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NESTED GROWTH CURVE MODELS

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SUMMARY. In this paper, we consider nested growth curve models introduced by Srivastava and Khatri (1979). We show that many testing problems in multivariate analysis of variance and growth curve models can be written as nested models. We provide maximum likelihood estimates of the parameters, likelihood ratio tests and their null distributions for various hypotheses in these nested models.

1. **Introduction**

Consider the growth curve model in which the $p \times N$ observation matrix Y has the following form :

$$Y = B\xi X + \Sigma^{\frac{1}{2}}\varepsilon \quad ,$$

where $B : p \times q$ and $X : m \times N$ are matrices of known constants and ξ is a $q \times m$ matrix of unknown parameters, often referred to as mean or regression parameters. It is assumed that the N columns of the matrix Y are independently normally distributed with a $p \times p$ positive definite covariance matrix $\Sigma = (\Sigma^{\frac{1}{2}})(\Sigma^{\frac{1}{2}})'$ and the mean matrix of Y is given by $B\xi A$. Thus all the pN elements of the $p \times N$ matrix ε are *iid* $N(0, 1)$. For the sake of simplicity of presentation, we shall assume that the rank of B, $\rho(B) = q \leq p$ and $\rho(X) = m$. This model was introduced by Potthoff and Roy (1964) and when $m = 1$ and X is an $1 \times N$ row vector of ones, it was introduced and analyzed by Rao (1959). However, for the general case the maximum likelihood estimation and likelihood ratio test procedure for the general linear hypotheses was given by Khatri (1966). Further properties and analysis of the above model was obtained by Rao (1961, 1965, 1966,

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1967), Grizzle and Allen (1969), Gleser and Olkin (1970), and Srivastava and Khatri (1979). The missing observation situation was considered by Kleinbaun (1973), Srivastava and McDonald (1974), Srivastava (1985) and Tsai and Kozail (1988). For an extensive bibliography, the reader is referred to von Rosen (1991), Kshirsagar and Smith (1995) and Srivastava and von Rosen (1999).

In most of the above references, the general linear hypotheses are tested but no direction is available as to what are the new parameters and how to estimate them and proceed for further reduction or elimination of parameters. With this objective in mind, Srivastava and Khatri (1979, pp 196-197) introduced a nested model and gave an outline as to how to obtain the estimates of the parameters in the nested model. The estimation problem has been reconsidered by Kariya (1985) and von Rosen (1989), but these solutions appear to be somewhat complicated for practical use. In addition the problem of testing any hypothesis in these models has not been considered.

The nested growth curve models also arise in a natural way, such as in comparing the effect of a treatment with a placebo. Suppose y_{1ti} is the response of the treatment on the i-th subject at time $t, i = 1, \ldots, N_1, t = 1, 2, 3, 4$, and y_{0tj} is the response of the placebo on the j-th subject at time t, $j = 1, \ldots, N_2, t = 1, 2, 3, 4$, where it is assumed that all the $N = N_1 + N_2$ subjects are the representative of the same population (independently distributed). If we assume that the response of the treatment is quadratic in time and that of the placebo is constant over time, then we can write the model as

$$E(y_{1ti}) = \beta_{01} + \beta_{11}t + \beta_{21}t^2 \ , i = 1, \ldots, N_1$$

for the treatment and,

$$E(y_{0tj}) = \beta_{00} \ , \ j = 1, \ldots, N_2 \ ,$$

for the placebo, where $t = 1, 2, 3, 4$. Thus, if we write

$$\begin{aligned} Y &= \begin{pmatrix} y_{111} & \cdots & y_{11N_1} & y_{011} & \cdots & y_{01N_2} \\ \vdots & & & \vdots & & \\ y_{141} & \cdots & y_{14N_1} & y_{041} & \cdots & y_{04N_2} \end{pmatrix} \\ B_1 &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \ , \ \xi = \begin{pmatrix} \beta_{01} & \beta_{00} \\ \beta_{11} & 0 \\ \beta_{21} & 0 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_3 \\ \xi_2 & \xi_4 \end{pmatrix} \ , \\ A &= \begin{pmatrix} \mathbf{1}'_{N_1} & \mathbf{0}' \\ \mathbf{0}' & \mathbf{1}'_{N_2} \end{pmatrix} = \begin{pmatrix} A'_1 \\ A'_2 \end{pmatrix} \ , \end{aligned}$$

where $\mathbf{1}_N$ is an N-vector of ones and $\xi_4' = (0,0)$, we get

$$E(Y) = B_1\xi A = B_1\eta_1 A_1 + B_2\eta_2 A_2 \ , \tag{1.1}$$

where

$$\eta_1{=}(\beta_{01},\beta_{11},\beta_{21})',\ \eta_2{=}\beta_{00}\ A_1{=}(\mathbf{1}'_{N_1},\mathbf{0}')\ A_2{=}(\mathbf{0}',\mathbf{1}'_{N_2})\ \text{ and }\ B_2{=}(1,1,1,1)'$$

a subset of B_1. We shall write it as $B_2 \subset B_1$. The model (2.1) when $B_2 \subset B_1$ is called a 'Nested Model' and was introduced by Srivastava and Khatri (1979). With a change of notation, $B_1 \to \tilde{B}$ and $A \to \tilde{A}_1$, the model (1.1) can be rewritten as

$$E(Y) = \tilde{B}\xi\tilde{A}_1 = \tilde{B}_1\tilde{\eta}_1\tilde{A}_1 + \tilde{B}_2\tilde{\eta}_2\tilde{A}_2 \ ,$$

where $\tilde{A}_2' \subset \tilde{A}_1'$, $\tilde{A}_1' = (\tilde{A}_2', \tilde{A}')$, $\tilde{A}_2 = (\mathbf{1}'_{N_1}, \mathbf{0}')$, $\tilde{B} = (\tilde{B}_1, \tilde{B}_2)$, $\tilde{B}_1 = (1,1,1)'$, $\eta_1 = (\xi_1, \xi_3)$ and $\tilde{\eta}_2 = \xi_2$. Thus, it is also a nested model where nesting is achieved through the matrix $\tilde{A}_1$ which was written A in the model (1.1). Hence, the methods of this paper can be applied to this model as well. We shall, however, consider estimation and testing problems for model (1.1) in Section 2.

An extension of the above model is of the form :

$$E(Y) = B_1\eta_1 A_1 + B_2\eta_2 A_2 + B_3\eta_3 A_3 \ \ , \tag{1.2}$$

where $B_3 \subset B_2 \subset B_1$. Estimation and testing in this model will be given in Section 4.

In Section 3, we show that the general linear hypotheses in the growth curve model reduces to the nested model of Section 2. Thus, if the hypothesis is tenable, the estimates of the parameters are available. In Section 5, we show that many testing problems considered in the literature can also use the results from the ' nested growth curve model'. For an alternative approach to the solution of some of these problems, see Srivastava (1997). Nested models have also been considered by Anderson, Marden and Perlman (1993).

2. Estimation and Testing in Nested Model

In this section, we consider the problem of obtaining the maximum likelihood estimates of the unknown parameters, the matrices of mean parameters and the covariance matrix. We also consider the problem of ascertaining that it is a 'nested growth curve model' as opposed to the 'full parameter' growth curve model (GCM). We begin with estimation.

2.1 Estimation of parameters in the nested model. Consider the nested model in which the columns of the matrix B_2 are a subset of the columns of the matrix B_1 written as $B_2 \subset B_1$ and

$$Y = B_1\eta_1 A_1 + B_2\eta_2 A_2 + \Sigma^{\frac{1}{2}}\varepsilon \ , \tag{2.1}$$

$B_1{:}p \times q, B_1{=}(B_2, B_3), B_2{:}p \times q_1, \eta_1{:}q \times m_1, \eta_2{:}q_1 \times m_2, A_1{:}m_1 \times N, A_2{:}m \times N$

and the elements of the error matrix ε are *iid* $N(0,1)$. For simplicity of presentation, we shall assume that B_1, B_2, A_1 and A_2 are of full ranks.

It is well known that the MLE of Σ for the model (2.1) is given by

$$N\hat{\Sigma}_2 = (Y - B_1\hat{\eta}_1 A_1 - B_2\hat{\eta}_2 A_2)(Y - B_1\hat{\eta}_1 A_1 - B_2\hat{\eta}_2 A_2)' \tag{2.2}$$

where $\hat{\eta}_1$ and $\hat{\eta}_2$ are the MLE of η_1, η_2. From Lemma A.2, the MLE of η_1 and η_2 are obtained by minimizing the determinant

$$d_2 \equiv d(\eta_1, \eta_2) = |(Y - B_1\eta_1 A_1 - B_2\eta_2 A_2)(\quad)'|$$

with respects to η_1 and η_2, where $(\quad)'$ is the transpose of the expression on the left. We shall first minimize d_2 with respect to η_1 keeping η_2 fixed. Let

$$\begin{aligned} Y_{1\eta} &= Y - B_2\eta_2 A_2 \\ U_{1\eta} &= Y_{1\eta} H_1 Y_{1\eta}' \\ U_\eta &= Y_{1\eta} K_1 Y_{1\eta}' \\ P_\eta &= B_1(B_1' U_\eta^{-1} B_1)^{-1} B_1' U_\eta^{-1} \\ K_1 &= I - H_1 \\ H_1 &= A_1'(A_1 A_1')^{-1} A_1 \\ T_\eta &= U_\eta + (I - P_\eta) U_{1\eta} (I - P_\eta)' \end{aligned} \tag{2.3}$$

Then, from Lemma A.2 given in the appendix

$$d_2 = |(Y_{1\eta} - B_1\eta_1 A_1)(Y_{1\eta} - B_1\eta_1 A_1)'| \geq |T_\eta|$$

and the equality holds if and only if

$$B_1\hat{\eta}_1 A_1 = P_\eta Y_{1\eta} H_1$$

That is at $\hat{\eta}_1$,

$$\begin{aligned} d(\hat{\eta}_1, \eta_2) &= |T_\eta| \quad = \quad |U_\eta + (I - P_\eta) U_{1\eta} (I - P_\eta)'| \\ &= |U_\eta||I + U_{1\eta}(I - P_\eta)' U_\eta^{-1} (I - P_\eta)| \\ &= |U_\eta||I + U_{1\eta}(U_\eta^{-1} - U_\eta^{-1} P_\eta)| \\ &= |U_\eta||I + U_{1\eta} B_0 (B_0' U_\eta B_0)^{-1} B_0'| \\ &= |U_\eta||I + (B_0' U_{1\eta} B_0)(B_0' U_\eta B_0)^{-1}| \ , \end{aligned}$$

where (B_1, B_0) is nonsingular and $B_0' B_1 = 0$, see Lemma A.1 . Since $B_0' B_1 = B_0'(B_2, B_3) = 0$ gives $B_0' B_2 = 0$, we get

$$(Y' - A_2' \eta_2' B_2') B_0 = Y' B_0 \ .$$

Hence,

$$B_0' U_\eta B_0 = B_0' Y K_1 Y' B_0 \ ,$$

$$B_0' U_{1\eta} B_0 = B_0' Y H_1 Y' B_0 \ .$$

Thus at $\hat{\eta}_1$

$$d(\hat{\eta}_1, \eta_2) = |U_\eta||T_1| \ ,$$

where

$$T_1 = I + (B_0' Y H_1 Y' B_0)(B_0' Y K_1 Y' B_0)^{-1} \tag{2.4}$$

and depends neither on η_1 nor on η_2. Next we minimize

$$|U_\eta| = |Y_{1\eta} K_1 Y_{1\eta}'| = |(Y K_1 - B_2 \eta_2 A_2 K_1)(Y K_1 - B_2 \eta_2 A_2 K_1)'|$$

with respect to η_2. Using Lemma A.2 , we get

$$B_2 \hat{\eta}_2 A_2 K_1 = P_2 Y K_1 H_2 = P_2 Y H_2 \ , \tag{2.5}$$

where

$$\begin{aligned} H_2 &= K_1 A_2' (A_2 K_1 A_2')^{-1} A_2 K_1, \ K_2 = I - H_2 \ , && (2.6) \\ P_2 &= B_2 (B_2' V^{-1} B_2)^{-1} B_2' V^{-1} && (2.7) \\ V &= Y K_1 K_2 K_1 Y' \quad = \quad Y(I - H) Y' \ , && \\ H &= A'(AA')^{-1} A \ , && (2.8) \end{aligned}$$

since, $K_1^2 = K_1, \ K_1 H_2 K_1 = H_2$ and

$$K_1 K_2 K_1 = K_1 - H_2 = I - H_1 - H_2 = I - H \ ,$$

as shown in the Appendix, Lemma A.3 . We also note that, since

$$\begin{aligned} A_2K_1K_2 &= A_2K_1 - A_2K_1A_2'(A_2K_1A_2')^{-1}A_2K_1 = 0 \ , \\ U_{\hat{\eta}} &= (YK_1 - B_2\hat{\eta}_2A_2K_1)(\quad)' \\ &= (YK_1 - B_2\hat{\eta}_2A_2K_1)K_2(YK_1 - B_2\hat{\eta}_2A_2K_1)' \\ &\quad + (YK_1 - B_2\hat{\eta}_2A_2K_1)H_2(YK_1 - B_2\hat{\eta}_2A_2K_1)' \\ &= YK_1K_2K_1Y' + (YH_2 - P_2YH_2)(\quad)' \\ &= V + (I - P_2)U_2(I - P_2)' \ , \end{aligned} \tag{2.9}$$

where

$$U_2 = YH_2Y' \ . \tag{2.10}$$

Since,

$$|U_{\hat{\eta}}| = |V||I + U_2(I - P_2)'V^{-1}(I - P_2)| = |V + U_2(I - P_2')| \ , \tag{2.11}$$

and since from (2.2) and (2.4) the determinant of the MLE of Σ is given by

$$|N\hat{\Sigma}_2| = d(\hat{\eta}_1, \hat{\eta}_2) = |U_{\hat{\eta}}||T_1| \ ,$$

where

$$|T_1| = \frac{|B_0'YK_1Y'B_0 + B_0'YH_1Y'B_0|}{|B_0'YK_1Y'B_0|} = \frac{|B_0'YY'B_0|}{|B_0'YK_1Y'B_0|} \ ,$$

it follows that

$$|N\hat{\Sigma}_2| = \frac{|B_0'YY'B_0|}{|B_0'YK_1Y'B_0|}|V + U_2[I - V^{-1}B_2(B_2'V^{-1}B_2)^{-1}B_2']| \tag{2.12}$$

Thus, we get the following

THEOREM 2.1 *For the nested model defined in (2.1), the maximum likelihood estimates of* η_2, η_1 *and* Σ *are given by*

(a) $B_2\hat{\eta}_2A_2K_1 = P_2YH_2$

(b) $B_1\hat{\eta}_1A_1 = P_{\hat{\eta}}(Y - B_2\hat{\eta}_2A_2)H_1$

(c) $N\hat{\Sigma}_2 = (Y - B_1\hat{\eta}_1A_1 - B_2\hat{\eta}_2A_2)(Y - B_1\hat{\eta}_1A_1 - B_2\hat{\eta}_2A_2)' \ ,$

where

$$P_{\hat{\eta}} = B_1(B_1' U_{\hat{\eta}}^{-1} B_1)^{-1} B_1' U_{\hat{\eta}}^{-1} ,$$

and H_1, P_2, H_2 *and* $U_{\hat{\eta}}$ *have been defined earlier. The determinant of* $N\hat{\Sigma}_2$ *is given in (2.12).*

2.2. *Likelihood ratio test for 'nested model' vs GCM.* Consider the growth curve model in which

$$Y = B_1 \eta A + \Sigma^{\frac{1}{2}} \varepsilon \tag{2.13}$$

where $q \times m$ matrix η is given by

$$\eta = \begin{array}{c} \\ q_1 \\ q_2 \end{array} \begin{array}{c} \begin{array}{cc} m_1 & m_2 \end{array} \\ \begin{pmatrix} \eta_{11} & \eta_2 \\ \eta_{21} & \eta_{22} \end{pmatrix} \end{array} = \left(\eta_1 \left| \begin{array}{c} \eta_2 \\ \eta_{22} \end{array} \right. \right)$$

the matrices $B_1 : p \times q$ and $A : m \times N$ are known and of full ranks . All the elements of ε are *iid* $N(0,1)$ and $\Sigma = \Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})'$ is an unknown positive definite matrix. Under the hypothesis that the $q_2 \times m_2$ matrix $\eta_{22} = 0$, it becomes the nested GCM given by

$$Y = B_1 \eta_1 A_1 + B_2 \eta_2 A_2 + \Sigma^{\frac{1}{2}} \varepsilon, \; B_2 \subset B_1 \; , \tag{2.14}$$

where

$$B_1 = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} B_2, & B_3 \end{pmatrix} \end{array} , \quad A' = \begin{array}{c} \begin{array}{cc} m_1 & m_2 \end{array} \\ \begin{pmatrix} A_1', & A_2' \end{pmatrix} \end{array} .$$

In this section, we derive the likelihood ratio test for testing (2.14) vs (2.13).

Under the GCM model (2.13), the MLE of Σ is given by

$$N\hat{\Sigma}_1 = (Y - B_1 \hat{\eta} A)(Y - B_1 \hat{\eta} A)'$$

where $\hat{\eta}$ is the MLE of η. This $\hat{\eta}$ minimizes the determinant d_1

$$d_1 = |(Y - B_1 \eta A)(Y - B_1 \eta A)'| \; .$$

with respect to η. From Lemma A.2 given in the appendix, the minimum value of d_1 is obtained at

$$\hat{\eta} = (B_1' V^{-1} B_1)^{-1} B_1' V^{-1} Y A' (AA')^{-1} \tag{2.15}$$

where

$$V = Y(I - H)Y' \; , \qquad H = A'(AA')^{-1} A \; .$$

With this value of $\hat{\eta}$, $N\hat{\Sigma}$ simplifies to

$$N\hat{\Sigma}_1 = V + (I - P)V_1(I - P)' \tag{2.16}$$

where

$$V_1 = YHY',\ P = B_1(B_1'V^{-1}B_1)^{-1}B_1'V^{-1} \ .$$

By using repeatedly the property that $|I_p + AB| = |I_q + BA|$ for any $A : p \times q$ and $B : q \times p$ and Lemma A.1, we obtain the minimum value of d as

$$\begin{aligned}
|N\hat{\Sigma}_1| &= |V + V_1(I - V^{-1}B_1(B_1'V^{-1}B_1)^{-1}B_1')| \\
&= |V||I + V_1(V^{-1} - V^{-1}B_1(B_1'V^{-1}B_1)^{-1}B_1'V^{-1})| \\
&= |V||I + (B_0'V_1B_0)(B_0'VB_0)^{-1}| \\
&= |V||B_0'(V + V_1)B_0|/|B_0'VB_0| \\
&= |V||B_0'YY'B_0|/|B_0'VB_0| \qquad (2.17)
\end{aligned}$$

where $B_0'B_1 = 0$ and (B_0, B_1) is nonsingular. The MLE's of η and Σ as given in (2.15) and (2.16) respectively were obtained by Khatri (1966). It can be shown that $\hat{\eta}$ is an unbiased estimate of η but $\hat{\Sigma}_1$ is a biased estimator of Σ, see Srivastava and von Rosen (1999).

From Corollary A.1, we can rewrite (2.17) as

$$|N\hat{\Sigma}_1| = |V + V_1|\frac{|B_1'(V + V_1)^{-1}B_1|}{|B_1'V^{-1}B_1|} \tag{2.18}$$

Next, we need to obtain the determinant of $N\hat{\Sigma}_2$, where $\hat{\Sigma}_2$ is the MLE of Σ under the nested model (2.14). This is already given in (2.12), namely

$$|N\hat{\Sigma}_2| = |T_2||B_0'YY'B_0|/|B_0'YK_1Y'B_0| \ ,$$

where

$$\begin{aligned}
T_2 &= V + U_2[I - V^{-1}B_2(B_2'V^{-1}B_2)^{-1}B_2'] \\
V &= Y[I - H]Y',\ H = A'(AA')^{-1}A \\
U_2 &= YH_2Y' \\
H_2 &= A'(AA')^{-1}C_2[C_2'(AA')^{-1}C_2]^{-1}C_2'(AA')^{-1}A \\
C_2' &= (0, I_{m_2}) : m_2 \times m, m = m_1 + m_2 \\
K_1 &= I - H_1, H_1 = A_1'(A_1A_1')^{-1}A_1 \ .
\end{aligned}$$

We note that

$$\begin{aligned}
|T_2| &= |V||I + U_2(V^{-1} - V^{-1}B_2(B_2'V^{-1}B_2)^{-1}B_2'V^{-1})| \\
U &\equiv YK_1Y' \ = \ Y(I - H_1 - H_2 + H_2)Y' \\
&= Y(I - H)Y' + YH_2Y', H_1 + H_2 = H \ = \ V + U_2 \ .
\end{aligned}$$

Thus, the likelihood ratio test for testing the hypothesis that it is a nested model (2.14) against the alternative that it is GCM, (2.13), is based on the statistic (see Corollary A.1)

$$\begin{aligned}
\lambda &= |N\hat{\Sigma}_1|/|N\hat{\Sigma}_2| \\
&= \frac{|B_0'UB_0|}{|B_0'VB_0|}|I + U_2(V^{-1} - V^{-1}B_2(B_2'V^{-1}B_2)^{-1}B_2'V^{-1})|^{-1} \ , \\
&= \frac{|I + U_2(V^{-1} - V^{-1}B_1(B_1'V^{-1}B_1)^{-1}B_1'V^{-1})|}{|I + U_2(V^{-1} - V^{-1}B_2(B_2'V^{-1}B_2)^{-1}B_2'V^{-1}|) \ ,} && (2.19) \\
&= \frac{|B_1'(V+U_2)^{-1}B_1|}{|B_1'V^{-1}B_1|}\frac{|B_2'V^{-1}B_2|}{|B_2'(V+U_2)^{-1}B_2|} \ . && (2.20)
\end{aligned}$$

2.3 *An alternative expression for the LRT.* To write the likelihood ratio test in an alternative form resembling LRT for MANOVA, we proceed as follows.

Let

$$E_1 = (I_{q_1}, 0) : q_1 \times q \text{ and } E_2 = (0, I_{q_2}) : q_2 \times q, q_1 + q_2 = q.$$

Then, since $B_1 = (B_2, B_3)$. We get

$$B_2 = B_1E_1' \ .$$

Hence

$$\begin{aligned}
V^{-1}P_2 &= V^{-1}B_2(B_2'V^{-1}B_2)^{-1}B_2'V^{-1} \\
&= V^{-1}B_1E_1'[E_1B_1'V^{-1}B_1E_1']^{-1}E_1B_1'V^{-1} \\
&= V^{-1}B_1[(B_1'V^{-1}B_1)^{-1} \\
&\quad -(B_1'V^{-1}B_1)^{-1}E_2'(E_2(B_1'V^{-1}B_1)^{-1}E_2')^{-1}E_2(B_1'V^{-1}B_1)^{-1}]B_1'V^{-1}.
\end{aligned}$$

For notational convenience, let

$$\tilde{Y} = YA'(AA')^{-1} \ .$$

Then

$$U_2 = YH_2Y' = \tilde{Y}C_2[C_2'(AA')^{-1}C_2]^{-1}C_2'\tilde{Y}' \ ,$$

and

$$\begin{aligned}
q &\equiv |I + U_2(V^{-1} - V^{-1}P_2)| \\
&= |I + C_2'\tilde{Y}'(V^{-1} - V^{-1}P_2)\tilde{Y}C_2[C_2'(AA')^{-1}C_2]^{-1}| \\
&= |C_2'(AA')^{-1}C_2|^{-1}|C_2'(AA')^{-1}C_2 + C_2'\tilde{Y}'(V^{-1} - V^{-1}P_2)\tilde{Y}C_2| \\
&= |C_2'(AA')^{-1}C_2|^{-1}|C_2'RC_2 + C_2'\hat{\eta}'E_2'[E_2(B_1'V^{-1}B_1)^{-1}E_2']^{-1}E_2\hat{\eta}C_2| \ ,
\end{aligned}$$

where

$$
\begin{aligned}
R &= (AA')^{-1} + \tilde{Y}'(V^{-1} - V^{-1}B_1(B_1'V^{-1}B_1)^{-1}B_1'V^{-1})\tilde{Y} \\
\hat{\eta} &= (B_1'V^{-1}B_1)^{-1}B_1'V^{-1}\tilde{Y} \ . \qquad (2.21)
\end{aligned}
$$

Simplifying q further, we obtain

$$
\begin{aligned}
q &= \frac{|C_2'RC_2|}{|C_2'(AA')^{-1}C_2|}|I{+}C_2'\hat{\eta}'E_2'[E_2(B_1'V^{-1}B_1)^{-1}E_2']^{-1}E_2\hat{\eta}C_2(C_2'RC_2)^{-1}| \\
&= \frac{|C_2'RC_2|}{|C_2'(AA')^{-1}C_2|}\frac{|E_2(B_1'V^{-1}B_1)^{-1}E_2'{+}E_2\hat{\eta}C_2(C_2'RC_2)^{-1}C_2'\hat{\eta}E_2'|}{|E_2(B_1'V^{-1}B_1)^{-1}E_2'|} \\
&\equiv \frac{|C_2'RC_2|}{|C_2'(AA')^{-1}C_2|}\frac{|P+Q|}{|P|}
\end{aligned}
$$

Now

$$
\begin{aligned}
|C_2'RC_2| &= |C_2'(AA')^{-1}C_2 + C_2'\tilde{Y}'B_0(B_0'VB_0)^{-1}B_0'\tilde{Y}C_2| \\
&= |C_2'(AA')^{-1}C_2||I{+}C_2'\tilde{Y}'B_0(B_0'VB_0)^{-1}B_0'\tilde{Y}C_2[C_2'(AA')^{-1}C_2]^{-1}| \\
&= |C_2'(AA')^{-1}C_2||I{+}(B_0'VB_0)^{-1}B_0'\tilde{Y}C_2[C_2'(AA')^{-1}C_2]^{-1}C_2'\tilde{Y}'B_0| \\
&= |C_2'(AA')^{-1}C_2||I{+}(B_0'VB_0)^{-1}B_0'U_2B_0| \\
&= |C_2'(AA')^{-1}C_2||B_0'(V{+}U_2)B_0|/|B_0'VB_0|
\end{aligned}
$$

Hence

$$
q = \frac{|B_0'(V+U_2)B_0|}{|B_0'VB_0|}\frac{|P+Q|}{|P|}
$$

Thus,

$$
\lambda = \frac{|B_0'(V{+}U_2)B_0|}{|B_0'VB_0|}/q = \frac{|P|}{|P{+}Q|} \ , \qquad (2.22)
$$

where

$$
\begin{aligned}
P &= E_2(B_1'V^{-1}B_1)^{-1}E_2' \\
Q &= E_2\hat{\eta}C_2(C_2'RC_2)^{-1}C_2'\hat{\eta}E_2' \\
\tilde{Y} &= YA'(AA')^{-1} \ ,
\end{aligned}
$$

and R and $\hat{\eta}$ are given in (2.21)

2.4 *Null distribution of the likelihood ratio test statistic.* Since B_1 is a $p \times q$ matrix of rank q, we can write

$$
B_1 = (\Gamma_{11}', \Gamma_{12}')\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{pmatrix} \equiv \Gamma_1\Lambda
$$

where $\Gamma'_{11} : p \times q_1,\ \Gamma_{11}\Gamma'_{11} = I_{q_1},\ \Gamma'_{12} : p \times q_2,\ \Gamma_{12}\Gamma'_{12} = I_{q_2},\ q_1 + q_2 = q$. Let

$$\Gamma = (\Gamma'_{11}, \Gamma'_{12}, \Gamma'_3) \equiv (\Gamma'_1, \Gamma'_3) = (\Gamma'_{11}, \Gamma'_{(2)})$$

be an orthogonal matrix, where $\Gamma'_1 = (\Gamma'_{11}, \Gamma'_{12}) : p \times (p - q)$ and $\Gamma'_{(2)} = (\Gamma'_{12}, \Gamma'_3) : p \times (p - q_1)$. Then

$$B_1 = \Gamma_1 \Lambda \ , \ \text{and } B_2 = \Gamma'_{11}\Lambda_{11} \ .$$

Hence,

$$\begin{aligned} V^{-1} - V^{-1}B_2(B'_2V^{-1}B_2)^{-1}B'_2V^{-1} &= V^{-1} - V^{-1}\Gamma'_{11}(\Gamma_{11}V^{-1}\Gamma'_{11})^{-1}\Gamma_{11}V^{-1} \\ &= \Gamma'_{(2)}(\Gamma_{(2)}V\Gamma'_{(2)})^{-1}\Gamma_{(2)} \\ &= \Gamma'_{(2)}W^{-1}_{(22)}\Gamma_{(2)} \ , \end{aligned}$$

where

$$\Gamma' V \Gamma = W = \begin{pmatrix} W_{(11)} & W_{(12)} \\ W_{(21)} & W_{(22)} \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix} \ .$$

Hence,

$$W_{(22)} = \Gamma_{(2)}V\Gamma'_{(2)} \ , \ W_{33} = \Gamma_3 V \Gamma'_3 \ ,$$

where

$$W \sim W_p(\Omega, n),\ n = N - m,\ \Omega = \Gamma'\Sigma\Gamma \ ,$$

$$\Omega = \begin{pmatrix} \Omega_{(11)} & \Omega_{(12)} \\ \Omega_{(21)} & \Omega_{(22)} \end{pmatrix} = \begin{pmatrix} \Omega_{(11)} & \Omega_{(12)} & \Omega_{(13)} \\ \Omega_{(21)} & \Omega_{(22)} & \Omega_{(23)} \\ \Omega_{(31)} & \Omega_{(32)} & \Omega_{(33)} \end{pmatrix} \ .$$

Hence

$$W_{(22)} \sim W_{p-q_1}(\Omega_{(22)}, n) \ , \ \Omega_{(22)} = \Gamma_{(2)}\Sigma\Gamma'_{(2)}$$

and

$$W_{33} \sim W_{p-q}(\Omega_{33}, n) \ , \ \Omega_{33} = \Gamma_3\Sigma\Gamma'_3 \ .$$

Choosing B_0 as Γ'_3, we find that

$$B'_1 B_0 = \Lambda'\Gamma_1\Gamma'_3 = 0$$

Hence

$$B_0(B'_0 V B_0)^{-1}B'_0 = \Gamma'_3(\Gamma_3 V \Gamma'_3)^{-1}\Gamma_3 = \Gamma'_3 W^{-1}_{33}\Gamma_3$$

Now

$$U_2 = Y H_2 Y'$$

and

$$\Gamma U_2 \Gamma' = \Gamma Y H_2 Y' \Gamma' \ .$$

Under the hypothesis,

$$\begin{aligned} E(\Gamma Y H_2) &= \Gamma[B_1 \eta_1 A_1 + B_2 \eta_2 A_2] H_2 &= \Gamma B_2 \eta_2 A_2 H_2 \\ &= \Gamma \Gamma'_{11} \Delta_{11} \eta_2 A_2 H_2 &= \begin{pmatrix} \Lambda_{11} \eta_2 A_2 H_2 \\ 0 \end{pmatrix} . \end{aligned}$$

Since H_2 is an idempotent matrix of rank m_2, we can write

$$\Gamma U_2 \Gamma' = \Gamma Y H_2 Y' \Gamma' = \Gamma Z Z' \Gamma'$$

where

$$\Gamma_{(2)} Z \sim N_{p-q_1, m_2}(0, \Omega_{(22)}, I_{m_2}).$$

Let

$$\Gamma_{(2)} Z = Z_{(2)} = \begin{array}{c} q_2 \\ p-q \end{array} \begin{pmatrix} Z_2 \\ Z_3 \end{pmatrix}, \text{ where } Z_3 = \Gamma_3 Z.$$

From (2.19), the likelihood ratio test statistic λ is given by

$$\lambda = \frac{|I + U_2 \Gamma'_3 W_{33}^{-1} \Gamma_3|}{|I + U_2 \Gamma'_{(2)} W_{(22)}^{-1} \Gamma_{(2)}|} = \frac{|W_{(22)}|}{|W_{33}|} \frac{|W_{33} + Z_3 Z'_3|}{|W_{(22)} + Z_{(2)} Z'_{(2)}|} \ .$$

We note that V and U_2 are independently distributed since $(I - H) H_2 = (I - H_1 - H_2) H_2 = H_2 - H_2 = 0$, as $H_1 H_2 = 0$.

To obtain the distribution of λ, we first note that the distribution of λ does not depend on Σ. Thus, without any loss of generality, we shall assume that $\Sigma = I$. Next, we make the following transformations.

$$\begin{aligned} W_{(22)} + Z_{(2)} Z'_{(2)} &= \tilde{K} \tilde{K}' \ , \\ \text{and } Z_{(2)} &= \tilde{K} U \ , \end{aligned}$$

where $\tilde{K}$ is an upper triangular matrix. Then $J(W_{(22)}, Z_{(2)} \to \tilde{K}, U) = 2^p \prod k_{ii}^{i+m_2}$. The joint density of $W_{(22)}$ and $Y_{(2)}$ is given by

$$C.|W_{(22)}|^{\frac{n-p+q_1-1}{2}} \text{etr} - \frac{1}{2}(W_{(22)} + Z_{(2)} Z'_{(2)})$$

Hence, the joint pdf of $\tilde{K}$ and U is given by

$$C 2^p |\tilde{K}|^{(n-p+q_1-1)} \prod_{i=1}^{p-q_1} k_{ii}^{i+m_2} |I - UU'|^{\frac{n-p+q_1-1}{2}} \text{etr} - \frac{1}{2} \tilde{K} \tilde{K}' \ .$$

Thus $\tilde{K}$ and U are independently distributed. The pdf of U is given by

$$C_1|I - UU'|^{\frac{n-p+q_1-1}{2}} \ .$$

Writing

$$\tilde{K} = \begin{pmatrix} \tilde{K} & \tilde{K}_{12} \\ 0 & \tilde{K}_2 \end{pmatrix} \text{ and } U = \begin{array}{cc} & m_2 \\ \begin{array}{c} q_2 \\ p-q \end{array} & \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \end{array}$$

we find that

$$\begin{aligned} W_{33} + Z_3Z_3' &= \tilde{K}_2\tilde{K}_2' \\ W_{33} &= \tilde{K}(I - U_2U_2')\tilde{K}_2' \end{aligned}$$

Hence

$$\lambda = \frac{|I - UU'|}{|I - U_2U_2'|} = \frac{|I - U_1'U_1 - U_2'U_2|}{|I - U_2'U_2|} \tag{2.23}$$

Let

$$R_1' = (I - U_2'U_2)^{\frac{1}{2}}U_1'$$

Then

$$\lambda = |I - R_1'R_1|$$

and the pdf of R_1 is given by

$$Const.|I - R_1'R_1|^{\frac{n-p+q-q_2-1}{2}}$$

Hence, λ is distributed as $U_{q_2,m_2,n-p+q}$. Thus, we get the following

THEOREM 2.2 *The likelihood ratio test rejects the hypothesis that it is a nested model given in (2.1) against the alternative that it is a GCM model given in (2.13) for small values of the statistic λ given in (2.20)/(2.22)/(2.23). The null distribution of λ is the U-statistic $U_{q_2,m_2,n-p+q}$, $n = N - m$. An asymptotic distribution of λ is given by*

$$\begin{aligned} &P\{-[n - p + q - \frac{1}{2}(q_2 - m_2 + 1)\ log\ U_{q_2,m_2,n-p+q} \geq C_0\} = \\ &\qquad P(\chi_f^2 \geq C_0) + n^{-2}\gamma_2\{P(\chi_{f+4}^2 \geq C_0) - P\{\chi_f^2 \geq C_0\} + O(n^{-3}) \ . \end{aligned}$$

where

$$f = q_2m_2, \ \textit{and}\ \gamma_2 = f[q_2^2 + m_2 - 5]/48 \ .$$

3. General Linear Hypothesis in GCM

Consider the growth curve model in which

$$E(Y) = B\xi X \ , \tag{3.1}$$

where $B : p \times q$ and $X : m \times N$ are matrices of known constants and ξ is a $q \times m$ matrix of unknown parameters. For the sake of simplicity of presentation, we shall assume that the rank of $B, \rho(B) = q,$ and $\rho(X) = m.$ The general linear hypotheses in the growth curve model involves testing of the hypothesis

$$H : F\xi G = 0 \ \text{ vs } \ A \neq H \ , \tag{3.2}$$

where $F : q_2 \times q, \ \rho(F) = q_2$ and $G : m \times m_2, \ \rho(G) = m_2$. Since $F\xi G = 0$ is equivalent to

$$O = (FF')^{-\frac{1}{2}} F\xi G (G'G)^{-\frac{1}{2}} \equiv L_2 \xi M_2 \ ,$$

we may rewrite (3.2) as

$$H : L_2\xi M_2 = 0 \ \text{ vs } \ A \neq H \ ,$$

where $L_2 L_2' = I_{q_2}$ and $M_2' M_2 = I_{m_2}$.
Define orthogonal matrices

$$L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} : p \times p \text{ and } M = (M_1, M_2) : m \times m \ .$$

Then

$$E(Y) = B\xi X = BL'L\xi MM'X = B_1 \eta A \ , \tag{3.3}$$

where

$$\begin{aligned} B_1 &= BL' = (BL_1', BL_2') \equiv (B_2, B_3), \ B_2 : p \times q_1 & (3.4) \\ A &= M'X = \begin{pmatrix} M_1'X \\ M_2'X \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} , & (3.5) \end{aligned}$$

and

$$\eta = L\xi M = \begin{pmatrix} L_1\xi M_1 & L_1\xi M_2 \\ L_2\xi M_1 & L_2\xi M_2 \end{pmatrix} = \begin{pmatrix} \eta_{11} & \eta_2 \\ \eta_{12} & \eta_{22} \end{pmatrix} \tag{3.6}$$

Under the hypothesis $H, \ \eta_{22} = 0$. Hence, under H, the model becomes

$$E_H(Y) = B_1 \eta_1 A_1 + B_2 \eta_2 A_2 \ , \tag{3.7}$$

where

$$\eta_1 = \begin{pmatrix} \eta_{11} \\ \eta_{12} \end{pmatrix} : q \times m_1, \ \eta_2 : q_1 \times m_2 \ ,$$

and the columns of B_2 are a subset of the columns of B_1 which we shall write as $B_2 \subset B_1$. The model (3.7) is now the 'Nested Growth curve model'. Thus, the problem of testing the general linear hypothesis in GCM is equivalent to that of testing that the model is (3.7) against the alternative that the model is the GCM model (3.3), and the results of the previous section apply in this case.

To write the LRT in terms of the original variables, we note that

$$\begin{aligned} B_1 &= BL' = (BL_1', BL_2'), \ L_2 = (FF')^{-\frac{1}{2}}F \\ B_2 &= BL_1' \\ A &= M'X, \ A_1 = M_1'X, \ A_2 = M_2'X, \ M_2 = G(G'G)^{-\frac{1}{2}} \ , \end{aligned}$$

where L and M are orthogonal matrices. Hence,

$$\begin{aligned} H &= A'(AA')^{-1}A \ = \ X'(XX')^{-1}X \\ H_1 &= A_1'(A_1A_1')^{-1}A_1 \ = \ X'M_1[M_1'XX'M_1]^{-1}M_1'X \\ &= X'[(XX')^{-1} - (XX')^{-1}M_2[M_2'(XX')^{-1}M_2]^{-1}M_2'(XX')^{-1}]X \\ &= H - X'(XX')^{-1}G[G'(XX')^{-1}G]^{-1}G'(XX')^{-1}X \ . \end{aligned}$$

Hence,

$$H_2 = X'(XX')^{-1}G[G'(XX')^{-1}G]^{-1}G'(XX')^{-1}X \ .$$

Also,

$$\begin{aligned} &B_2(B_2'V^{-1}B_2)^{-1}B_2' = BL_1'(L_1B'V^{-1}BL_1')^{-1}L_1B' \\ &= B[(B'V^{-1}B)^{-1} - (B'V^{-1}B)^{-1}L_2'(L_2(B'V^{-1}B)^{-1}L_2')^{-1}L_2(B'V^{-1}B)^{-1}]B' \\ &= B(B'V^{-1}B)^{-1}B' - B(B'V^{-1}B)^{-1}F'(F(B'V^{-1}B)^{-1}F')^{-1}F(B'V^{-1}B)^{-1}B', \end{aligned}$$

and

$$B_1(B_1'V^{-1}B_1)^{-1}B_1' = B(B'V^{-1}B)^{-1}B' \ .$$

Hence, from (2.19), the likelihood ratio test for the general linear hypotheses $H : F\xi G = 0$ in the GCM model in which $E(Y) = B\xi X$ is given by

$$\lambda = \frac{|\tilde{P}|}{|\tilde{P} + \tilde{Q}|} \ ,$$

where

$$\begin{aligned}
\tilde{P} &= I + U_2(V^{-1} - V^{-1}B(B'V^{-1}B)^{-1}B'V^{-1}) \\
\tilde{Q} &= U_2V^{-1}B(B'V^{-1}B)^{-1}F'[F(B'V^{-1}B)^{-1}F']^{-1}F(B'V^{-1}B)^{-1}B'V^{-1} \\
U_2 &= YH_2Y' \;=\; YX'(XX')^{-1}G[G'(XX')^{-1}G]^{-1}G'(XX')^{-1}XY'
\end{aligned}$$

Alternatively, we obtain from (2.22) with $E_2L' \to F$, $MC_2 = G$, $B_1 \to BL$, $LL' = I_q$, $A = MX$, $MM' = I_m$.

$$\begin{aligned}
\lambda &= \frac{|P|}{|P+Q|} \;, \\
P &= F(B'V^{-1}B)^{-1}F' \;, \\
Q &= (F\hat{\xi}G)(G'RG)^{-1}(F\hat{\xi}G)' \;, \\
R &= (XX')^{-1} \\
&\quad +(XX')^{-1}XY'[V^{-1} - V^{-1}B(B'V^{-1}B)^{-1}B'V^{-1}]YX'(XX')^{-1} \;, \\
\hat{\xi} &= (B'V^{-1}B)^{-1}B'V^{-1}YX'(XX')^{-1}, \; V = Y[I - X'(XX')^{-1}X]Y' \;.
\end{aligned}$$

The asymptotic distribution of λ is given by Theorem 2.2.

4. Generalized Nested Models

Consider the GCM model in which

$$E(X) = B_1\xi A$$

$B_1 : p \times q$, $\xi : q \times m$, $A : m \times N$; matrices A and B_1 are full rank. Now, we consider the case when

$$\xi = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} m_1 & m_2 & m_3 \end{array} \\ \begin{pmatrix} \xi_1 & \xi_4 & \xi_7 \\ \xi_2 & \xi_5 & \xi_8 \\ \xi_3 & \xi_6 & \xi_9 \end{pmatrix} \end{array}$$

where $\xi_6 = 0$, $\xi_8 = 0$, $\xi_9 = 0$. Let

$$\eta_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \; \eta_2 = \begin{pmatrix} \xi_4 \\ \xi_5 \end{pmatrix}, \; \eta_3 = (\xi_7) \;,$$

$$B_1 = \Big(\overset{q_1+q_2}{B_2}, \quad \overset{q_3}{\tilde{B}} \Big) = \Big(\overset{q_1}{B_3}, \; \overset{q_2}{B_{22}}, \; \overset{q_3}{\tilde{B}} \Big), \; A' = \Big(\overset{m_1}{A_1'}, \; \overset{m_2}{A_2'}, \; \overset{m_3}{A_3'} \Big) \;,$$

$q_1 + q_2 + q_3 = q$ and $m_1 + m_2 + m_3 = m$.
Then

$$B_1 \xi A = B_1 \eta_1 A_1 + B_2 \eta_2 A_2 + B_3 \eta_3 A_3 \tag{4.1}$$

where $B_3 \subset B_2 \subset B_1$. This is the case when some more parameters in the model (2.1) were felt to be redundant. In the next section, we give the MLE of parameters under the assumption of normality.

4.1 *Maximum likelihood estimates of* η_1, η_2, η_3 *and* Σ. From the likelihood function, it is known that the MLE of Σ is given by

$$N\hat{\Sigma}_3 = (Y - B_1\hat{\eta}_1 A_1 - B_2\hat{\eta}_2 A_2 - B_3\hat{\eta}_3 A_3)(Y - B_1\hat{\eta}_1 A_1 - B_2\hat{\eta}_2 A_2 - B_3\hat{\eta}_3 A_3)' \tag{4.2}$$

where $\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3$ are the MLE of η_1, η_2, η_3 respectively. The MLE of η_1, η_2 and η_3 are the values of η_1, η_2 and η_3 that minimizes the determinant

$$\begin{aligned} d_3 &\equiv d(\eta_1, \eta_2, \eta_3) \\ &= |(Y - B_1\eta_1 A_1 - B_2\eta_2 A_2 - B_3\eta_3 A_3)(Y - B_1\eta_1 A_1 - B_2\eta_2 A_2 - B_3\eta_3 A_3)'| \end{aligned}$$

We minimize d_2 first with respect to η_1 for fixed η_2 and η_3. Using Lemma A.2, we obtain

$$B_1\hat{\eta}_1 A_1 = P_{1\eta} Y_{1\eta} H_1$$

where

$$\begin{aligned} Y_{1\eta} &= (Y - B_2\eta_2 A_2 - B_3\eta_3 A_3) \ , \\ H_1 &= A_1'(A_1 A_1')^{-1} A_1, \ K_1 = I - H_1 \ , \\ P_{1\eta} &= B_1(B_1' U_\eta^{-1} B_1)^{-1} B_1' U_\eta^{-1} \ , \\ U_\eta &= Y_{1\eta} K_1 Y_{1\eta}' \ . \end{aligned} \tag{4.3}$$

Let B_0 be a matrix such that (B_1, B_0) is nonsingular and $B_0' B_1 = 0$. Then at $\hat{\eta}_1$

$$d_3 = |U_\eta||B_0' Y K_1 Y' B_0|^{-1}|B_0' Y Y' B_0| \tag{4.4}$$

The last two expressions do not depend on η_2 and η_3. Thus, we need to minimize only $|U_\eta|$ with respect to η_2 and η_3. For fixed η_3, we can minimize $|U_\eta|$ with respect to η_2 by using Lemma A.2. This gives

$$B_2\hat{\eta}_2 A_2 K_1 = P_{2\eta} Y_{2\eta} H_2 \ ,$$

where

$$\begin{aligned} Y_{2\eta} &= (Y K_1 - B_3\eta_3 A_3 K_1) \ , \\ H_2 &= K_1 A_2'(A_2 K_1 A_2')^{-1} A_2 K_1, \ K_2 = I - H_2 \ , \\ P_{2\eta} &= B_2(B_2' W_\eta^{-1} B_2)^{-1} B_2' W_\eta^{-1} \ , \\ W_\eta &= Y_{2\eta} K_2 Y_{2\eta}' \ . \end{aligned} \tag{4.5}$$

Let $\tilde{B}_0$ be a matrix such that $(B_2, \tilde{B}_0)$ is nonsingular and $\tilde{B}_0' B_2 = 0$. Then at $\hat{\eta}_2$

$$|U_\eta| = |W_\eta||\tilde{B}_0' Y K_1 Y' \tilde{B}_0||\tilde{B}_0' Y K_1 K_2 K_1 Y' \tilde{B}_0|^{-1} \ . \tag{4.6}$$

Now it remains to minimize

$$|W_\eta| = |(YK_1K_2 - B_3\eta_3A_3K_1K_2)(YK_1K_2 - B_3\eta_3A_3K_1K_2)'|$$

with respect to η_3. From Lemma A.2, the minimum occurs at $\hat{\eta}_3$ where

$$\begin{aligned} B_3\hat{\eta}_3A_3K_1K_2 &= P_3YK_1K_2H_{(3)} \ , \ = \ P_3YK_1H_{(3)} && (4.7) \\ H_{(3)} &= K_2K_1A_3'(A_3K_1K_2K_1A_3')^{-1}A_3K_1K_2 \ , \\ K_{(3)} &= I - H_{(3)} \ , H_3 = K_1H_{(3)}K_1 \ , \\ P_3 &= B_3(B_3'V^{-1}B_3)^{-1}B_3'V^{-1} \ , \\ V &= YK_1K_2K_1K_{(3)}K_1K_2K_1Y' \ = \ Y[I - H_1 - H_2 - H_3]Y' \\ &= Y[I - A'(AA')^{-1}A]Y' \ , && (4.8) \end{aligned}$$

from Lemma A.4. Further, we note that $K_2H_{(3)} = H_{(3)}$ and $K_1H_{(3)}K_1 = H_3$, we get at $\eta_3 = \hat{\eta}_3$,

$$\begin{aligned} W_{\hat{\eta}} &= (YK_1K_2 - P_3YK_1H_{(3)})K_{(3)}(YK_1K_2 - P_3YK_1H_{(3)})' \\ &\quad +(YK_1K_2 - P_3YK_1H_{(3)})H_{(3)}(YK_1K_2 - P_3K_1YH_{(3)})' \\ &= YK_1K_2K_{(3)}K_2K_1Y' + (I - P_3)YH_3Y'(I - P_3)' \\ &= V + (I - P_3)V_3(I - P_3)' && (4.9) \end{aligned}$$

where

$$V_3 = YH_3Y' \ . \tag{4.10}$$

Let

$$\begin{aligned} W &= YK_1K_2K_1Y' = Y(I - H_1 - H_2)Y' \ , \\ W_2 &= YH_2Y' \ , \\ \tilde{P}_2 &= B_2(B_2'W^{-1}B_2)^{-1}B_2'W^{-1} \ . \end{aligned}$$

Then, using Corollary A.1, we find from (4.4), (4.6) and (4.9) that since $YK_1Y' = YK_1K_2K_1Y' + YH_2Y' = W + W_2$,

$$|N\hat{\Sigma}_3| = \frac{|B_0YY'B_0|}{|B_0'YK_1Y'B_0|}|I_p + W_2(W^{-1} - W^{-1}\tilde{P}_2)||V||I_p + V_3(V^{-1} - V^{-1}P_3)| \ . \tag{4.11}$$

Hence we get the following theorem.

THEOREM 4.1 *The MLE of* η_1, η_2, η_3 *in the model* (4.2) *are respectively given by*

(a) $B_3\hat{\eta}_3 A_3 K_1 K_2 = P_3 Y H_{(3)}$

(b) $B_2\hat{\eta}_2 A_2 K_1 = P_{2\hat{\eta}} Y_{2\hat{\eta}} H_2$

(c) $B_1\hat{\eta}_1 A_1 = P_{1\hat{\eta}} Y_{1\hat{\eta}} H_1$.

The determinant of $N\hat{\Sigma}_3$ *is given by (4.11).*

4.2 *Testing of hypothesis in nested model (4.1).* In the nested model (4.1) the matrix of mean parameters ξ is given by

$$\xi = \begin{array}{cc} & \begin{array}{ccc} m_1 & m_2 & m_3 \end{array} \\ \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} & \begin{pmatrix} \xi_1 & \xi_4 & \xi_7 \\ \xi_2 & \xi_5 & 0 \\ \xi_3 & 0 & 0 \end{pmatrix} \end{array} . \tag{4.12}$$

This model has less number of parameters than the nested model considered in Section 2. In order to ascertain if this reduced parameter model is tenable compared to the nested model in (2.1), we may wish to test the hypothesis that ξ is given by (4.12) against the alternative that

$$\xi = \begin{array}{cc} & \begin{array}{ccc} m_1 & m_2 & m_3 \end{array} \\ \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} & \begin{pmatrix} \xi_1 & \xi_4 & \xi_7 \\ \xi_2 & \xi_5 & 0 \\ \xi_3 & \xi_6 & 0 \end{pmatrix} \end{array} = q \begin{array}{cc} \begin{array}{cc} m_1+m_2 & m_3 \end{array} & \\ \left(\begin{array}{c|c} & \delta_3 \\ \delta_1 & 0 \\ & 0 \end{array}\right) & \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \end{array}$$

where $\delta_3 = \xi_7$. Thus, the nested model in the alternative becomes

$$E(Y) = B_1\delta_1 A_{(1)} + B_3\delta_3 A_3 \ , \tag{4.13}$$

where

$$A' = (A_1', A_2', A_3') \equiv (A_{(1)}', A_3') \text{ and } B_1 = (B_3, B_{22}, \tilde{B}) = (B_2, \tilde{B}) \ .$$

The MLE of Σ under the alternative denoted by $\hat{\Sigma}_2$ can be obtained from the results of Section 2 by noting the correspondence between the two models : $B_2 \to B_3, A_2 \to A_3, A_1 \to A_{(1)}, K_1 \to K_{(1)}, P_2 \to P_3, U_2 \to V_3$, and $H_2 \to H_3$ (see Lemma A.5). Thus the determinant of the MLE of Σ under the alternative is given by

$$|N\hat{\Sigma}_2| = \frac{|B_0'YY'B_0|}{|B_0'YK_{(1)}Y'B_0|}|V + V_3[I - V^{-1}B_3(B_3'V^{-1}B_3)^{-1}B_3']| \ , \tag{4.14}$$

Hence, the likelihood ratio test rejects the hypothesis that it is a nested model (4.1)/(4.12) against the alternative that it is the nested model (4.13) for small values of the statistic

$$\lambda = \frac{|N\hat{\Sigma}_2|}{|N\hat{\Sigma}_3|} \tag{4.15}$$

where $|N\hat{\Sigma}_3|$ is given in (4.11) and $|N\hat{\Sigma}_2|$ is given in (4.14). From Lemma A.3 - A.5, we note that

$$K_1 K_2 K_1 = I - H_1 - H_2 = I - H_{(1)} = K_{(1)} \ .$$

Hence,

$$W = YK_{(1)}Y', \ W + W_2 = YK_1Y'$$

Thus,

$$\lambda = \frac{|B_0'YK_1Y'B_0|}{|B_0'YK_{(1)}Y'B_0|}|I_p + W_2(W^{-1} - W^{-1}\tilde{P}_2)|^{-1}$$

Using Corollary A.1, we find that in terms of the original variables,

$$\begin{aligned} \lambda &= \frac{|I + W_2(W^{-1} - W^{-1}B_1(B_1'W^{-1}B_1)^{-1}B_1'W^{-1})|}{|I + W_2(W^{-1} - W^{-1}B_2(B_2'W^{-1}B_2)^{-1}B_2'W^{-1})|} \\ &= \frac{|B_1'(W+W_2)^{-1}B_1|}{|B_1'W^{-1}B_1|}\frac{|B_2'W^{-1}B_2|}{|B_2'(W+W_2)^{-1}B_2|} \end{aligned} \tag{4.16}$$

Following as in Section 2.4, it can be shown that the null distribution of λ is that of $U_{q3,m_2,n+m_3-p+q}$. Hence, we get the following

THEOREM 4.2 *The likelihood ratio test rejects the hypothesis that it is a nested model given in (4.1)/(4.12) against the alternative that it is a nested model (4.13) for small values of the statistic λ given in (4.16). The null distribution of λ is the U-statistic $U_{q3,m_2,n+m_3-p+q,n=N-m}$. The asymptotic distribution of λ is given by*

$$\begin{aligned} &P\{-[n + m_1 - p + q - \frac{1}{2}(q_3 - m_2 + 1)logU_{q_3,m_2,n+m_3-p+q} \geq C_0\} \\ &\quad = P\{\chi_f^2 \geq C_0\} + n^{-2}\gamma_2\{P(\chi_{f+4}^2 \geq C_0) - P(\chi_f^2 \geq C_0)\} + O(n)^{-3} \end{aligned}$$

where

$$f = q_3 m_2, \ \text{and } \gamma_2 = f[q_3^2 + m_2 - 5]/48 \ .$$

5. Some Related Testing of Hypotheses Problems

In this section we consider several testing of hypotheses problems, some of them are connected with the nested model. We begin with the problem of testing the hypothesis that it is a GMANOVA model against the alternative that it is a MANOVA model. The likelihood ratio test for this problem has been given by Srivastava and Carter (1983, p.184).

5.1 *Testing GMANOVA versus MANOVA.* In the MANOVA model, we assume that for a $p \times N$ matrix Y,

$$E(Y) = \beta X \tag{5.1}$$

where β is a $p \times m$ matrix of regression parameters and X is a $m \times N$ matrix of known constants. The N columns of Y are assumed to be independently normally distributed with covariance matrix Σ. For this model the maximum likelihood estimate of Σ, denoted by $\hat{\Sigma}_A$ is given by

$$N\hat{\Sigma}_A = V = Y(I-H)Y' \ , \ H = X'(XX')^{-1}X \ . \tag{5.2}$$

We may wish to test the hypothesis that

$$E(Y) = B\xi X \tag{5.3}$$

for some known $p \times q$ matrix B. We shall assume that B is of rank $q \leq p$. That is, it is a growth curve model or a generalized MANOVA model, GMANOVA. For this model, the MLE of Σ, denoted by $N\hat{\Sigma}_H$ can be obtained from (2.16) with $B_1 \to B$, and $A \to X$. It is given by

$$N\hat{\Sigma}_H = V + (I-P)V_1(I-P)' \ , \tag{5.4}$$

where

$$P = B(B'V^{-1}B)^{-1}B'V^{-1} \ , V_1 = YHY' \ .$$

Using (2.17) and (2.18), the likelihood ratio test for testing that it is GMANOVA model against the alternative that it is a MANOVA model is given by

$$\lambda_1 = \frac{|B_0'VB_0|}{|B_0'(V+V_1)B_0|} = \frac{|V|}{|V+V_1|}\frac{|B'V^{-1}B|}{|B'(V+V_1)^{-1}B|} \ , \tag{5.5}$$

where (B, B_0) is nonsingular and $B_0'B = 0$. The distribution of λ_1 is $U_{p-q,m,n}$, $n = N - m$. That is, we have the following

THEOREM 5.1 *For testing the hypothesis that it is a GMANOVA model (5.3) against the alternative that it is a MANOVA model (5.1), the likelihood ratio test is based on the statistic* λ_1 *given in (5.5). The null distribution of* λ_1 *is* $U_{p-q,m,n}, n = N - m$*; its asymptotic distribution is given by*

$$\begin{aligned} &P\{-[n - \frac{1}{2}(p - q - m + 1)ln\lambda_1 > z]\} \\ &\quad = P\{\chi_f^2 \geq z\} + \gamma_2 n^{-2}(P(\chi_{f+4}^2 \geq z) - P(\chi_f^2 \geq z)) + O(n^{-4}) \ , \end{aligned}$$

where

$$f = (p - q)m, \ \gamma_2 = f[(p - q)^2 + m - 5)/48 \ .$$

5.2. *Testing Bilinear hypothesis in MANOVA model.* Consider the MANOVA model described in (5.1). Suppose we wish to test the hypothesis

$$H : F\beta G = 0 \ , \tag{5.6}$$

where $F : (p - q) \times p$ of rank $(p - q)$ and $G : m \times m_2$ of rank m_2 are known matrices. By defining

$$L_2 = (FF')^{-\frac{1}{2}}F \text{ and } M_2 = G(G'G)^{-\frac{1}{2}} \ ,$$

the above hypothesis (5.6) is equivalent to

$$H : L_2\beta M_2 = 0 \ . \tag{5.7}$$

where $L_2L_2' = I_{p-q}$ and $M_2'M_2 = I_{m_2}$. We introduce two semi-orthogonal matrices $L_1 : q \times p$, $L_1L_1' = I_q$ and $M_1' : m_1 \times m$, $M_1'M_1 = I_{m_1}$ such that $L_1L_2' = 0$ and $M_1'M_2 = 0$. Thus,

$$L' = (L_1', L_2') \text{ and } M = (M_1, M_2) \tag{5.8}$$

are orthogonal matrices of orders $p \times p$ and $m \times m$ respectively. Define

$$A = M'X = (A_1', A_2')' \ ,$$

$$L_1\beta M_1 = \eta_1, \ L_1\beta M_2 = \eta_2, \ L_2\beta M_2 = \eta_3, \ L_2\beta M_2 = \eta_4,$$

$$\eta_{(1)} = (\eta_1', \eta_3')' \ .$$

Then, under the hypothesis H that $L_2\beta M_2 = 0$,

$$\begin{aligned} \beta X &= L'L\beta M'MX = L' \begin{array}{c} \begin{array}{cc} m_1 & m_2 \end{array} \\ \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & 0 \end{pmatrix} \end{array} A \\ &= L'\Big(\eta_{(1)}\Big|\begin{matrix} \eta_2 \\ 0 \end{matrix}\Big)A = L'\eta_{(1)}A_1 + L_1'\eta_2 A_2 \ . \end{aligned} \tag{5.9}$$

This is a nested model for which the estimation of parameters has been carried out in Section 2. Thus, from Theorem 2.1, with $B_1 = L',\ B_2 = L_1'$,

$$\begin{aligned} H_1 &= A_1'(A_1A_1')^{-1}A_1 = X'M_1(M_1'XX'M_1)^{-1}M_1'X\ , \\ H_2 &= X'(XX')^{-1}M_2[M_2'(XX')^{-1}M_2]^{-1}M_2'(XX')^{-1}X\ , \\ &= X'(XX')^{-1}G[G'(XX')^{-1}G]^{-1}G'(XX')^{-1}X\ , \\ K_1 &= I - H_1,\ P_{\hat{\eta}} = I\ , \end{aligned}$$

the MLE of η_2 and $\eta_{(1)}$ are given by

(a) $L_1'\hat{\eta}_2M_2'XK_1 = P_2YH_2$

(b) $L'\hat{\eta}_{(1)}M_1'X = (Y - L_1'\hat{\eta}_2M_2'X)H_1$,

where

$$\begin{aligned} P_2 &= L_1'(L_1V^{-1}L_1')^{-1}L_1V^{-1}\ , \\ V &= Y[I - A'(AA')^{-1}A]Y' = Y[I - X'(XX')^{-1}X]Y'\ . \end{aligned}$$

Since in this case L is a $p \times p$ matrix, there is no B_0 and thus, the first term in (2.12) is one. Hence

$$|N\hat{\Sigma}_H| = |V + U_2[I - V^{-1}L_1'(L_1V^{-1}L_1)^{-1}L_1]|\ , \tag{5.10}$$

where

$$\begin{aligned} U_2 &= YH_2Y' = \hat{\beta}G[G'(XX')^{-1}G]^{-1}(\hat{\beta}G)'\ , \\ \hat{\beta} &= YX'(XX')^{-1}\ , \end{aligned}$$

and $\hat{\Sigma}_H$ denotes the MLE of Σ under the hypothesis. The MLE of Σ under MANOVA model is $N^{-1}V$. Hence, the likelihood ratio test for testing the hypothesis $H : F\beta G = 0$ in the MANOVA model $E(Y) = \beta X$ is based on the statistic

$$\begin{aligned} \lambda_2 &= \frac{|V|}{|V + U_2(I - V^{-1}L_1'(L_1V^{-1}L_1')^{-1}L_1)|} \qquad (5.11) \\ &= \frac{1}{|I + U_2L_2'(L_2VL_2')^{-1}L_2|} \\ &= \frac{|L_2VL_2'|}{|L_2(V + U_2)L_2'|} = \frac{|FVF'|}{|FVF' + (F\hat{\beta}G)[G'(XX')^{-1}G]^{-1}(F\hat{\beta}G)'|}\ . \end{aligned}$$

Since under $H,\ A_1H_2 = M_1'XH_2 = 0$ and $L_2L_1' = 0$,

$$E(L_2YH_2) = L_2L'\eta_1A_1H_2 + L_2L_1'\eta_2A_2H_2 = 0\ ,$$

it follows that $L_2U_2L_2'$ has a central Wishart distribution $W_{p-q}(L_2\Sigma L_2', m_2)$. Also, U_2 and V are independently distributed.

Thus, the distribution of λ_2 is $U_{p-q,m_2,n}$. Hence, we have the following asymptotic distribution

THEOREM 5.2 *For testing the bilinear hypothesis (5.6) against the alternative that it is a MANOVA model (5.1), the likelihood ratio test is based on the statistic* λ_2 *which under the hypothesis is distributed as* $U_{p-q,m_2,n}, n = N - m$. *Its asymptotic distribution is given by*

$$\begin{aligned}
&P\{-[n - \frac{1}{2}(p - q - m_2 + 1)]ln\lambda_2 \geq z\} \\
&\quad = P\{\chi_f^2 \geq z\} + n^{-2}\gamma_2\{P(\chi_{f+4}^2 \geq z) - P(\chi_f^2 \geq z)\} + O(n^{-3}) \ ,
\end{aligned}$$

where

$$f = (p - q)m_2, \ \gamma_2 = f[(p - q)^2 + m_2 - 5]/48 \ .$$

5.3. *Testing in Nested Model (5.9).* Consider the model (5.9) in which the columns of Y are independently normally distributed with covariance matrix Σ and

$$E(Y) = L' \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & 0 \end{pmatrix} A \equiv L'\eta_{(1)}A_1 + L_1'\eta_2 A_2 \ ,$$

a nested model. Suppose we now wish to test the hypothesis that $\eta_3 = 0$. That is,

$$E(Y) = L' \begin{pmatrix} \eta_1 & \eta_2 \\ 0 & 0 \end{pmatrix} A = L_1'\eta_1 A_1 + L_1'\eta_2 A_2 \equiv L_1'\xi A \ , \qquad (5.12)$$

a growth curve model. The MLE of Σ under both the models are available in previous subsections. The determinant of the MLE under the (alternative) model (5.9) is given by (5.10) except that now the notation $\hat{\Sigma}_H$ for the MLE of Σ under (5.9) may not be appropriate but our interest is in the expression given in (5.10), and not its notation. The MLE of Σ under the model (5.12) can be obtained from (2.16) or (5.4) with $B \to L_1'$ and hence $B_0 \to L_2'$. Using (2.17) the likelihood ratio test for testing the hypothesis that it is the model (5.12) against the alternative that it is the model (5.9) is based on the statistic, see (5.11),

$$\begin{aligned}
\lambda_3 &= \frac{|V + U_2(I - V^{-1}L_1'(L_1V^{-1}L_1')^{-1}L_1]||L_2VL_2'|}{|V||L_2(V + V_1)L_2'|} \\
&= \frac{|L_2(V + U_2)L_2'|}{|L_2(V + V_1)L_2'|} = \frac{|L_2(V + U_2)L_2'|}{|L_2[V + U_2 + (V_1 - U_2)]L_2'|} \ ,
\end{aligned}$$

where

$$\begin{aligned} V &= Y[I-H]Y',\ H = A'(AA')^{-1}A \\ V_1 &= YHY',\ U_2 = YH_2Y' \\ H_2 &= K_1A_2'(A_2K_1A_2')^{-1}A_2K_1,\ K_1 = I - H_1,\ H_1 = A_1'(A_1A_1')^{-1}A_1 \\ H &= H_1 + H_2\ . \end{aligned}$$

Hence,

$$V_1 = YH_1Y' + YH_2Y'$$

and

$$YH_1Y' = V_1 - U_2\ .$$

Thus, under model (5.12),

$$\lambda_3 \sim U_{p-q,m_1,n+m_2}\ .$$

Hence, we get the following

THEOREM 5.3 *The likelihood ratio test for testing the hypothesis that the model is (5.12) against the alternative it is model (5.9) is based on the statistic* λ_3*. The distribution of* λ_3 *under model (5.12) is* $U_{p-q,m_1,n+m_2}$ *; an asymptotic null distribution of* λ_3 *is given by*

$$\begin{aligned} &P\{-[n-(p-q-m_1+1)]ln\lambda_3 \geq z\} \\ &\qquad = P\{\chi_f^2 \geq z\} + n^{-2}\gamma_2\{P(\chi_{f+4}^2 \geq z) - P(\chi_f^2 \geq z)\} + O(n^{-3})\ , \end{aligned}$$

where

$$f = (p-q)m_1,\ \gamma_2 = f[(p-q)^2 + m_1 - 5]/48\ .$$

Appendix

The following Lemmas have been used in obtaining the maximum likelihood tests and estimators.

LEMMA A.1 *Let* $B : p \times q$ *and* $B_0 : p \times m$ *be two matrices such that* $\rho(B_0) = p - \rho(B)$ *and* $B'B_0 = 0$*. Then for any* $p \times p$ *symmetric and positive definite matrix* V*,*

$$V^{-1} = V^{-1}B(B'V^{-1}B)^{-}B'V^{-1} + B_0(B_0'VB_0)^{-}B_0.$$

This is Lemma 1.9.2 of Srivastava and Khatri (1979, p.19).

COROLLLARY A.1 *Let* (B, B_0) *be a* $p \times p$ *nonsingular matrix such that* $B'B_0 = 0,\ B : p \times q,\ \rho(B) = q,$ *and* $B_0 : p \times (p-q),\ \rho(B_0) = p - q$. *Then for any* $p \times p$ *positive definite symmetric matrix* V *and any* $p \times p$ *positive semi-definite symmetric* V_1,

$$\begin{aligned}
\frac{|B_0'VB_0 + B_0'V_1B_0|}{|B_0'VB_0|} &= |I_p + V_1(V^{-1} - V^{-1}B(B'V^{-1}B)^{-1}B'V^{-1})| \\
&= \frac{|V+V_1|}{|V|}\frac{|B'(V+V_1)^{-1}B|}{|B'V^{-1}B|}
\end{aligned}$$

PROOF. The left-side denoted by LS is given by

$$\begin{aligned}
LS &= |I_{p-q} + (B_0'VB_0)^{-1}B_0'V_1B_0| \\
&= |I_p + B_0(B_0'VB_0)^{-1}B_0'V_1| \\
&= |I_p + (V^{-1} - V^{-1}B(B'V^{-1}B)^{-1}B'V^{-1})V_1| \\
&= |V^{-1}||V + V_1 - B(B'V^{-1}B)^{-1}B'V^{-1}V_1)| \\
&= |V^{-1}||V + V_1 - B(B'V^{-1}B)^{-1}B'V^{-1}(V + V_1 - V)| \\
&= |V^{-1}||V + V_1||I_p - B(B'V^{-1}B)^{-1}B'V^{-1}[I - V(V+V_1)^{-1}]| \\
&= |V^{-1}||V + V_1||I_{p-q} - (B'V^{-1}B)^{-1}B'V^{-1}[I - V(V+V_1)^{-1}]B| \\
&= |V^{-1}||V + V_1||(B'V^{-1}B)^{-1}B'(V+V_1)^{-1}B|\ .
\end{aligned}$$

LEMMA A.2 LET $Y : p \times N,\ B : p \times q$ AND $A : m \times N$ BE MATRICES SUCH THAT $(Y - B\xi A)(Y - B\xi A)'$ IS POSITIVE DEFINITE FOR EVERY $q \times m$ MATRIX ξ. THEN

$$|(Y - B\xi A)(Y - B\xi A)'| \geq |T| \text{ for all } \xi, \qquad (A.1)$$

where

$$\begin{aligned}
T &= V + [I-P]V_1[I-P]' \\
P &= B(B'V^{-1}B)^{-1}B'V^{-1} \\
V_1 &= YHY',\ H = A'(AA')^{-1}A \\
V &= YKY',\ K = I - H,\ V + V_1 = YY'\ .
\end{aligned}$$

The equality in (A.1) holds if and only if

$$B\xi A = PYH$$

This is Lemma 1.10.3 of Srivastava and Khatri (1979, p.24). Thus, V is positive definite if $\rho(K) \geq p$.

Lemma A.3. *Let* $A' = (A_1', A_2')$, $H_1 = A_1'(A_1A_1')^{-1}A_1$, $K_1 = I - H_1$, $H_2 = K_1A_2'(A_2K_1A_2')^{-1}A_2K_1$, $K_2 = I - H_2$, *and* $H = A'(AA')^{-1}A$ *where* $A_1 : m_1 \times N$ *and* $A_2 : m_2 \times N$, $m_1 + m_2 = m$. *Then for* $C_2' = (0, I_{m_2}) : m_2 \times m$

(a) $H_2 = A'(AA')^{-1}C_2[C_2'(AA')^{-1}C_2]^{-1}C_2'(AA')^{-1}A$,

(b) $H = H_1 + H_2$.

(c) $K_1K_2K_1 = K_1 - H_2 = I - H_1 - H_2$.

Proof. Let $C_1' = (I_{m_1}, 0) : m_1 \times m$. Then $C_1'C_2 = 0$, $A_1 = C_1'A$ and $A_2 = C_2'A$. Hence, from Lemma A.1

$$\begin{aligned} H_1 &= A'C_1[C_1'AA'C_1]^{-1}C_1'A \\ &= A'[(AA')^{-1} - (AA')^{-1}C_2[C_2'(AA')^{-1}C_2]^{-1}C_2'(AA')^{-1}]A \\ &= H - A'(AA')^{-1}C_2[C_2'(AA')^{-1}C_2]^{-1}C_2'(AA')^{-1}A \ , \end{aligned}$$

and

$$\begin{aligned} K_1A_2' &= (I - H_1)A_2' \quad = \quad (I - H_1)A'C_2 \\ &= [I - H + A'(AA')^{-1}C_2[C_2'(AA')^{-1}C_2]^{-1}C_2'(AA')^{-1}A]A'C_2 \\ &= A'(AA')^{-1}C_2[C_2'(AA')^{-1}C_2]^{-1}, \text{ since } C_2'C_2 = I_{m_2} \ . \end{aligned}$$

Hence

$$\begin{aligned} A_2K_1A_2' &= C_2'AA'(AA')^{-1}C_2[C_2'(AA')^{-1}C_2]^{-1} \\ &= [C_2'(AA')^{-1}C_2]^{-1} \ , \\ H_2 &= A'(AA')^{-1}C_2[C_2'(AA')^{-1}C_2]^{-1}C_2'(AA')^{-1}A \ , \end{aligned}$$

and

$$H_1 + H_2 = H \ .$$

Lemma A.4. *Let* $H = A'(AA')^{-1}A$, $H_1 = A_1'(A_1A_1')^{-1}A_1$, $K_1 = I - H_1$, $H_2 = K_1A_2'(A_2K_1A_2')^{-1}A_2K_1$, $K_2 = I - H_2$, *and* $H_3 = K_1K_2K_1A_3'$ $(A_3K_1K_2K_1A_3')^{-1}A_3K_1K_2K_1$, *where* $A' = (A_1', A_2', A_3')$. *Then*

$$H = H_1 + H_2 + H_3, \ H_iH_j = 0, \ i \neq j \ .$$

and H_i*'s are idempotent matrices.*

PROOF. We first write

$$A = \begin{pmatrix} A_1 \\ A_{(2)} \end{pmatrix} \text{ where } A_{(2)} = \begin{pmatrix} A_2 \\ A_3 \end{pmatrix}$$

Then from Lemma A.3

$$H = H_1 + H_{(2)},\ H_1 H_{(2)} = 0\ ,$$

where

$$H_{(2)} = K_1 A'_{(2)} (A_{(2)} K_1 A'_{(2)})^{-1} A_{(2)} K_1\ ,$$

and

$$(A_{(2)} K_1 A'_{(2)}) = \begin{pmatrix} A_2 K_1 A'_2 & A_2 K_1 A'_3 \\ A_3 K_1 A'_2 & A_3 K_1 A'_3 \end{pmatrix}\ .$$

Let

$$\begin{aligned} A_{2.1} &= A_3 K_1 A'_3 - A_3 K_1 A'_2 (A_2 K_1 A'_2)^{-1} A_2 K_1 A'_3 \\ &= A_3 [K_1 - K_1 A'_2 (A_2 K_1 A'_2)^{-1} A_2 K_1] A'_3 \\ &= A_3 K_1 (I - H_2) K_1 A'_3 \ = \ A_3 K_1 K_2 K_1 A'_3\ . \end{aligned}$$

Then, from Srivastava and Khatri (1979, p.8)

$$(A_{(2)} K_1 A'_{(2)})^{-1} = \begin{pmatrix} (A_2 K_1 A'_2)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -(A_2 K_1 A'_2)^{-1} A_2 K_1 A'_3 \\ I \end{pmatrix} A_{2.1}^{-1} \begin{pmatrix} -(A_2 K_1 A'_2)^{-1} A_2 K_1 A'_3 \\ I \end{pmatrix}'$$

Hence

$$\begin{aligned} H_{(2)} \ &= \ K_1 (A'_2, A'_3)(A_{(2)} K_1 A'_{(2)})^{-1} \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} K_1 \\ &= K_1 A'_2 (A_2 K_1 A'_2)^{-1} A_2 K_1 \\ &\quad + K_1 [-K_1 A'_2 (A_2 K_1 A'_2)^{-1} A_2 K_1 A'_3 + K_1 A'_3] A_{2.1}^{-1} (-H_2 K_1 A'_3 + K_1 A'_3)' K_1 \\ &= H_2 + K_1 K_2 K_1 A'_3 (A_3 K_1 K_2 K_1 A'_3)^{-1} A_3 K_1 K_2 K_1 \\ &= H_2 + H_3 \end{aligned}$$

Thus,

$$H = H_1 + H_{(2)} = H_1 + H_2 + H_3\ .$$

It may be noted that $H = H_1 + H_{(2)}$ can also be established by the method of this lemma. Furthermore, $K_1 K_2 K_1 = K_1 - K_1 H_2 K_1 = I - H_1 - H_2$.

Lemma A.5. *Let* $A'_{(1)} = (A'_1, A'_2)$, $H_{(1)} = A'_{(1)}(A_{(1)}A'_{(1)})^{-1}A_{(1)}$, $K_{(1)} = I - H_{(1)}$, $A' = (A'_{(1)}, A'_3)$ *and* K_1, K_2 *as defined in Lemma A.4. Then* H_3 *defined in Lemma A.4,*

$$H_3 = K_1K_2K_1A'_3(A_3K_1K_2K_1A'_3)^{-1}A_3K_1K_2K_1 \qquad (A.2)$$

can also be written as

$$H_3 = K_{(1)}A'_3(A_3K_{(1)}A'_3)^{-1}A_3K_{(1)} \qquad (A.3)$$

Proof From Lemma A.3 (b),

$$H = H_{(1)} + K_{(1)}A'_3(A_3K_{(1)}A'_3)^{-1}A_3K_{(1)} \ .$$

Proceeding as in Lemma A.4,

$$H_{(1)} = A'_1(A_1A'_1)^{-1}A_1 + K_1A'_2(A_2K_1A'_2)^{-1}A_2K_1 = H_1 + H_2$$

Hence

$$H = H_1 + H_2 + K_{(1)}A'_3(A_3K_{(1)}A'_3)^{-1}A_3K_{(1)}$$

But from Lemma A.4

$$H = H_1 + H_2 + K_1K_2K_1A'_3(A_3K_1K_2K_1A'_3)^{-1}A_3K_1K_2K_1$$

Thus, the expressions on the right side of (A.2) and (A.3) are equal.

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References

Anderson, S.A. Marden, J.J. and Perlman, M.D. (1993). Totally ordered multivariate linear models. *Sankhyā Ser. A*, **55**, 370-394.

Gleser, L.J. and Olkin, I. (1970). Linear models in multivariate anaysis. In: *Essays in Probability and Statistics*, R.C. Bose, I.M. Chakravarti, P.C. Mahalanobis, C.R. Rao, and K.J.C. Smith, eds., University of North Carolina Press, Chapel Hill, 267-292.

Grizzle, J.E. and Allen, D.M. (1969). Analysis of growth and dose response curves. *Biometrics.* **25**, 357-381.

Kariya, T. (1985). *Testing in the Multivariate General Linear Model.* Kinokuniya, Tokyo.

KHATRI, C.G. (1966). A note on a MANOVA model applied to problems in growth curve. *Ann. Inst. Statist. Math.,* **18**, 75-86.

KLEINBAUM, D.G. (1973). A generalization of the growth curve model which allows missing data. *J. Multivariate Anal.*, **3**, 117-124.

KSHIRSAGAR, A.M. and SMITH, W.B. (1995). *Growth Curves.* Marcel Dekker, New York.

POTTHOFF, R.F. and ROY, S.N. (1964). A generalized multivariate analysis of variance model useful especially for growth curve problems. *Biometrika,* **51**, 313-326.

RAO, C.R. (1959). Some problems involving linear hypotheses in multivariate analysis. *Biometrika*, **46**, 49-58.

RAO, C.R. (1961). Some observations on multivariate statistical methods in anthropological research. *Bull. Inst. Internat. Statist.,* **38**, 99-109.

RAO, C.R. (1965). The theory of least squares when the parameters are stochastic and its application to the analysis of growth curves. *Biometrika,* **52**, 447-458.

RAO, C.R. (1966). Covariance adjustment and related problems in multivariate analysis. In: *Multivariate Analysis.*, P.R. Krishnaiah, ed., Academic Press, New York, 87-103.

RAO, C.R. (1967). Least squares theory using an estimated dispersion matrix and its application to measurement of signals. In: *Proceedings of the Fifth Berkeley Symposium on Mathematical and Statistical Problems,* **1**, L. LeCam and J. Neyman, eds., University of California Press, Berkeley, 355-372.

SRIVASTAVA, J.N. and MCDONALD, L.L. (1974). Analysis of growth curves under the hierarchical models. *Sankhyā Ser. A,* **36**, 251-260.

SRIVASTAVA, M.S. (1985). Multivariate data with missing observations. *Comm. Stat. Theory Methods,* **14**, 775-792.

SRIVASTAVA, M.S. (1997). Generalized multivariate analysis of variance models. Technical Report No 9718, University of Toronto.

SRIVASTAVA, M.S. and CARTER E.M. (1983). *An Introduction to Applied Multivariate Statistics.* North-Holland, New York.

SRIVASTAVA, M.S. and KHATRI, C.G. (1979). *An Introduction to Multivariate Statistics.* North-Holland, New York.

SRIVASTAVA, M.S. and VON ROSEN, D. (1999). Growth Curve Models In *Multivariate Analysis, Design of Experiments and Survey Sampling.* S. Ghosh, ed., Marcel Dekker, New York, 547-578.

TSAI, K.T. and KOZIOL, J.A. (1988). Score and Wald tests for the multivariate growth curve model with missing data. *Ann. Inst. Statist. Math.* **40**, 179-186.

VON ROSEN, D. (1989). Maximum likelihood estimators in multivariate linear normal models. *J. Multivariate Anal.,* **31**, 187-200.

VON ROSEN, D. (1991). The growth curve model : A review. *Comm. Statist. Theory Methods,* **20**, 2791-2822.

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