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THE SMIRNOV CLASS ON COMPACT GROUPS WITH ORDERED DUALS*

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SUMMARY. On the circle group, the Smirnov class $\mathcal{A}$ is the largest extension of the spaces H^p to which the calculus of Fourier series can be extended (Helson, 1990a,b,c). This note explores a definition of the class on the compact abelian groups studied in Helson, (1975).

1.

On the circle, $\mathcal{A}$ is the set of quotients $k = f/g$ where f, g are in H^2 and g is outer. If k has this form, then f and g can be chosen to be bounded functions. In Helson (1990a) it was shown that a nonnull function k is in $\mathcal{A}$ if and only if $\log |k|$ is summable, and k has this property: whenever g is in H^2 and kg is in L^2, kg is in H^2. This characterization seems closer to the properties of analytic functions than the definition as a quotient.

Some proofs will rely on results from Helson (1975). This should not prevent a reader unacquainted with Helson (1975) (that is, almost everyone) from following the arguments and understanding the theorems.

2.

Let K be a compact abelian group whose dual Γ is a dense subgroup of the real line R, endowed with the discrete topology. For x in K and λ in Γ, $x(\lambda)$ is the value of the character x at λ; and $\chi_\lambda(x)$ is the same number, regarding λ as a character of K. Normalized Haar measure on K is σ. Fourier coefficients of a summable function f are

$$a_\lambda(f) = \int f\,\overline{\chi}_\lambda\, d\sigma \qquad (\lambda \text{ in } \Gamma). \tag{1}$$

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*To M.G. Nadkarni, in appreciation of our long friendship.

The spaces $L^p(K)$ are constructed for $p > 0$. The subspace $H^p(K)$ is the closure of the analytic trigonometric polynomials (linear combinations of χ_λ with $\lambda \geq 0$ in Γ) in the metric of $L^p(K)$. For $p \geq 1$ these are the functions of $L^p(K)$ whose coefficients vanish for $\lambda < 0$; and $H_0^p(K)$ is the subset consisting of functions for which also $a_0 = 0$.

In K we define the one-parameter subgroup K_0 consisting of the elements (e_t), where for each real t, e_t is the element defined by $e_t(\lambda) = \exp it\lambda \quad (\lambda \text{ in } \Gamma)$. In $L^2(K)$ define the unitary operators

$$T_t f(x) = f(x + e_t). \tag{2}$$

On the circle group, $\log|g|$ must be summable for every nonnull element g of any space H^p. This is not so on K; instead, the necessary and sufficient condition for w, a positive function in some class $L^p(K)$, to be the modulus of a function in $H^p(K)$ is that

$$\int_{-\infty}^{\infty} \log w(x + e_t)\, \frac{dt}{t^2 + 1} \;>\; -\infty \text{ a.e.} \tag{3}$$

On the circle group this condition is the same as logarithmic summability, but on K it is weaker.

An *invariant subspace* is a closed subspace M of $L^2(K)$ such that for every f in M, each product $\chi_\lambda \cdot f$ with $\lambda > 0$ in Γ also belongs to M. The subspace is *simply invariant* or *proper* if this fails to be true also for negative λ. Each f in $L^2(K)$ is contained in a smallest invariant subspace, denoted by M_f. This subspace is proper if and only if $w = |f|$ satisfies (3). f is a *generator* for the subspace. The prototype of an invariant subspace is $H^2(K)$; and $H_0^2(K)$ is a related one. $H^2(K)$ is generated by the constant function 1; it has other generators, which are called *outer functions*. A main open question is whether every proper invariant subspace has a generator; and in particular, whether H_0^2 has. A generator of H_0^2, if there is one, should also be called an outer function.

Proper invariant subspaces M come in two *versions*. M_+ is the intersection of the subspaces $M_\lambda = \chi_\lambda \cdot M$ for negative λ in Γ; M_+ contains M. M_- is the closure of all M_λ for positive λ in Γ; this subspace is contained in M. Actually (Helson, 1975) M_- and M_+ are either identical or one dimension apart, so that M is the same as one or both of them. $H^2(K)$ and $H_0^2(K)$ are examples of versions, each of itself and of the other. In every case where the versions of M are unequal, the versions of M are $q \cdot H^2(K)$ and $q \cdot H_0^2(K)$ where q is a function of modulus 1 a.e. M_g equals $(M_g)_+ \neq (M_g)_-$ if and only if $\log|g|$ is summable; this is a *Beurling subspace*. Otherwise $M_g = (M_g)_-$, and it is not known whether $(M_g)_+$ can be larger.

To each proper invariant subspace M is associated a subspace $\widetilde{M}$, consisting of all g in $L^2(K)$ such that fg is in $H^1(K)$ for each f in M. ($\widetilde{M}$ is closely related to the orthogonal complement of M.) We always have $\widetilde{M} = (M_+)^\sim = (\widetilde{M})_+$, and $\widetilde{\widetilde{M}} = M_+$.

3.

Since a generator of $H^2(K)$ must have summable logarithm, the definition of the Smirnov class on K as quotients, in which the denominator is outer, will not be appropriate. Instead, the properties of the class are captured by a modification of the property mentioned above, leading to this definition: *the Smirnov class* $\mathcal{A}(K)$ *consists of the null function, and all functions* F *whose modulus* $w = |F|$ *makes* (3) *finite, and such that whenever* g *is in* $H^2(K)$ *and* Fg *is in* $L^2(K)$*, then also* Fg *belongs to* $H^2(K)$.

Theorem 1. *The Smirnov class consists of all quotients* f/g *in which* f, g *are in* $L^2(K)$*,* $w = |g|$ *satisfies (3), and* f *belongs to* M_g.

If f belongs merely to $(M_g)_+$, the quotient is still in $\mathcal{A}(K)$, because if r is any function in $H^\infty(K)$ such that $\log|r|$ is not summable, then rf belongs to $M_{rg} = (M_{rg})_-$.

Every nonnull element of a proper invariant subspace has modulus that satisfies (3), so f will have that property. In the definition there is no requirement that f and g should be analytic, but as we shall see, they can always be chosen to be.

Let F be a nonnull function in $\mathcal{A}(K)$; we shall show that it is a quotient of the given form. On account of the hypothesis on the modulus of F, there are nonnull functions g in $H^2(K)$ such that $f = Fg$ is square-summable, and therefore belongs to $H^2(K)$. We show that f belongs to M_g, or at least to $(M_g)_+$.

It will be sufficient to show that if k is any square-summable function such that kg is in $H^1(K)$, then kf is also in $H^1(K)$. Now $kf = Fkg$. If k is bounded, this product is in $H^2(K)$ by the definition of $\mathcal{A}(K)$, as required. The set of all such square-summable k constitutes an invariant subspace, and bounded functions are dense in every such subspace (Helson, 1975); this completes the proof.

In the other direction, let $F = f/g$ be a quotient of the sort described; we show that F is in $\mathcal{A}(K)$. Multiplying f and g by the same bounded function if necessary, we can assume that they are bounded functions. Let h be any function of $H^2(K)$ such that $Fh = fh/g$ is square-summable; we are to verify that this quotient is analytic. First we assume that h/g is square-summable.

Since f is in M_g, there is a sequence (P_n) of analytic trigonometric polynomials such that $P_n g$ converges to f in $L^2(K)$. By the Schwarz inequality, $(P_n g)h/g$ converges in $L^1(K)$ to fh/g. But $P_n h$ is analytic, so the limit is too. This shows that Fh is in $H^2(K)$.

Now we give up the assumption that h/g is square-summable; we merely suppose that fh/g is square-summable. Let r be any function in $H^2(K)$ such that r/g is square-summable. Since f is bounded, it is easy to see that fh is in M_g. By what we have just proved, fhr/g is in $H^2(K)$. If the set of such r is dense in $H^2_0(K)$, then fh/g must be in $H^2(K)$.

Write $r = gk$; the set of all r is the set of all gk where k is square-summable and belongs to $\widetilde{\widetilde{M}}_g$. Now $g \cdot \widetilde{\widetilde{M}}_g$ is dense in $H^2_0(K)$ (Theorems 18 and 20 of Helson (1975)); this finishes the proof.

There is a kind of converse statement.

Theorem 2. *If F is in $\mathcal{A}(K)$, and if g and $f = Fg$ are in $L^2(K)$, then f is in $(M_g)_+$.*

Take h any bounded function in $\widetilde{\widetilde{M}}_g$. Then gh is in $H^2(K)$ and $Fgh = fh$ in $L^2(K)$. Since F is in $\mathcal{A}(K)$, fh is in $H^2(K)$. Bounded functions are dense in any invariant subspace; hence the same is true for any h in $\widetilde{M}_g$. Thus f is in $\widetilde{\widetilde{M}}_g = (M_g)_+$ as we wished to show.

Corollary. *Let h belong to $H^2(K)$. h is outer in the generalized sense that M_h is $H^2(K)$ or $H_0^2(K)$ if and only if $1/h$ belongs to $\mathcal{A}(K)$.*

By Theorems 1 and 2, 1 is in $(M_h)_+$ if and only if $1/h$ is in $\mathcal{A}(K)$; there is nothing to prove.

Thus h is a generator for $H_0^2(K)$ if and only if h belongs to that space, and $1/h$ is in $\mathcal{A}(K)$. We do not know whether there are any such functions.

Corollary. *If p is a non-constant inner function on K, then $1 - p$ is outer.*

We must show that $1/(1-p)$ is in $\mathcal{A}(K)$. The modulus w of any analytic function satisfies (3). Now let g be a function of $H^2(K)$ such that $g/(1-p)$ is square-summable. If $0 < r < 1$, $|1-p| \le 2|1-rp|$. By the dominated convergence theorem, the Fourier coefficients of $g/(1-p)$ are limits of the coefficients of $g/(1-rp)$ as r increases to 1. This fraction is analytic, so $g/(1-p)$ is in $H^2(K)$, which shows that $1/(1-p)$ is in $\mathcal{A}(K)$, as we wanted to prove.

4.

Theorem 3. *$\mathcal{A}(K)$ is closed under addition and multiplication.*

Let $F = f/g$, $G = h/k$ be functions in $\mathcal{A}(K)$. We assume, as we may, that f, g, h, k are bounded functions. Then it is easy to see that fk belongs to M_{gk} (using the fact that f is in M_g and k is bounded); and gh is also in M_{gk}. Therefore

$$F + G = (fk + gh)/gk \tag{4}$$

is in $\mathcal{A}$(K).

$FG = fh/gk$. If (P_n) and (Q_n) are analytic trigonometric polynomials such that $P_n g$ tends to f and $Q_n k$ tends to h in $L^2(K)$, then $P_n Q_n gk$ tends to fh in $L^1(K)$. Thus fh is in the L^1-closure of M_{gk}. But fh is bounded; this implies that it is in M_{gk} itself (Helson, 1975, p. 12), and the proof is finished.

Theorem 4. *For $p > 0$, $\mathcal{A}(K) \cap L^p(K) = H^p(K)$.*

The interesting case is $p < 1$; the result for such p implies the same for all p.

Theorem 4 of Helson (1975) states that $H^p(K)$ consists exactly of those f in $L^p(K)$ such that fg is in $H^2(K)$ for some bounded outer function g. Our theorem follows from this fact.

Theorem 5. *If W is any positive function on K satisfying*

$$\int_{-\infty}^{\infty} |\log W(x + e_t)| \frac{dt}{t^2+1} < \infty \qquad a.e., \tag{5}$$

then $W = |F|$ *for some* F *in* $\mathcal{A}(K)$.

By Theorem 23 of Helson (1975), (3) is the necessary and sufficient condition for w to be the modulus of an analytic function. Let g be a bounded analytic function whose modulus is $\min(1, W)$, and h one with modulus $\min(1, 1/W)$. Then $W = |g/h|$. By Theorem 16 of Helson (1975), there is a function q of modulus 1 in M_h, and qg is also in M_h. Thus $F = qg/h$ is a function in $\mathcal{A}(K)$ with modulus W.

5.

An investigation was made in Helson (1990a,b) of real functions in $\mathcal{A}$ on the circle group. The result was that these are precisely the functions of the form $i(p+q)/(p-q)$ where p, q are inner functions, and $p-q$ is outer. For example, if we take $p = 1$ we get a class of such functions.

These examples are not interesting on K because $\log|1-q|$ is summable (unless $q = 1$), and we want more exotic ones. Anyway it is interesting to find the general form of such functions on K.

THEOREM 6. *The real functions in* $\mathcal{A}(K)$ *are exactly the quotients* $i(p+q)/(p-q)$ *where* p, q *are inner, and where* $(M_{p-q})_+ = $ (*closure of* $M_p + M_q)_+$.

The condition, interpreted on the circle, means that $p-q$ is outer provided that p, q have no common inner factor. On the circle two inner functions have a greatest common inner divisor, but no such result is known on K.

Let F be a real function in $\mathcal{A}(K)$, with the representation f/g of the usual sort. Set $r = \overline{g}/|g|$, so that rg is positive. Then rf must be real. Replacing f, g by rf, rg, we can assume that g is positive and f real. We have

$$\frac{F+i}{F-i} = \frac{f+gi}{f-gi}\,. \tag{6}$$

Divide numerator and denominator on the right by $(f^2+g^2)^{\frac{1}{2}}$; now the fraction becomes p/q, where p, q have modulus 1.

Solving (6) for F gives

$$F = i\frac{p+q}{p-q}\,. \tag{7}$$

Let h be a function of unit modulus such that $h(p-q)$ is analytic. Then $h(p+q)$ is analytic too, because it belongs to the subspace generated by $h(p-q)$. It follows that hp and hq are themselves analytic, that is, they are inner functions. Replacing p, q by hp, hq means that the functions p, q in (7) can be chosen to be inner functions.

Let k be any function in $L^2(K)$ with $k(p-q)$ analytic. Then, as just observed, kp and kq must each be analytic. This implies that p, q belong to $(M_{p-q})_+$, which establishes one inclusion in the statement of the theorem.

It is trivial that $p-q$ belongs to $M_p + M_q$, and this gives the other inclusion.

6.

THEOREM 7. *The following properties of a function F in $\mathcal{A}(K)$ are equivalent:*
(a) $1/F$ is in $\mathcal{A}(K)$
(b) The set of all functions Fg, where g and Fg are in $H^2(K)$, is dense in $H^2(K)$ or in $H_0^2(K)$
(c) If g is in $L^2(K)$ and Fg is in $H^2(K)$, then g is in $H^2(K)$.

This result extends the first corollary of Theorem 2, and is analogous to a theorem of Helson (1990a). The properties (b, c) would be the natural definitions of outer function in $\mathcal{A}(K)$. The proof is not very interesting and is omitted.

7.

Let $\mathcal{W}$ be the class of all positive functions w on K, bounded by 1, and satisfying (3). In the space $L_w^2(K)$ based on the measure $wd\sigma$, let $H_w^2(K)$ be the closure of analytic trigonometric polynomials. The space $\mathcal{A}(\mathcal{K})$ is the union of all $H_w^2(K)$, w in $\mathcal{W}$. We can define invariant subspaces and their versions in $L_w^2(K)$. The unsolved problem referred to several times can be rephrased in this way: are the versions of $H_w^2(K)$ necessarily equal if $\log w$ is not summable?

The union of all $H_w^2(K)$ possesses a natural locally convex topology (Helson, 1990c). On the circle, topologies on $\mathcal{A}$ have been studied recently in a beautiful paper by John E. McCarthy (1992). We shall not investigate the extension of those results to the groups K.

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