymbol{\omega} = {}^{\mathbf{t}}(\omega_1', \ldots, \omega_n')$. Then $Y\boldsymbol{\omega}' = \alpha\boldsymbol{\omega}'$, and $\omega_i' \in I_X$ for all i. If we put $I_Y = \mathbb{Z}\omega_1' + \cdots + \mathbb{Z}\omega_n'$, then $I_Y \subset I_X$. Since $\gamma^{-1} \in \mathrm{GL}_n(\mathbb{Z})$ and $\boldsymbol{\omega} = \gamma^{-1}\boldsymbol{\omega}'$, we have $I_X \subset I_Y$, and hence $I_X = I_Y$.

Thus the ideal class $[I_X]$ depends only on the Γ-orbit ${}^{\Gamma}X$, and not on the choice of its representative X. Now we state a result due to Latimer and MacDuffee (1933); see Taussky (1949) for a simpler proof.

THEOREM 5.2. *The assignment* ${}^{\Gamma}X \mapsto [I_X]$ *is a one-to-one correspondence between the collection of* Γ*-orbits in* $V_P(\mathbb{Z})$ *and the collection of equivalence classes of ideals in* $\mathbb{Z}[\alpha]$.

Orders in $\mathbb{Q}(\alpha)$. A subring $\mathcal{O}$ of the number field $K = \mathbb{Q}(\alpha)$ is called an *order* in K, if its quotient field is K, $\mathcal{O} \cap \mathbb{Q} = \mathbb{Z}$, and its additive group is finitely generated.

A free $\mathbb{Z}$-submodule of K (additive) of rank $n = [K : \mathbb{Q}]$ is called a *lattice* in K; for example, any (nonzero) ideal of $\mathbb{Z}[\alpha]$ is a lattice in K. Two lattices $\mathfrak{M}$ and $\mathfrak{M}'$ in K are said to be *equivalent*, if $a\mathfrak{M} = b\mathfrak{M}'$ for some nonzero $a, b \in \mathbb{Q}(\alpha)$. Let $\bar{\mathfrak{M}}$ denote the class of lattices equivalent to $\mathfrak{M}$. For ideals I and J of $\mathbb{Z}[\alpha]$, we have $[I] = [J] \Leftrightarrow \bar{I} = \bar{J}$.

For a lattice $\mathfrak{M}$ in K,

$$\mathcal{O}(\mathfrak{M}) := \{\beta \in K : \beta\mathfrak{M} \subset \mathfrak{M}\} \tag{50}$$

is an order in K, it is called the order of $\mathfrak{M}$, and it depends only on the class $\bar{\mathfrak{M}}$.

Let $\mathcal{O}$ be an order in K. Then by the class number theorem (Koch (1997), Theorem 1.9), there are only finitely many classes of lattices in K with order $\mathcal{O}$. This number is called the *class number of* $\mathcal{O}$ and denoted by $h_{\mathcal{O}}$.

The ring $\mathcal{O}_K$ of algebraic integers in K is an order. Any order $\mathcal{O}$ in K is contained in $\mathcal{O}_K$, and $[\mathcal{O}_K : \mathcal{O}] < \infty$. Also $\mathbb{Z}[\alpha]$ is an order in K, and hence there are only finitely orders $\mathcal{O}$ in K with $\mathcal{O} \supset \mathbb{Z}[\alpha]$.

PROPOSITION 5.3. *The* Γ*-orbits in* $V_P(\mathbb{Z})$ *are in one-to-one correspondence with the classes of lattices in* K *whose orders contain* $\mathbb{Z}[\alpha]$.

In particular, each order $\mathcal{O}$ *containing* $\mathbb{Z}[\alpha]$ *is associated to* $h_{\mathcal{O}}$ *distinct* Γ*-orbits in* $V_P(\mathbb{Z})$*, and the number of distinct* Γ*-orbits in* $V_P(\mathbb{Z})$ *equals* $\sum_{\mathcal{O} \supset \mathbb{Z}[\alpha]} h_{\mathcal{O}}$.

PROOF. In view of Theorem 5.2, to any Γ-orbit ${}^{\Gamma}X$ in $V_P(\mathbb{Z})$, we associate the lattice class $\bar{I}_X$ of the ideal I_X in $\mathbb{Z}[\alpha]$. Then $\mathcal{O}(I_X) \supset \mathbb{Z}[\alpha]$.

Conversely, let $\mathfrak{M}$ be a lattice in K such that $\mathcal{O}(\mathfrak{M}) \supset \mathbb{Z}[\alpha]$. Then there exists a nonzero integer a such that $I := a\mathfrak{M}$ is an ideal of $\mathbb{Z}[\alpha]$. By Theorem 5.2, there exists $X \in V_P(\mathbb{Z})$, such that $[I] = [I_X]$. Therefore $\bar{\mathfrak{M}} = \bar{I}_X$, and hence $\bar{\mathfrak{M}}$ is associated to a unique orbit ${}^{\Gamma}X$, and $\mathcal{O}(\mathfrak{M}) = \mathcal{O}(I_X)$. This proves the one-to-one correspondence.

Now the second statement follows from the class number theorem for orders (Koch (1997), Theorem 1.9). $\square$

5.2 *Number theoretic 'realizations' of H and $H(\mathbb{Z})$.* Fix $X_0 \in V_P(\mathbb{Z})$ and let the notation be as before. Put

$$Z_{X_0} = \{Y \in \mathrm{M}_n(\mathbb{R}) : YX_0 = X_0Y\}.$$

Since $X_0 \in \mathrm{M}_n(\mathbb{Q})$, we have that Z_{X_0} is the real vector space defined over $\mathbb{Q}$; that is, $Z_{X_0}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} = Z_{X_0}$, where $Z_{X_0}(\mathbb{Q}) := Z_{X_0} \cap \mathrm{M}_n(\mathbb{Q})$.

Let $\boldsymbol{\omega} = {}^{\mathbf{t}}(\omega_1, \dots, \omega_n) \in \mathbb{Z}[\alpha]^n$, $\boldsymbol{\omega} \neq 0$, be such that $X_0\boldsymbol{\omega} = \alpha\boldsymbol{\omega}$. Since all eigenvalues of X_0 are distinct, there exists an $\mathbb{R}$-algebra homomorphism $\lambda : Z_{X_0} \to \mathbb{C}$ given by $Y \mapsto \lambda_Y$, such that $Y\boldsymbol{\omega} = \lambda_Y\boldsymbol{\omega}$. Now if $Y \in Z_{X_0}(\mathbb{Q})$ then $\lambda_Y \in \mathbb{Q}(\alpha)$.

Let $I_{X_0} = \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_n$. Then I_{X_0} is an ideal of $\mathbb{Z}[\alpha]$, and hence $I_{X_0} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}(\alpha)$. Therefore $\{\omega_1, \dots, \omega_n\}$ are linearly independent over $\mathbb{Q}$. Hence if $Y \in Z_{X_0}(\mathbb{Q})$ and $Y\boldsymbol{\omega} = 0$, then $Y = 0$. Thus

$$\ker \lambda \cap Z_{X_0}(\mathbb{Q}) = 0.$$

Let Y_β denote the matrix of the multiplication by $\beta \in \mathbb{Q}(\alpha)$ on the $\mathbb{Q}$-vector space $I_{X_0} \otimes_{\mathbb{Z}} \mathbb{Q}$, with respect to the basis $\{\omega_1, \dots, \omega_n\}$. The map $\beta \mapsto Y_\beta$ is a $\mathbb{Q}$-algebra homomorphism of $\mathbb{Q}(\alpha)$ into $\mathrm{M}_n(\mathbb{Q})$. Since $Y_\alpha = X_0$, we have that $Y_\beta \in Z_{X_0}(\mathbb{Q})$. Also $\lambda_{Y_\beta} = \beta$. Hence $\lambda : Z_{X_0}(\mathbb{Q}) \to \mathbb{Q}(\alpha)$ is an isomorphism between the $\mathbb{Q}$-algebras. In particular,

$$Z_{X_0}(\mathbb{Q}) = \mathbb{Q}[X_0] \text{ and } Z_{X_0} = \mathbb{R}[X_0].$$

Note that for $Y \in Z_{X_0}(\mathbb{Q})$, $\lambda_Y I_{X_0} \subset I_{X_0} \Leftrightarrow Y \in \mathrm{M}_n(\mathbb{Z})$. Therefore

$$Z_{X_0}(\mathbb{Z}) := Z_{X_0} \cap \mathrm{M}_n(\mathbb{Z}) = \{Y \in Z_{X_0}(\mathbb{Q}) : \lambda_Y \in \mathcal{O}(X_0)\}, \tag{51}$$

where $\mathcal{O}(X_0)$ denotes the order of I_{X_0} (see (50)).

Equality of Determinant and Norm. Recall the Notation 3.1. Define $\sigma_i(\boldsymbol{\omega}) := {}^{\mathbf{t}}(\sigma_i(\omega_1), \dots, \sigma_i(\omega_n))$. Then $X_0\sigma_i(\boldsymbol{\omega}) = \sigma_i(\alpha)\sigma_i(\boldsymbol{\omega})$. Let

$$g_1 = (\sigma_1(\boldsymbol{\omega}), \dots, \sigma_n(\boldsymbol{\omega})) \in \mathrm{M}_n(\mathbb{C}).$$

Then

$${g_1}^{-1}X_0g_1 = \mathrm{diag}(\sigma_1(\alpha), \dots, \sigma_n(\alpha)),$$

and $\sigma_i(\alpha) \neq \sigma_j(\alpha)$, if $i \neq j$. Therefore ${g_1}^{-1}Z_{X_0}g_1$ consists of diagonal matrices. We define functions D_i on Z_{X_0} by

$${g_1}^{-1}Yg_1 = \mathrm{diag}(D_1(Y), \dots, D_n(Y)), \qquad \forall\, Y \in Z_{X_0}.$$

Since $Z_{X_0} = \mathbb{R}[X_0]$ and the D_i's are $\mathbb{R}$-algebra homomorphisms, for all $Y \in Z_{X_0}$, we have $D_i(Y) \subset \mathbb{R}$ for $1 \leq i \leq r_1$, and by (8),

$$D_{r_1+r_2+i}(Y) = \bar{D}_{r_1+i}(Y), \qquad (1 \leq i \leq r_2).$$

Therefore

$$\det(Y) = \prod_{i=1}^{n} |D_i(Y)| = \prod_{i=1}^{r_1+r_2} |D_i(Y)|^{\nu_i}, \qquad \forall\, Y \in Z_{X_0}, \tag{52}$$

where $\nu_i = 1$ if $i \leq r_1$, and $\nu_i = 2$ if $i > r_1$. It is straightforward to verify that $D_i(Y) = \sigma_i(\lambda_Y)$, $\forall\, Y \in Z_{X_0}(\mathbb{Q})$. We have proved the following

LEMMA. $\det(Y) = N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\lambda_Y)$ *for all* $Y \in Z_{X_0}(\mathbb{Q})$.

Note that $H = \{Y \in Z_{X_0} : |\det(Y)| = 1\}$. Therefore by (51),

$$\begin{aligned} H(\mathbb{Z}) &= H \cap Z_{X_0}(\mathbb{Z}) \\ &= \{Y \in Z_{X_0}(\mathbb{Q}) : |N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\lambda_Y)| = 1,\ \lambda_Y \in \mathcal{O}(X_0)\} \\ &= \{Y \in Z_{X_0}(\mathbb{Q}) : \lambda_Y \in \mathcal{O}(X_0)^\times\} \\ &\cong \mathcal{O}(X_0)^\times; \end{aligned} \tag{53}$$

here $\mathcal{O}(X_0)^\times$ denotes the multiplicative group of the order $\mathcal{O}(X_0)$ which is the same as the multiplicative group of unit norm elements in $\mathcal{O}(X_0)^\times$.

Dirichlet's Unit theorem and compactness of $H/H(\mathbb{Z})$.

THEOREM 5.4. $H/H(\mathbb{Z})$ *is compact.*

PROOF. Define $l : H \to \mathbb{R}^{r_1+r_2}$ as (see (52))

$$l(h) = (\nu_1 \log|D_1(h)|, \ldots, \nu_{r_1+r_2} \log|D_{r_1+r_2}(h)|),\ \forall\, h \in H.$$

Let

$$E = \{(x_1, \ldots, x_{r_1+r_2}) \in \mathbb{R}^{r_1+r_2} : x_1 + \cdots + x_{r_1+r_2} = 0\}.$$

Since $Z_{X_0} = \mathbb{R}[X_0]$ is an $\mathbb{R}$-algebra and D_i's are $\mathbb{R}$-algebra homomorphisms, by (52) and (53), $l : H \to E$ is a surjective homomorphism.

By (20), $H_1 = \Sigma \cdot K_1 \cdot C$ is a direct product decomposition; let $p : H_1 \to C$ denote the associated projection. We define $l_1 : C \to E$ by

$$l_1(\boldsymbol{c}) = (\log c_1, \ldots, \log c_{r_1+r_2}), \qquad \text{(see (18))}$$

and extend it to H_1 by $l_1(h) = l_1(p(h))$, $\forall\, h \in H_1$.

We note that $l_1(g_0 h {g_0}^{-1}) = l(h)$ for all $h \in H$. Therefore

$$\ker\, l = {g_0}^{-1}(\ker\, l_1) g_0 = {g_0}^{-1} \Sigma K_1 g_0.$$

Hence $\ker(l)$ is compact.

We define $\ell : \mathcal{O}(X_0)^\times \to E$, by

$$\ell(\lambda) = (\nu_1 \log|\sigma_1(\lambda)|, \ldots, \nu_{r_1+r_2} \log|\sigma_{r_1+r_2}(\lambda)|),\ \forall\, \lambda \in \mathcal{O}(X_0)^\times. \tag{54}$$

Clearly, $l(Y) = \ell(\lambda_Y)$ for all $Y \in H(\mathbb{Z})$. By Dirichlet unit theorem (Koch (1997), Theorem 1.13), $\ell(\mathcal{O}(X_0)^\times)$ is a lattice in E. Therefore $l(H)/l(H(\mathbb{Z}))$ is compact. Since $\ker(l)$ is compact, this completes the proof. □

Computation of $\nu(H/H(\mathbb{Z}))$. Let $\mathrm{pr} : R^{r_1+R_2} \to R^{r_1+r_2-1}$ be the projection on the first $r_1 + r_2 - 1$ coordinate space. We choose a measure m on $E \subset \mathbb{R}^{r_1+r_2}$ such that its image under pr is the standard Lebesgue measure on $\mathbb{R}^{r_1+r_2-1}$. Let $\bar{m}$ denote the associated measure on $E/\ell(\mathcal{O}(X_0)^\times)$. We note that $l_1 : C \to E$ preserves the choices of the Haar integrals $d\boldsymbol{c}$ and dm.

Let $\tilde{K}_1 = \Sigma K_1$. In view of (19) and (27), let $\tilde{\theta}$ be the Haar measure on $\tilde{K}_1$ such that

$$\tilde{\theta}(\tilde{K}) = \#(\Sigma)\theta(K_1) = 2^{r_1}. \tag{55}$$

Define $q : \tilde{K}_1 \backslash H_1 \to C$ as $q(\tilde{K}_1 h) = p(h)$ for all $h \in H_1$. Then by (29), q is a measure preserving homeomorphism.

Therefore $l_1 \circ q : \tilde{K}_1 \backslash H_1 \to E$ is a group isomorphism and preserves the chosen Haar measures on both sides. Note that $H \cap \Gamma = H(\mathbb{Z})$, and

$$l_1(H_1 \cap \Gamma_1) = l(H \cap \Gamma) = l(H(\mathbb{Z})) = \ell(\mathcal{O}(X_0)^\times).$$

Therefore we have an isomorphism,

$$\tilde{K}_1 \backslash H_1/(H_1 \cap \Gamma_1) \cong E/\ell(\mathcal{O}(X_0)^\times)$$

preserving the invariant measures on both sides. Now by Theorem B.1 (stated and proved in Appendix B),

$$\nu_1(H_1/(H_1 \cap \Gamma_1)) = \frac{\tilde{\theta}(\tilde{K}_1)}{\#(\tilde{K}_1 \cap (H_1 \cap \Gamma_1))} \cdot \bar{m}(E/\ell(\mathcal{O}(X_0)^\times)). \tag{56}$$

By the Dirichlet's unit theorem, let $\{\epsilon_1, \ldots, \epsilon_{r_1+r_2-1}\}$ be a set of generators of $\mathcal{O}(X_0)^\times$ modulo the group of roots of unity. Then

$$\ell(\mathcal{O}(X_0)^\times) = \bigoplus_{j=1}^{r_1+r_2-1} \mathbb{Z}\,\ell(\epsilon_j).$$

Hence, by (54),

$$\bar{m}(E/\ell(\mathcal{O}(X_0)^\times)) = \left|\det\left((\nu_i \log|\sigma_i(\epsilon_j)|)_{i,j=1}^{r_1+r_2-1}\right)\right| =: R_{\mathcal{O}(X_0)}, \tag{57a}$$

which is called the *regulator* of the the order $\mathcal{O}(X_0)$ (see Koch (1997), Sect. 1.3).

We note that ${g_0}^{-1}(\tilde{K}_1 \cap (H_1 \cap \Gamma_1))g_0 = \ker(l) \cap H(\mathbb{Z}) \cong \ker(\ell)$, which is the group of roots of unity in $\mathcal{O}(X_0)$, and its order is denoted by $w_{\mathcal{O}(X_0)}$. Therefore,

$$\#(\tilde{K}_1 \cap (H_1 \cap \Gamma_1)) = w_{\mathcal{O}(X_0)}. \tag{57b}$$

Now from (55)–(57b) we obtain the following:

THEOREM 5.5. *Let $\mathcal{O}(X_0)$ be the order of the ideal I_{X_0} of $\mathbb{Z}[\alpha]$ which is associated to X_0 as in Theorem 5.2. Then*

$$\nu(H/H \cap \Gamma) = \nu_1(H_1/H_1 \cap \Gamma_1) = 2^{r_1} R_{\mathcal{O}(X_0)}/w_{\mathcal{O}(X_0)}.$$

5.3 *Volume of $G/\mathrm{GL}_n(\mathbb{Z})$.* To use the volume computation of $G/\mathrm{GL}_n(\mathbb{Z})$ due to Siegel, one needs to compare the Haar measure on G chosen for his computation with the one chosen in (25). Instead of doing that, we will find it more convenient to use similar volume computations as in Terras (1988, Section 4.4.4), which also uses Siegel's formula.

The space $\mathcal{P}_n$ of positive $n \times n$ matrices. Let $\mathcal{P}_n$ be the space of $n \times n$ real positive symmetric matrices. Then $\mathrm{GL}_n(\mathbb{R})$ acts transitively on $\mathcal{P}_n$ by

$$(g, Y) \mapsto {}^{\mathbf{t}}gYg,\ \forall\ (g, Y) \in \mathrm{GL}_n(\mathbb{R}) \times \mathcal{P}_n.$$

We consider a $\mathrm{GL}_n(\mathbb{R})$-invariant measure μ_n on $\mathcal{P}_n$ defined as follows: If we write $Y \in \mathcal{P}_n$ as $Y = (y_{ij})$, $y_{ij} = y_{ji}$, $y_{ij} \in \mathbb{R}$, then

$$d\mu_n(Y) = |\det(Y)|^{-(n+1)/2} \prod_{i \le j} dy_{ij}.$$

Let $\mathcal{SP}_n = \{Y \in \mathcal{P}_n : \det(Y) = 1\}$. Then G acts transitively on $\mathcal{SP}_n$, and preserves the invariant integral dW on $\mathcal{SP}_n$ which is defined as follows: If we write $Y \in \mathcal{P}_n$ as $Y = t^{1/n}W$, $(t > 0,\ W \in \mathcal{SP}_n)$, then

$$d\mu_n(Y) = (dt/t)dW. \tag{58}$$

Volume of Minkowski fundamental domain. Let $\mathcal{SM}_n$ denote the Minkowski fundamental domain for the action of $\mathrm{GL}_n(\mathbb{Z})$ on $\mathcal{SP}_n$. We have chosen $d\mu_n$ and dW such that by Terras, (1988, Section 4.4.4, Theorem 4, p. 168), we have the following:

$$\mathrm{Vol}(\mathcal{SM}_n) := \int_{\mathcal{SM}_n} 1\ dW = \prod_{k=2}^{n} \pi^{-k/2}\Gamma(k/2)\zeta(k). \tag{59}$$

Comparing volume forms. Now we want to compare the volume forms $d\boldsymbol{n}\, d\boldsymbol{a}$ on $\mathrm{O}(n)\backslash G$ and dW on $\mathcal{SP}_n$ with respect to the map $\mathrm{O}(n)g \mapsto {}^{\mathbf{t}}gg$.

Put $D = \{\boldsymbol{b} = \mathrm{diag}(b_1, \ldots, b_n) : b_i > 0\}$. Choose the Haar integral $d\boldsymbol{b} = \prod_{i=1}^{n} db_i/b_i$ on D. Then

$$d\boldsymbol{b} = (dt/t)\, d\boldsymbol{a}, \text{ where } \boldsymbol{b} = t^{1/n}\boldsymbol{a},\ t > 0,\ \boldsymbol{a} \in A. \tag{60}$$

By direct computation of the Jacobian of the map

$$(\boldsymbol{n}, \boldsymbol{b}) \mapsto Y := {}^{\mathbf{t}}(\boldsymbol{n}\boldsymbol{b})(\boldsymbol{n}\boldsymbol{b})$$

from $N \times D \to \mathcal{P}_n$, one has (Terras (1988, Sec.4.1, Ex.24, p.23))

$$d\mu_n(Y) = 2^n d\boldsymbol{n}\, d\boldsymbol{b}. \tag{61}$$

By (58), (60) and (61), for $\boldsymbol{n} \in N$ and $\boldsymbol{a} \in A$, we have

$$dW = 2^{n-1} d\boldsymbol{n}\, d\boldsymbol{a}, \text{ where } W = {}^{\mathbf{t}}(\boldsymbol{n}\boldsymbol{a})(\boldsymbol{n}\boldsymbol{a}). \tag{62}$$

If $d(\bar{g})$ denotes the Haar integral on $\mathrm{O}(n)\backslash G \cong AN$ associated to the Haar integrals dg and dk, then by (25),

$$d\bar{g} = d\boldsymbol{n}\, d\boldsymbol{a}, \text{ where } \bar{g} = O(n)\boldsymbol{n}\boldsymbol{a},\ \boldsymbol{n} \in N,\ \boldsymbol{a} \in A. \tag{63}$$

Now for any $f \in \mathrm{C_c}(\mathcal{SP}_n)$, by (62) and (63), we have

$$\int_{\mathcal{SP}_n} f(W)\, dW = 2^{n-1} \int_{\mathrm{O}(n)\backslash G} f({}^{\mathbf{t}}gg)\, d\bar{g}. \tag{64}$$

Relating $\mathrm{Vol}(\mathcal{SM}_n)$ *and* $\mathrm{Vol}(G/\mathrm{GL}_n(\mathbb{Z}))$. By (64), the map $\mathrm{O}(n)g \mapsto {}^{\mathbf{t}}gg$ from $\mathrm{O}(n)\backslash G$ to $\mathcal{SP}_n$ is a right G-equivariant diffeomorphism, and it preserves the invariant integrals $2^{n-1}d\bar{g}$ and dW. We also note that $\mathrm{O}(n)\backslash G$ is connected, and $Z(G)$ is the largest normal subgroup of G contained in K. Therefore by Theorem B.1 (stated and proved in Appendix B),

$$2^{n-1}\mu(G/\mathrm{GL}_n(\mathbb{Z})) = \frac{\mathrm{Vol}(\mathrm{O}(n))}{\#(Z(G)\cap \mathrm{GL}_n(\mathbb{Z}))}\,\mathrm{Vol}(\mathcal{SM}_n).$$

By (24), $\mathrm{Vol}(\mathrm{O}(n)) = 2$, and $\#(Z(G)\cap \mathrm{GL}_n(\mathbb{Z})) = 2$. Also $\Gamma = \mathrm{GL}_n(\mathbb{Z})$. Thus by (59), we have the following:

THEOREM 5.6.

$$\mu(G/\Gamma) = 2^{-(n-1)} \prod_{k=2}^{n} \pi^{-k/2}\Gamma(k/2)\zeta(k).$$

5.4 PROOF OF THEOREM 5.1. By Proposition 5.3, there exists a finite set $\mathcal{F} \subset V_P(\mathbb{Z})$, such that $V_P(\mathbb{Z})$ is a disjoint union of the orbits ${}^{\Gamma}X_0$, $X_0 \in \mathcal{F}$. By Theorem 2.2, (1), and (4),

$$C_P = \sum_{X_0\in\mathcal{F}} C_{X_0}.$$

By Theorem 2.4,

$$C_{X_0} = c_\eta \cdot \frac{\nu(H/H\cap\Gamma)}{\mu(G\cap\Gamma)}.$$

Let $\mathcal{O}(X_0)$ denote the order in $\mathbb{Q}(\alpha)$ associated to the Γ-orbit ${}^{\Gamma}X_0$ as in Proposition 5.3. Then by (45), Theorem 5.5, and Theorem 5.6,

$$\begin{aligned} C_{X_0} &= \frac{(2\pi)^{r_2}\,\mathrm{Vol}(B^{n(n-1)/2})}{2^{n-1}D^{1/2}_{\mathbb{Q}(\alpha)/\mathbb{Q}}} \cdot \frac{2^{r_1}R_{\mathcal{O}(X_0)}/w_{\mathcal{O}(X_0)}}{2^{-(n-1)}\prod_{k=2}^{n}\pi^{-k/2}\Gamma(k/2)\zeta(k)} \\ &= \frac{(2\pi)^{r_2}2^{r_1}R_{\mathcal{O}(X_0)}}{w_{\mathcal{O}(X_0)}D^{1/2}_{\mathbb{Q}(\alpha)/\mathbb{Q}}} \cdot \frac{\mathrm{Vol}(B^{n(n-1)/2})}{\mathrm{Vol}(\mathcal{SM}_n)}. \end{aligned}$$

This shows that C_{X_0} depends only on $\mathcal{O}(X_0)$. We recall that $\mathcal{O}(X_0) \supset \mathbb{Z}[\alpha]$. By Proposition 5.3, for each order $\mathcal{O}$ in K containing $\mathbb{Z}[\alpha]$, there exist exactly $h_{\mathcal{O}}$ number of $X_0 \in \mathcal{F}$, such that $\mathcal{O}(X_0) = \mathcal{O}$. Therefore

$$C_P = \sum_{\mathcal{O}\supset\mathbb{Z}[\alpha]} \frac{(2\pi)^{r_2}2^{r_1}h_{\mathcal{O}}R_{\mathcal{O}}}{w_{\mathcal{O}}D^{1/2}_{\mathbb{Q}(\alpha)/\mathbb{Q}}} \cdot \frac{\mathrm{Vol}(B^{n(n-1)/2})}{\mathrm{Vol}(\mathcal{SM}_n)}. \qquad \square$$

Proof of Theorem 1.2. By our hypothesis $\mathbb{Z}[\alpha]$ is the integral closure of $\mathbb{Z}$ in $K = \mathbb{Q}(\alpha)$, and hence $\mathbb{Z}[\alpha]$ is the maximal order $\mathcal{O}_K$ in K. Now the theorem follows immediately from Theorem 5.1. □

Appendix A
Decompositions of Haar integrals on $\mathrm{SL}_2(\mathbb{R})$

Let

$$\begin{aligned} h(t) &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, && \forall\, t \in \mathbb{R} \\ a(\lambda) &= \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}, && \lambda > 0. \\ k(\theta) &= \begin{pmatrix} \cos(2\pi\theta) & -\sin(2\pi\theta) \\ \sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}, && \theta \in \mathbb{R}/\mathbb{Z}. \end{aligned}$$

First will compare the decompositions of Haar integrals on $\mathrm{SL}_2(\mathbb{R})$ with respect to the Iwasawa decomposition and the Cartan decomposition.

Proposition A.1. *For any* $f \in \mathrm{C_c}(\mathrm{SL}_2(\mathbb{R}))$,

$$\begin{aligned} &\int_{(\mathbb{R}/\mathbb{Z})\times\mathbb{R}\times\mathbb{R}_{>0}} f(k(\theta_1)h(t)a(\lambda))\, d\theta_1\, dt\, \tfrac{d\lambda}{\lambda} \\ =\ & (\pi/2)\int_{(\mathbb{R}/\mathbb{Z})\times\mathbb{R}_{>0}\times(\mathbb{R}/\mathbb{Z})} f(k(\theta_2)a(\alpha)k(\theta))\, |\alpha^2-\alpha^{-2}|\, d\theta_2\, \tfrac{d\alpha}{\alpha}\, d\theta. \end{aligned} \tag{65}$$

Proof. Suppose $g = k(\theta_1)h(t)a(\lambda) = k(\theta_2)a(\alpha)k(\theta)$. Then

$$^{\mathbf{t}}gg = a(\lambda)\, ^{\mathbf{t}}h(t)h(t)a(\lambda) = k(-\theta)a(\alpha^2)k(\theta). \tag{66}$$

Substituting $\beta := \alpha^2$, $\mu := \lambda^2$, and $\phi = 2\pi\theta$, from (66) we get,

$$\begin{aligned} \mu &= (1/2)(\beta+\beta^{-1}) + (1/2)(\beta-\beta^{-1})\cos(2\phi) \\ t &= -(1/2)(\beta-\beta^{-1})\sin(2\phi). \end{aligned} \tag{67}$$

Therefore

$$|\partial(\mu,t)/\partial(\beta,\phi)| = \frac{|\beta-\beta^{-1}|}{\beta}\mu.$$

Hence

$$|\partial(\lambda,t)/\partial(\alpha,\theta)| = 2\pi\frac{|\alpha^2-\alpha^{-2}|}{\alpha}\lambda. \tag{68}$$

Then by (66) and (67) the map

$$(\theta_2,\alpha,\theta) \mapsto (\theta_1,t,\lambda), \tag{69}$$

is surjective if $0 \le \theta < 1/2$, and $\alpha \ge 1$, and it is injective if $0 \le \theta < 1/2$ and $\alpha > 1$. Therefore the map given by (69) from $\mathbb{R}/\mathbb{Z}\times\mathbb{R}_{>0}\times\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}\times\mathbb{R}\times\mathbb{R}_{>0}$

is differentiable, it is surjective, its degree at regular points is 4, and its Jacobian is given by (68). This gives (65). □

Next, we will show that $\mathrm{SL}_2(\mathbb{R}) = \mathrm{SO}(2)h(\mathbb{R}_+)\mathrm{SO}(2)$, and express the Haar integral on $\mathrm{SL}_2(\mathbb{R})$ with respect to this decomposition.

Proposition A.2. *For any* $f \in \mathrm{C_c}(\mathrm{SL}_2(\mathbb{R}))$,

$$\begin{aligned} &\int_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}_+\times\mathbb{R}/\mathbb{Z}} f(k(\phi')h(t)k(\phi))\,d\phi'\,dt^2\,d\phi \\ = &\int_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}>0\times\mathbb{R}/\mathbb{Z}} f(k(\theta')a(\alpha)k(\theta))\,|\alpha^2-\alpha^{-2}|\,d\theta'\,\tfrac{d\alpha}{\alpha}\,d\theta. \end{aligned} \tag{70}$$

Proof. If we write $g = k(\phi')h(t)k(\phi) = k(\theta')a(\alpha)k(\theta)$, then

$$^{\mathbf{t}}gg = k(\phi)\,^{\mathbf{t}}h(t)h(t)k(\phi) = k(\theta)a(\alpha^2)k(\theta). \tag{71}$$

Therefore,

$$\mathrm{trace}(\,^{\mathbf{t}}gg) = 1+t^2 = \alpha^2+\alpha^{-2}. \tag{72}$$

Consider the change of variables $s := t^2$, and $\beta := \alpha^2$. Then

$$\partial s = \frac{\beta-\beta^{-1}}{\beta}\partial\beta.$$

Clearly, $\partial\phi/\partial\theta = 1$, and $\partial t/\partial\theta = 0$. Therefore

$$|\partial(s,\phi)/\partial(\beta,\theta)| = \frac{|\beta-\beta^{-1}|}{\beta},$$

and hence

$$|\partial(s,\phi)/\partial(\alpha,\theta)| = \frac{2|\alpha^2-\alpha^{-2}|}{\alpha}. \tag{73}$$

By (71) and (72), we have that the map

$$(\theta',\alpha,\theta)\to(\phi',s,\phi)$$

is surjective if $\alpha \geq 1$, and it is one-one if $\alpha > 1$. Therefore the map is a differentiable, surjective, its degree at regular points is 2, and its Jacobian is given by (73). This gives (70). □

From Proposition A.1 and Proposition A.2, we obtain the following:

Proposition A.3. *For any* $f \in \mathrm{C_c}(\mathrm{SL}_2(\mathbb{R}))$,

$$\begin{aligned} &\int_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}\times\mathbb{R}_{>0}} f(k(\theta)h(s)a(\lambda))\,d\theta\,ds\,\frac{d\lambda}{\lambda} \\ = \;&(\pi/2)\int_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}_+\times\mathbb{R}/\mathbb{Z}} f(k(\phi')h(t)k(\phi))\,d\phi'\,dt^2\,d\phi. \end{aligned}$$

Appendix B

A Lemma on volumes of two sided quotients

Let G be a Lie group and Γ a lattice in G. Assume that we are given a Haar measure on G, and we want to find the volume of G/Γ. In many cases one can find a compact subgroup K of G such that $E = K\backslash G$ is diffeomorphic to a Euclidean space, and construct a fundamental domain, say $\mathcal{F}$, for the right Γ-action on E. The following result expresses the volume of G/Γ in terms of the volume of $\mathcal{F}$.

Theorem B.1. *Let G be a Lie group and K be a compact subgroup of G such that $K\backslash G$ is connected. Let Γ be a discrete subgroup of G. Let $\tilde{\mu}$ (resp. ν) be a Haar measures on G (resp. K). Let η (resp. μ) be the corresponding G-invariant measure on $K\backslash G$ (resp. G/Γ). Let $\mathcal{F}$ be a measurable fundamental domain for the right Γ-action on $K\backslash G$; in other words, $\mathcal{F}$ is measurable and it is the image of a measurable section of the canonical quotient map $K\backslash G \to K\backslash G/\Gamma$. Then*

$$\mu(G/\Gamma) = \frac{\nu(K)}{\#(K_0 \cap \Gamma)} \cdot \eta(\mathcal{F}), \tag{74}$$

where K_0 is the largest normal subgroup of G contained in K.

To prove this result, we need the the following two observations.

Lemma B.2. *For $\gamma \in G$, put*

$$X_\gamma = \{\omega \in G : \omega\gamma\omega^{-1} \in K\}.$$

Then either X_γ is a finite union of strictly lower dimensional analytic subvarieties of G, or $\gamma \in K_0$.

Proof. Because the map $\omega \mapsto \omega\gamma\omega^{-1}$ on G is an analytic map, and K is a Lie subgroup of G, we have that X_γ is a finite union of analytic subvarieties of G. Note that $KX_\gamma = X_\gamma$ and $KG^0 = G$. Therefore either X_γ is strictly lower dimensional, or $X_\gamma = G$.

Put $K' = \{\gamma \in G : X_\gamma = G\}$. Then K' is a normal subgroup of G, and $K' \subset K$. Hence $K' \subset K_0$. This completes the proof. □

Lemma B.3. *Let Γ be a discrete subgroup of G. Define*

$$K(g) = K \cap g\Gamma g^{-1} \quad \text{and} \quad f(g) = \#(K(g)), \;\; \forall\, g \in G.$$

Then for $\tilde{\mu}$–a.e. $g \in G$, we have

$$K(g) = g(K_0 \cap \Gamma)g^{-1} \quad \text{and} \quad f(g) = \#(K_0 \cap \Gamma). \tag{75}$$

Proof. We put $n_0 = \#(K_0 \cap \Gamma)$. Since K_0 is normal in G and $K_0 \subset K$,

$$K(g) \supset K_0 \cap g\Gamma g^{-1} = g(K_0 \cap \Gamma)g^{-1}, \;\forall\, g \in G. \tag{76}$$

Take any $g \in G$. Since K is compact and Γ is discrete, there exists an open neighbourhood Ω of e in G such that

$$\Omega K \Omega^{-1} \cap g\Gamma g^{-1} = K \cap g\Gamma g^{-1}.$$

Therefore

$$K(\omega g) = \omega(\omega^{-1} K \omega \cap g\Gamma g^{-1})\omega^{-1} \subset \omega K(g)\omega^{-1}, \ \forall \ \omega \in \Omega. \tag{77}$$

First suppose, $f(g) \leq n_0$. Then by (76) $n = n_0$, and by (77),

$$K(\omega g) = \omega K(g)\omega^{-1} = \omega g(K_0 \cap \Gamma)g^{-1}\omega^{-1}, \ \forall \ \omega \in \Omega.$$

In particular, $f(\omega g) = n_0$ for all $\omega \in \Omega$.

Now suppose $f(g) > n_0$. Then by (77)

$$\begin{aligned} \Omega g \cap f^{-1}(f(g)) &= \{\omega g \in \Omega g : K(\omega g) = \omega g(g^{-1}Kg \cap \Gamma)g^{-1}\omega^{-1}\} \\ &\subset \cap_{\gamma \in g^{-1}Kg \cap \Gamma} X_\gamma. \end{aligned}$$

Now, by Lemma B.2, either there exists $\gamma \in g^{-1}Kg \cap \Gamma$ such that X_γ is a finite union of strictly lower dimensional analytic subvarieties of G, or $g^{-1}Kg \cap \Gamma \subset K_0$. In the latter case, by (76), $K(g) = g(K_0 \cap \Gamma)g^{-1}$, and hence $f(g) = n_0$, which is a contradiction.

Thus we have shown that (75) holds for all $g \in f^{-1}(n_0)$, and $\cup_{n \neq n_0} f^{-1}(n)$ is contained in a countable union of strictly lower dimensional analytic subvarieties of G, and hence $\tilde{\mu}(\cup_{n\neq n_0} f^{-1}(n)) = 0$. This completes the proof. □

Proof of Theorem B.1. Consider the natural quotient map $\psi : G/\Gamma \to K\backslash G/\Gamma$. For any $g \in G$ and $x = g\Gamma \in G/\Gamma$, we have

$$\psi^{-1}(Kg\Gamma) = Kx \cong K/K \cap (g\Gamma g^{-1}) = K/K(g).$$

Since $K(kg) = K(g)$, $\forall \ k \in K$, we can define $f(Kg) = f(g)$, $\forall \ g \in G$. Now by Fubini's theorem,

$$\mu(G/\Gamma) = \int_{Kg \in \mathcal{F}} \nu(K)/f(Kg) \, d\eta(Kg). \tag{78}$$

By Lemma B.3, $f(g) = \#(K_0 \cap \Gamma)$ for $\tilde{\mu}$–a.e. $g \in G$. Hence $f(Kg) = \#(K_0 \cap \Gamma)$ for η–a.e. $Kg \in K\backslash G$. Now (74) follows from (78). □

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