

MEASURING INCOHERENCE

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SUMMARY. The degree of incoherence, when previsions are not made in accordance with a probability measure, is measured by the rate at which an incoherent bookie can be made a sure loser. Each bet is rescaled by one of several normalizations to account for the overall sizes of bets. For each normalization, the sure loss for incoherent previsions is divided by the normalization to determine the rate of incoherence. We study several properties of normalizations and degrees of incoherence and present some examples. Potential applications include the degree of incoherence of classical statistical procedures.

1. Introduction

A familiar argument for modeling uncertainty with subjective probability is given by the Dutch Book theorem. In its simplest form it says that (in the role of the bookie) if you are willing to accept finite combinations of either side of bets, each of which you judge to be at fair odds, then either

- (a) your fair betting odds are coherent, that is, they satisfy the axioms of probability, or
- (b) a gambler betting against you, knowing your fair odds, can arrange a finite set of bets so that you are a sure loser, in which case your fair odds are incoherent.

Excellent introductions to this classic result can be found in Shimony (1955), Freedman and Purves (1969) and de Finetti (1974, Section 3.3).

Paper received December 2001; revised March 2002.

AMS (1991) subject classification. Primary 62A15; secondary 62F03.

Keywords and phrases. Bookie, coherence, escrow, gambler.

As a theoretical matter, it has been argued by many, e.g. Good (1952), Smith (1961), Levi (1974), Kyburg (1978), Seidenfeld, Schervish and Kadane (1990) and Walley (1991) that it is excessive to require, even normatively, that a decision maker has determinate fair odds for an arbitrary event. Instead, these authors relax the theory of coherence to allow for one-sided betting. This corresponds to allowing a difference between a maximum price for buying and a minimum price for selling a commodity, without demanding that there be a single price at which the decision maker is willing both to buy or to sell the commodity. In this approach, coherent one-sided odds correspond to the lower or upper probabilities taken from a set of probabilities.

As a practical matter, even when a decision maker has determinate degrees of belief, it is very difficult to structure one's fair odds, and more generally, to specify one's expectations for sets of random variables (what deFinetti calls a set of previsions) in such a way that these both reflect one's beliefs and are coherent. See, for example, Kadane and Wolfson (1998). In that sense, even an "ideal" coherent agent may be able to report her previsions only to a degree of precision before they become incoherent: they are imprecisely known previsions.

Yet, the dichotomy between coherent and incoherent sets of previsions, or even between coherent and incoherent one-sided previsions, does not allow for discussion of what sets are "very" incoherent or just "slightly" incoherent. This paper explores a remedy for that problem by studying indices for how effectively an incoherent bookie can be forced to lose money on a single round of wagers. A larger magnitude in one of our indices is associated with a greater degree of incoherence.

The problem as stated so far requires some normalization. Suppose that a particular combination of gambles yields a sure loss y for the bookie. Then multiplying each gamble by the same constant $k > 0$ will create a combination of gambles that yields sure loss ky . In this paper we explore how to perform the normalization from three different perspectives: the bookie's perspective, the gambler's perspective and a neutral perspective. We discuss normalization in general in Section 4.

To fix ideas, consider that the bookie has finite resources. A prudent gambler wants to be sure that the bookie can cover the bets as contracted. One way to do this is to require that, for each wager, the bookie escrows the maximum amount that might be lost. An illustration of one of our indices is to use the sum of these escrow amounts as the normalization. Thus, we can ask how much the incoherent bookie can be forced to lose for sure, given a specified level of total escrow that the bookie has the resources to cover. Alternatively, we can recognize that the gambler, too, has limited resources.

Then, we can ask how much the gambler can be guaranteed to win, for given level of the gambler's total escrow. A third perspective considers how to measure the size of sure gains from a neutral point of view, which in the case of simple bets on events corresponds to the total stake of the wagers.

2. Gambles and incoherence

Think of a random variable X as a function from some space S of possibilities to the real numbers. We assume that, for a bounded random variable X , a bookie might announce some values of x such that he/she finds acceptable all gambles whose net payoff to the bookie is $\alpha(X - y)$ for $\alpha > 0$ and $y < x$. Each such x will be called a *lower prevision for X* . In addition, or alternatively, the bookie may announce some values of x such that the gamble $\alpha(X - y)$ is acceptable when $\alpha < 0$ and $y > x$. These x will be called *upper previsions for X* . We allow that the bookie might announce only upper previsions or only lower previsions or both. For example, if X is the indicator I_A of an event A the bookie might announce that he/she finds acceptable all gambles of the form $\alpha(I_A - x)$ for $x < p$ if $\alpha > 0$ but no other gambles, in particular, not for $x = p$. It will turn out not to matter for any of our results whether or not the bookie finds the gamble $\alpha(I_A - p)$ acceptable. In the special case in which x is both an upper prevision and a lower prevision, we call x a *prevision of X* and denote it $P(X)$. Readers interested in a thorough discussion of upper and lower previsions should refer to Walley (1991).

A collection x_1, \dots, x_n of upper and/or lower previsions for X_1, \dots, X_n respectively is *incoherent* if there exists $\epsilon > 0$ and a collection of acceptable gambles $\{\alpha_i(X_i - y_i)\}_{i=1}^n$ such that

$$\sup_{s \in S} \sum_{i=1}^n \alpha_i(X_i(s) - y_i) < -\epsilon, \quad (2.1)$$

in which case we say that a *Dutch Book* has been made against the bookie. Of course, we would need $\alpha_i > 0$ and $y_i < x_i$ if x_i is a lower prevision for X_i and we would need $\alpha_i < 0$ and $y_i > x_i$ if x_i is an upper prevision for X_i . When a collection of upper and/or lower previsions is incoherent, we would like to be able to measure how incoherent they are. As noted earlier, the ϵ in (2.1) is not a good measure because we could make ϵ twice as big by multiplying all of the α_i in (2.1) by 2, but the previsions would be the same. Instead, we determine measures of the sizes of the gambles and then consider the left-hand side of (2.1) relative to the total size of the combination of gambles, in Sections 3 and 4.

3. General Measure of Incoherence

Although incoherence is defined in terms of finitely many gambles, it is more convenient to discuss incoherence in the presence of arbitrary collections of random variables and previsions.

EXAMPLE 1. Let $\{A_i\}_{i=1}^\infty$ be a countable partition. Suppose that a bookie assigns lower prevision $1/2$ to every A_i . Then every subcollection of at least three previsions chosen from this infinite collection leads to Dutch book. However, there is a sense in which the size of the Dutch book can be increased the more random variables one bets on.

Let S be the set of possible states and let \mathcal{X} be a set of uniformly bounded functions $X : S \rightarrow \mathbb{R}$. That is, there exists M such that $|X(s)| \leq M$ for all s and all X . The set \mathcal{X} is the set of random variables that are available for gambles. Let $\mathcal{X}_P = \mathcal{X} \times \mathbb{R}$. A point $(X, p) \in \mathcal{X}_P$ is a random variable together with a possible prevision. To avoid unbounded gambles, assume that there exists M' such that $|p| \leq M'$ for all $(X, p) \in \mathcal{X}_P$. The domain of our measure of incoherence is the collection of subsets of \mathcal{X}_P . Let \mathcal{H} be a collection of nonempty subsets of \mathcal{X}_P . To gamble, choose finitely many elements $(X_1, p_1), \dots, (X_n, p_n)$ of \mathcal{X}_P together with finitely many appropriate coefficients $\alpha_1, \dots, \alpha_n$. Here, "appropriate" means that $\alpha_i \geq 0$ if p_i is a lower prevision for X_i and $\alpha_i \leq 0$ if p_i is an upper prevision for X_i . Let \aleph be a set of functions $\alpha : \mathcal{X}_P \rightarrow \mathbb{R}$ such that $\alpha(X, p) = 0$ for all but finitely many (X, p) and such that $\alpha(X, p)$ is an appropriate coefficient for each (X, p) . In order to be sure that combinations of gambles remain bounded, some type of bounding on elements of \aleph is needed. Hence require that there exists K such that $\sum_{\text{All } (X, p)} |\alpha(X, p)| \leq K$. For each $H \in \mathcal{H}$ and $\alpha \in \aleph$, let $N(\alpha, H)$ stand for a normalization for the combination of gambles $\sum_{(X, p) \in H} \alpha(X, p)[X(s) - p]$. Normalizations are discussed in Section 4. A measure of incoherence relative to the normalization N is defined as a function from \mathcal{H} to $[0, \infty)$ by

$$\mathcal{I}(H) = \max \left\{ 0, - \inf_{\{\alpha: N(\alpha, H) \leq 1\}} \sup_s \sum_{(X, p) \in H} \alpha(X, p)[X(s) - p] \right\}.$$

If we knew that $N(\alpha, H) > 0$ whenever α is not identically 0, then we might define

$$\mathcal{I}(H) = \max \left\{ 0, - \inf_\alpha \sup_s \frac{\sum_{(X, p) \in H} \alpha(X, p)[X(s) - p]}{N(\alpha, H)} \right\}.$$

The conditions put on the normalization N in Section 4 will guarantee that these two definitions are equivalent when it is known that $N(\alpha, H) > 0$ for nonzero α .

4. Normalizations

Start with a single acceptable gamble such as $Y = \alpha(X - p)$. A normalization for Y should measure its size in some sense. There are a number of possible ways to measure the size of Y . For example $\sup_t |Y(t)|$ or $\sup_t -Y(t)$ might be suitable normalization. Since all random quantities under consideration are uniformly bounded, another possible normalization might be $|\alpha|$.

Consideration of more than one gamble simultaneously requires normalization of the entire collection. The normalization for a collection of gambles is taken to be some function of the normalizations of the individual gambles that make up the collection. To be precise, let α be a constant, X a random variable and p a possible prevision for X . The normalization for the gamble $Y = \alpha(X - p)$ will be denoted $e(\alpha, X, p)$, where e is a nonnegative function satisfying

$$e(c\alpha, X, p) = |c|e(\alpha, X, p), \text{ for all real } c \text{ and all } \alpha, X, p. \quad (4.1)$$

Notice that e is required to be homogeneous of degree 1 as a function of α , not as a function of (X, p) . Although coherence requires that the prevision of cX be cp when the prevision of X is p , this is not the case for potentially incoherent previsions. Hence, we want to allow for the possibility that the prevision of cX is something other than cp even when the prevision of X is p . Hence, the behavior of e is not controlled when X and/or p get rescaled.

We focus our attention on three particular choices of e , although some of the theorems are stated in more general terms. One choice of e might be called *the bookie's escrow*. This is defined by

$$e_1(\alpha, X, p) = \max\{0, -\inf_s \alpha(X, p)[X(s) - p]\}.$$

The bookie's escrow is the most that the bookie could lose from the single gamble. This would be the amount that the bookie would have to prove that he/she had available in order to be able to cover the bet. With this normalization, we refer to the index \mathcal{I} as a *rate of loss* because it measures the rate at which the bookie can be forced to lose relative to the amount needed to cover the bets. Similarly, define *the gambler's escrow* is by

$$e_2(\alpha, X, p) = \max\{0, \sup_s \alpha(X, p)[X(s) - p]\}.$$

This is the most that the gambler could lose from the bet. With this normalization, we refer to the index \mathcal{I} as a *rate of profit* because it is the rate at which the gambler can extract funds from the bookie relative to the amount needed to cover the bets. The third normalization is neutral between the bookie and gambler and is defined simply as $e_3(\alpha, X, p) = |\alpha(X, p)|$. The notation $e(\alpha, X, p)$ is used when the statement refers to an arbitrary normalization.

Each of the three normalizations is continuous in the sense to be defined next. Consider two elements of \mathcal{X}_P , (X, p) and (X', p') . We will measure the difference between these by $\sup_s |X(s) - X'(s)| + |p - p'|$.

DEFINITION 1. Let $(X, p) \in \mathcal{X}_P$ and let $d = 1$ if p is a lower prevision and $d = -1$ if p is an upper prevision. We say that e is *continuous at* (X, p) if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|e(d, X, p) - e(d, X', p')| < \epsilon$ for every (X', p') such that p' is the same type of prevision as p (i.e., upper or lower) and such that $\sup_s |X(s) - X'(s)| + |p - p'| < \delta$.

We require that the normalization for individual gambles be continuous at each element of \mathcal{X}_P .

For n gambles $\{\alpha_i(X_i - p_i)\}_{i=1}^n$, the normalization for the combination will be assumed to be of the form

$$f_n(e(\alpha_1, X_1, p_1), \dots, e(\alpha_n, X_n, p_n)), \quad (4.2)$$

In the notation of Section 3, (4.2) equals $N(\alpha, H)$ when $\{(X, p) \in H : \alpha(X, p) \neq 0\} = \{(X_1, p_1), \dots, (X_n, p_n)\}$. In order for a function f_n to be an appropriate normalization, we have a few requirements.

The first requirement extends (4.1).

$$f_n(cx_1, \dots, cx_n) = cf_n(x_1, \dots, x_n), \quad \text{for all } c > 0, x_1, \dots, x_n. \quad (4.3)$$

Equation (4.3) says that the function f_n must be homogeneous of degree 1 in its arguments so that scaling up all the gambles by the same amount will scale the normalization by that amount as well. Second, since we are not concerned with the order in which gambles are made, we require

$$f_n(x_1, \dots, x_n) = f_n(y_1, \dots, y_n), \quad \text{for all } n, x_1, \dots, x_n \quad (4.4)$$

and all permutations (y_1, \dots, y_n) of (x_1, \dots, x_n) .

Third, in keeping with the use of normalization to measure the sizes of bets, we require that, if a gamble is replaced by one with higher size, the total size should not go down:

$$f_n(x_1, \dots, x_n) \text{ is nondecreasing in each of its arguments.} \quad (4.5)$$

If a gamble has 0 size, we assume that the total size is determined by the other gambles:

$$f_{n+1}(x_1, \dots, x_n, 0) = f_n(x_1, \dots, x_n), \text{ for all } x_1, \dots, x_n \text{ and all } n. \quad (4.6)$$

Part of the definition of a norm (in normed linear spaces) is that the whole be no larger than the sum of its parts. We also impose such a requirement, namely

$$f_n(x_1, \dots, x_n) \leq \sum_{i=1}^n x_i, \quad \text{for all } n \text{ and all } x_1, \dots, x_n. \quad (4.7)$$

Small changes in the component gambles should only produce small changes in the size, so we require that

$$f_n \text{ is continuous for every } n. \quad (4.8)$$

Finally, since the arguments of f_n are the sizes of the individual gambles, we require

$$f_1(x) = x. \quad (4.9)$$

So, if $\{\alpha_i(X_i - p_i)\}_{i=1}^n$ is a collection of gambles, an $f_n(e(\alpha_1, X_1, p_1), \dots, e(\alpha_n, X_n, p_n))$ satisfying (4.3)–(4.9) is called a *normalization for the collection of gambles*. Every sequence of functions $\{f_n\}_{n=1}^\infty$, each of which satisfies (4.3)–(4.9), together with a function e leads to its own way of defining normalization. Such a sequence is called a *normalizing sequence*. Each function in the sequence is a *normalizing function*.

We can find a fairly simple form for all normalizing sequences. Combining (4.9), (4.5) and (4.6), we see that $f_n(x_1, \dots, x_n) \geq \max\{x_1, \dots, x_n\}$. From (4.4), we conclude that f_n is a function of the ordered values $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ of x_1, \dots, x_n . That is, $f_n(x_1, \dots, x_n) = f_n(x_{(1)}, \dots, x_{(n)})$. Combining these results with (4.7), we get

$$0 \leq f_n(x_{(1)}, \dots, x_{(n)}) - x_{(n)} \leq \sum_{i=1}^{n-1} x_{(i)}. \quad (4.10)$$

Let $\lambda_n(x_{(1)}, \dots, x_{(n)}) = (f_n(x_{(1)}, \dots, x_{(n)}) - x_{(n)})/x_{(n)}$ so that

$$f_n(x_{(1)}, \dots, x_{(n)}) = x_{(n)} (1 + \lambda_n(x_{(1)}, \dots, x_{(n)})). \quad (4.11)$$

In order to satisfy (4.6), we need $\lambda_n(0, x_{(2)}, \dots, x_{(n)}) = \lambda_{n-1}(x_{(2)}, \dots, x_{(n)})$. In order to satisfy (4.3), λ_n must be invariant under common scale changes for all of its arguments. That is

$$\lambda_n(cx_{(1)}, \dots, cx_{(n)}) = \lambda_n(x_{(1)}, \dots, x_{(n)}).$$

Every such function can be written as

$$\lambda_n(x_{(1)}, \dots, x_{(n)}) = \gamma_n \left(\frac{x_{(1)}}{x_{(n)}}, \dots, \frac{x_{(n-1)}}{x_{(n)}} \right).$$

In order to satisfy (4.5), we must have γ_n nondecreasing in each of its arguments. In order to satisfy (4.10), we must have

$$0 \leq \gamma_n(y_1, \dots, y_{n-1}) \leq \sum_{i=1}^{n-1} y_i.$$

In summary, every normalizing sequence satisfies

$$f_n(x_1, \dots, x_n) = x_{(n)} \left[1 + \gamma_n \left(\frac{x_{(1)}}{x_{(n)}}, \dots, \frac{x_{(n-1)}}{x_{(n)}} \right) \right], \quad (4.12)$$

for some sequence $\gamma_1, \gamma_2, \dots$ of continuous functions where $\gamma_1 \equiv 0$ and for $n > 1$ the functions satisfy the following properties:

- $\gamma_n(y_1, \dots, y_{n-1})$ is defined and continuous for $0 \leq y_1 \leq y_2 \leq \dots \leq y_{n-1} \leq 1$,
- γ_n is nondecreasing in each argument,
- $0 \leq \gamma_n(y_1, \dots, y_{n-1}) \leq \sum_{i=1}^{n-1} y_i$,
- $\gamma_n(0, y_2, \dots, y_{n-1}) = \gamma_{n-1}(y_2, \dots, y_{n-1})$,
- $x[1 + \gamma_n(y_1/x, \dots, y_{n-1}/x)]$ is nondecreasing in x for all $y_1 \leq \dots \leq y_{n-1} \leq x$.

(The last condition is equivalent to f_n being nondecreasing in $x_{(n)}$.) It is straightforward to show that every sequence that meets this description satisfies (4.3)-(4.9), hence we have characterized normalizing sequences.

One simple collection of normalizing sequences consists of all sequences in which $\gamma_n(y_1, \dots, y_{n-1}) = \gamma \sum_{i=1}^{n-1} y_i$ for some common constant $\gamma \in [0, 1]$. In this case, we get a family of normalizing functions:

$$f_{\gamma,n}(x_1, \dots, x_n) = x_{(n)} + \gamma \sum_{i=1}^{n-1} x_{(i)}, \quad (4.13)$$

for each $0 \leq \gamma \leq 1$. Another example is $\gamma_n(z_1, \dots, z_{n-1}) = z_{n-1}$ for $n > 1$. This one makes the total normalization equal to the sum of the two largest

individual gamble normalizations. Other functions are possible, but we focus on $f_{\gamma,n}$ for $0 \leq \gamma \leq 1$. It is easy to see that the two extreme normalizing functions correspond to $\gamma = 0$ and $\gamma = 1$:

$$\begin{aligned} f_{0,n}(x_1, \dots, x_n) &= x_{(n)}, \\ f_{1,n}(x_1, \dots, x_n) &= \sum_{i=1}^n x_i. \end{aligned}$$

Notice that, since every normalizing function f_n is between $f_{0,n}$ and $f_{1,n}$, the maximum and minimum possible rates of incoherence correspond to these two extreme normalizing sequences.

5. Continuity

This section provides conditions under which the extent of incoherence $\mathcal{I}(\cdot)$ is a continuous function of the gambles under consideration and their previsions.

Discussion of continuity of $\mathcal{I}(\cdot)$ requires a topology on the domain \mathcal{H} . We extend the difference measure that led to Definition 1 to a metric topology on \mathcal{H} . If $H_1, H_2 \in \mathcal{H}$ have different cardinalities, define $d'(H_1, H_2) = \infty$. If $H_1, H_2 \in \mathcal{H}$ have the same cardinality, let $\mathcal{P}(H_1, H_2)$ be the set of all one-to-one correspondences q between H_1 and H_2 . That is, $q \in \mathcal{P}(H_1, H_2)$ if and only if q is a one-to-one function from H_1 onto H_2 . For convenience, denote $q(X, p) = (q(X, p)_1, q(X, p)_2)$. Define

$$d'(H_1, H_2) = \inf_{q \in \mathcal{P}(H_1, H_2)} \sum_{(X,p) \in H_1} \left[\sup_s |X(s) - q(X, p)_1(s)| + |p - q(X, p)_2| \right].$$

Clearly, $d'(H_1, H_2) = \infty$ whenever the cardinality of $H_1 \Delta H_2$ is more than countable. (Here Δ denotes the symmetric difference between two sets.) Consider the potential metric

$$d(H_1, H_2) = \frac{d'(H_1, H_2)}{1 + d'(H_1, H_2)}. \tag{5.1}$$

If H_1 and H_2 have the same cardinality, if $q \in \mathcal{P}(H_1, H_2)$, and if $\alpha \in \mathbb{N}_0$, we define α_q as follows

$$\alpha_q(X, p) = \begin{cases} \alpha(q^{-1}(X, p)) & \text{if } (X, p) \in H_2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

LEMMA 1. *The function d defined in (5.1) is a metric.*

PROOF. Clearly, $d \geq 0$ and it is symmetric. Also, $d(H_1, H_2) = 0$ if and only if $H_1 = H_2$. The triangle inequality remains. It is well known that the triangle inequality holds for d if it holds for d' , so we will prove the triangle inequality for d' . Let $H_1, H_2, H_3 \in \mathcal{H}$. We must show $d'(H_1, H_2) \leq d'(H_1, H_3) + d'(H_2, H_3)$. If not all three subsets have the same cardinality, then the right-hand side of the inequality is ∞ . Indeed, if one of $H_i \Delta H_j$ has uncountable cardinality, then the right-hand side is ∞ . Assume then that all three subsets have the same cardinality and that $H_i \Delta H_j$ is countable for all i and j . If $H_i \cap H_j \neq \emptyset$, then it is easy to see that $\sum_{(X,p) \in H_i} [\sup_s |X(s) - q(X,p)_1(s)| + |p - q(X,p)_2|]$ is smaller when $q(X,p) = (X,p)$ for all $(X,p) \in H_i \cap H_j$ than otherwise. So, we may set $H = (H_1 \Delta H_2) \cup (H_1 \Delta H_3) \cup (H_2 \Delta H_3)$, which is a countable set. Then for all i, j

$$d'(H_i, H_j) = \inf_{q \in \mathcal{P}(H_i, H_j)} \sum_{(X,p) \in H} \left[\sup_s |X(s) - q(X,p)_1(s)| + |p - q(X,p)_2| \right].$$

Let $\epsilon > 0$, and for $i = 1, 2$, let $q_{i,\epsilon} \in \mathcal{P}(H_3, H_i)$ be such that

$$\sum_{(X,p) \in H} \left[\sup_s |X(s) - q_{i,\epsilon}(X,p)_1(s)| + |p - q_{i,\epsilon}(X,p)_2| \right] \leq d'(H_i, H_3) + \epsilon.$$

For each $(X,p) \in H$, we know that

$$\begin{aligned} & \sup_s |q_{1,\epsilon}(X,p)_1(s) - q_{2,\epsilon}(X,p)_1(s)| + |q_{1,\epsilon}(X,p)_2 - q_{2,\epsilon}(X,p)_2| \\ & \leq \sup_s |X(s) - q_{1,\epsilon}(X,p)_1(s)| + |p - q_{1,\epsilon}(X,p)_2| \\ & \quad + \sup_s |X(s) - q_{2,\epsilon}(X,p)_1(s)| + |p - q_{2,\epsilon}(X,p)_2|. \end{aligned}$$

We also know that

$$\begin{aligned} & d'(H_1, H_2) \\ & \leq \sum_{(X,p) \in H} \left[\sup_s |q_{1,\epsilon}(X,p)_1(s) - q_{2,\epsilon}(X,p)_1(s)| + |q_{1,\epsilon}(X,p)_2 - q_{2,\epsilon}(X,p)_2| \right]. \end{aligned}$$

Hence, $d'(H_1, H_2) \leq d'(H_1, H_3) + d'(H_2, H_3) + \epsilon$. Since this is true for all ϵ , the triangle inequality holds. \square

Let $e : \aleph \times \mathcal{X}_P \rightarrow [0, \infty)$ be a normalization for single gambles. Let $\{f_n\}_{n=1}^\infty$ be a normalizing sequence. For each $\alpha \in \aleph$ and $H \subseteq \mathcal{X}_P$, if $\{(X,p) \in$

$H : \alpha(X, p) \neq 0\} = \{(X_1, p_1), \dots, (X_n, p_n)\}$, assume that

$$N(\alpha, H) = f_n(e[\alpha(X_1, p_1), (X_1, p_1)], \dots, e[\alpha(X_n, p_n), (X_n, p_n)]). \quad (5.3)$$

Let $\alpha_0 \in \aleph$ be the constant function that equals 0 for all (X, p) . $\aleph_0 = \aleph \setminus \{\alpha_0\}$. Define the function $k_1 : S \times \mathcal{H} \times \aleph_0 \rightarrow \mathbb{R}$ by

$$k_1(s, H, \alpha) = \frac{\sum_{(X,p) \in H} \alpha(X, p)[X(s) - p]}{N(\alpha, H)}.$$

Also define the following two derived functions:

$$\begin{aligned} k_2(H, \alpha) &= \sup_s k_1(s, H, \alpha), \\ k_3(H) &= \inf_{\alpha \in \aleph_0} k_2(H, \alpha). \end{aligned}$$

If $N(\alpha, H) > 0$, we see that

$$\mathcal{I}(H) = \min\{0, -k_3(H)\}.$$

The main continuity result is the following theorem:

THEOREM 1. *For each $H \in \mathcal{H}$, define $\aleph(H) = \{\alpha : k_2(H, \alpha) < 0\}$. Let $H_1 \in \mathcal{H}$ and suppose that there exists $c > 0$ such that*

$$N(\alpha, H_1) \geq c \max_{(X,p) \in H_1} |\alpha(X, p)|, \text{ for all } \alpha \in \aleph(H_1). \quad (5.4)$$

Assume that $\aleph(H_1) \neq \emptyset$ and that

$$\sup_{\alpha \in \aleph(H_1)} \frac{\sum_{(X,p) \in H_1} |\alpha(X, p)|}{N(\alpha, H_1)} < \infty. \quad (5.5)$$

Also assume that, for every $\epsilon > 0$, there exists $\delta > 0$ such that if H_2 has the same cardinality as H_1 and if $q \in \mathcal{P}(H_1, H_2)$ and if

$$\sum_{(X,p) \in H_1} \left[\sup_s |X(s) - q(X, p)_1(s)| + |p - q(X, p)_2| \right] < \delta,$$

then for every $\alpha \in \aleph_0$

$$|N(\alpha, H_1) - N(\alpha_q, H_2)| < \epsilon \max_{(X,p) \in H_1} |\alpha(X, p)|, \quad (5.6)$$

where α_q is define in (5.2). Then k_3 is continuous at H_1 .

The proof of this theorem relies on the following lemma.

LEMMA 2. *Let g and h be bounded functions. If $\sup_x |g(x) - h(x)| \leq \epsilon$, then*

$$\begin{aligned} \left| \sup_x g(x) - \sup_x h(x) \right| &\leq \epsilon, \\ \left| \inf_x g(x) - \inf_x h(x) \right| &\leq \epsilon. \end{aligned}$$

PROOF. The inequality $\sup_x |g(x) - h(x)| \leq \epsilon$ implies $g(x) \leq h(x) + \epsilon$ and $h(x) \leq g(x) + \epsilon$ for all x . It follows that

$$\begin{aligned} \sup_x g(x) &\leq \sup_x h(x) + \epsilon, \\ \sup_x h(x) &\leq \sup_x g(x) + \epsilon. \end{aligned}$$

Together these imply $|\sup_x g(x) - \sup_x h(x)| \leq \epsilon$. The proof for inf is similar. \square

PROOF OF THEOREM 1. Let $L = k_3(H_1)$, which must be negative since $\aleph(H_1) \neq \emptyset$. Let $0 < \epsilon < -L/2$. Let M' be the supremum in (5.5). Let δ be the value that guarantees (5.6) for ϵ replaced by $\min\{c\epsilon/[6MM'], c/10\}$. Let $\epsilon' = \min\{c\epsilon/3, \delta/2\}$. Let H_2 be such that $d(H_1, H_2) < \epsilon'/(1 + \epsilon')$. Then $d'(H_1, H_2) < \epsilon'$. This implies that H_1 and H_2 have the same cardinality. Let $q \in \mathcal{P}(H_1, H_2)$ be such that

$$\sum_{(X,p) \in H_1} \left[\sup_s |X(s) - q(X,p)_1(s)| + |p - q(X,p)_2| \right] < d'(H_1, H_2) + \epsilon' < \delta.$$

For each $\alpha \in \aleph(H_1)$, define α_q by (5.2). We now have

$$\begin{aligned} &|k_1(s, H_1, \alpha) - k_1(s, H_2, \alpha_q)| \\ &\leq \frac{1}{N(\alpha, H_1)} \sum_{(X,p) \in H_1} \left[|\alpha(X,p)| \left\{ \sup_s |X(s) - q(X,p)_1(s)| + \right. \right. \\ &\quad \left. \left. |p - q(X,p)_2| \right\} \right] \\ &\quad + \left| \frac{1}{N(\alpha, H_1)} - \frac{1}{N(\alpha_q, H_2)} \right| \sum_{(X,p) \in H_2} |\alpha_q(X,p)| \sup_s |X(s) - p| \end{aligned} \tag{5.7}$$

For every $\alpha \in \aleph_0$, $|\alpha(X, p)|/N(\alpha, H_1) \leq 1/c$ for all $(X, p) \in H_1$ and

$$\left| \frac{1}{N(\alpha, H_1)} - \frac{1}{N(\alpha_q, H_2)} \right| = \frac{|N(\alpha, H_1) - N(\alpha_q, H_2)|}{N(\alpha, H_1)N(\alpha_q, H_2)} \leq \frac{\epsilon}{6MM'N(\alpha_q, H_2)}.$$

Since $\sup_s |X(s) - p| \leq 2M$, it follows that

$$\left| \frac{1}{N(\alpha, H_1)} - \frac{1}{N(\alpha_q, H_2)} \right| \sum_{(X,p) \in H_2} |\alpha_q(X, p)| \sup_s |X(s) - p| \leq \frac{\epsilon}{3}.$$

So, the expression on the right side of (5.7) is no greater than

$$\begin{aligned} & \frac{1}{c} \sum_{(X,p) \in H_1} \left[\sup_s |X(s) - q(X, p)_1(s)| + |p - q(X, p)_2| \right] + \frac{\epsilon}{3} \\ & \leq \frac{d'(H_1, H_2)}{c} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon. \end{aligned} \tag{5.8}$$

Now apply Lemma 2 with $g(s) = k_1(s, H_1, \alpha)$ and $h(s) = k_1(s, H_2, \alpha_q)$. Since (5.8) holds for all s , and $\alpha \in \aleph(H_1)$, we obtain. $|k_2(H_1, \alpha) - k_2(H_2, \alpha_q)| \leq \epsilon$ for all $\alpha \in \aleph(H_1)$.

Next, we show that

$$\inf_{\alpha \in \aleph(H_1)} k_2(H_2, \alpha_q) = \inf_{\beta \in \aleph(H_2)} k_2(H_2, \beta). \tag{5.9}$$

We do this by showing that each β that makes $k_2(H_2, \beta)$ sufficiently close to the infimum on the right of (5.9) must be equal to α_q for some $\alpha \in \aleph(H_1)$. Let $V = \inf_{\alpha \in \aleph(H_2)} k_2(H_2, \alpha)$. Since (5.8) holds with $\epsilon < -L/2$, we have that $V < -L/2$. Let $\beta \in \aleph_0$ satisfy $|k_2(H_2, \beta) - V| < -L/10$. Let q be as earlier in the proof, and let α be such that $\alpha_q = \beta$. We must show that $\alpha \in \aleph(H_1)$. We know that

$$\begin{aligned} & \left| \sup_s \sum_{(X,p) \in H_1} \alpha(X, p)[X(s) - p] - \sup_s \sum_{(X,p) \in H_2} \alpha_q(X, p)[X(s) - p] \right| \\ & \leq \max_{(X,p) \in H_1} |\alpha(X, p)| \frac{2c\epsilon}{3} \leq \frac{-L}{3} N(\alpha, H_1). \end{aligned}$$

We also know that

$$\sup_s \sum_{(X,p) \in H_2} \alpha_q(X, p)[X(s) - p] \leq \left(\frac{L}{2} - \frac{L}{10} \right) N(\alpha_q, H_2) = 0.4LN(\alpha_q, H_2),$$

and

$$N(\alpha_q, H_2) \geq N(\alpha, H_1) - \frac{c}{10} \max_{(X,p) \in H_1} |\alpha(X,p)| \geq 0.9N(\alpha, H_1).$$

Combining these last three inequalities (recall $L < 0$) yields

$$\sup_s \sum_{(X,p) \in H_1} \alpha(X,p)[X(s) - p] \leq 0.36LN(\alpha, H_1) - \frac{L}{3}N(\alpha, H_1) < 0.$$

It follows that $k_2(\alpha, H_1) < 0$ and $\alpha \in \mathfrak{N}(H_1)$. Hence (5.9) holds.

Finally, apply Lemma 2 with $g(\alpha) = k_2(H_1, \alpha)$ and $h(\alpha) = k_2(H_2, \alpha)$, which implies that $|k_3(H_1) - k_3(H_2)| \leq \epsilon$. \square

Continuity of \mathcal{I} will hold at all H where the conditions of Theorem 1 hold. Notice that $\mathfrak{N}(H_1) \neq \emptyset$ is equivalent to H_1 containing incoherent previsions, since $\alpha \in \mathfrak{N}(H_1)$ if and only if α makes Dutch book against H_1 . Also, (5.5) is immediate whenever the normalizing sequence is $f_n(x_1, \dots, x_n) = \sum_{i=1}^n x_i$. For smaller normalizing sequences, (5.5) continues to hold if (5.4) holds and if H_1 is a finite set.

For the other conditions, first, consider normalizations based on e_3 . These satisfy (5.4) with $c = 1$ and the left side of (5.6) is always 0. Next, consider normalizations based on e_1 . These satisfy (5.4) for each H such that for every $(X,p) \in H$, if p is a lower prevision then $\inf_s X(s) < p$ and if p is an upper prevision then $p < \sup_s X(s)$. In such cases, c can be taken as the minimum of the gaps between p and $\inf_s X(s)$ for lower previsions and between p and $\sup_s X(s)$ for upper previsions. For (5.6), δ can be taken to equal ϵ . Finally, consider normalizations based on e_2 . These satisfy (5.4) for each H such that for every $(X,p) \in H$, if p is a lower prevision then $p < \sup_s X(s)$ and if p is an upper prevision then $\inf_s X(s) < p$. In such cases, c can be taken as the minimum of the gaps between p and $\sup_s X(s)$ for lower previsions and between p and $\inf_s X(s)$ for upper previsions. For (5.6), δ can be taken to equal ϵ .

To summarize, assume either that $f_n(x_1, \dots, x_n) = \sum_{i=1}^n x_i$ or that H is a finite set. Then all normalizations based on e_3 are continuous at H . Normalizations based on e_1 are continuous at those H such that for every $(X,p) \in H$, if p is a lower prevision then $p < \sup_s X(s)$ and if p is an upper prevision then $\inf_s X(s) < p$. Normalizations based on e_2 are continuous at those H such that for every $(X,p) \in H$, if p is a lower prevision then $p < \sup_s X(s)$ and if p is an upper prevision then $\inf_s X(s) < p$.

6. Dominance

Let H and H' be two subsets of \mathcal{H} that differ only in the previsions assigned. That is, $(X, p) \in H$ if and only if there exists p' such that $(X, p') \in H'$. Throughout this section, whenever α is a strategy chosen by the gambler, we assume that $\alpha(X, p) = \alpha(X, p')$ whenever $(X, p) \in H$ and $(X, p') \in H'$.

DEFINITION 2. Say that H dominates H' with respect to α if for each state $s \in S$, $\sum_{(X,p) \in H} \alpha(X, p)[X(s) - p] > \sum_{(X,p') \in H'} \alpha(X, p')[X(s) - p']$.

DEFINITION 3. For an $\epsilon > 0$, say that H ϵ -dominates H' with respect to α if for each state $s \in S$, $\sum_{(X,p) \in H} \alpha(X, p)[X(s) - p] \geq \sum_{(X,p') \in H'} \alpha(X, p')[X(s) - p'] + \epsilon$.

Suppose that the previsions in H are incoherent and that H dominates H' with respect to α . Then the gambler is certain of a larger gain when the strategy α is employed against H' than when it is employed against H , regardless which state $s \in S$ occurs. Moreover, if H ϵ -dominates H' with respect to α , the gambler's sure gain playing α against H' is at least ϵ greater than the gain against H .

We are interested in determining which indices of incoherence respect dominance, as just defined. Recall that an index of incoherence for a set of previsions is defined by identifying a normalization N and then setting

$$\mathcal{I}(H) = \max \left\{ 0, - \inf_{\alpha: N(\alpha, H) \leq 1} \sup_s \sum_{(X,p) \in H} \alpha(X, p)[X(s) - p] \right\},$$

Consider the related quantity which we shall call a *partial index*:

$$I(H, \alpha) = \frac{\max \left\{ 0, - \sup_s \sum_{(X,p) \in H} \alpha(X, p)[X(s) - p] \right\}}{N(\alpha, H)}.$$

The partial index I is defined only when not both the numerator and denominator are 0.

None of our indices is suitable for making distinctions between coherent previsions. Indeed, the possibility of Dutch book must exist before we really care about how one set of previsions dominates another.

DEFINITION 4. Suppose that H dominates H' with respect to α and α makes Dutch book against H' . We say that the partial index I reflects the dominance of H over H' with respect to α if $I(H', \alpha) > I(H, \alpha)$. Say

that I reflects dominance if, for every α and every H and H' such that H dominates H' with respect to α and α makes Dutch book against H' , I reflects the dominance of H over H' with respect to α .

THEOREM 2. *Let the provisions in H and H' be incoherent, and suppose that H dominates H' with respect to α . Let $\mathcal{I}(H) = k > 0$. Assume that the partial index I reflects the dominance of H over H' with respect to α .*

(a) *Suppose that $I(H, \alpha) = k$, so that the gambler's strategy α achieves the infimum for the \mathcal{I} -index. Then $\mathcal{I}(H') > k$.*

(b) *Suppose that $\{\alpha_n\}_{n=1}^\infty$ is a sequence of strategies for the gambler with the property that*

$$\lim_{n \rightarrow \infty} I(H, \alpha_n) = k,$$

and that H ϵ -dominates H' with respect to each α_n . Then $\mathcal{I}(H') \geq k + \epsilon$.

PROOF. The two conclusions are immediate from the definitions above. For part (a) note that partial index I reflects dominance with respect to α . Since H dominates H' with respect to α , $I(H') \geq I(H', \alpha) > I(H, \alpha) = k$. Similarly for part (b), note that as $I(H', \alpha_n) \geq I(H, \alpha_n) + \epsilon$ then, since $\mathcal{I}(H') = \sup_\alpha I(H', \alpha)$, we have

$$\mathcal{I}(H') \geq \lim_{n \rightarrow \infty} I(H, \alpha_n) + \epsilon = \mathcal{I}(H) + \epsilon = k + \epsilon. \quad \square$$

The next result identifies conditions when each of our different partial indices reflects dominance. The example illustrate some conditions when several do not.

THEOREM 3. *Let I be the partial index of incoherence using any normalization based on e_3 . Then I reflects dominance.*

PROOF. Assume that H dominates H' with respect to α . Then, as

$$\sum_{(X,p) \in H} \alpha(X,p)[X(s) - p] > \sum_{(X,p') \in H'} \alpha(X,p')[X(s) - p'],$$

for each $s \in S$, then

$$\begin{aligned} I(H, \alpha) &= -\frac{\sup_s \sum_{(X,p) \in H} \alpha(X,p)[X(s) - p]}{N(\alpha, H)} \\ &< -\frac{\sup_s \sum_{(X,p') \in H'} \alpha(X,p')[X(s) - p']}{N(\alpha, H)} = I(H', \alpha). \quad \square \end{aligned}$$

THEOREM 4. *Let I be the “rate of loss” partial index using for its normalizer: $N(\alpha, H) = \sum_{(X,p) \in H} \max \{0, -\inf_s \alpha(X, p)[X(s) - p]\}$. Suppose that there is at least one $(X, p) \in H$ such that X is not a constant and $\alpha(X, p) \neq 0$. Also, assume that, for each $(X, p) \in H$, $\inf_s \alpha(X, p)[X(s) - p] < 0$. (That is, none of the gambles are guaranteed winners for the bookie.) If H dominates H' with respect to α and α makes Dutch book against H' and $N(\alpha, H) > 0$, then I reflects the dominance of H over H' with respect to α .*

We begin with a lemma about the partial index I .

LEMMA 3. *Suppose that $N(\alpha, H) > 0$ and that there exists at least one $(X, p) \in H$ such that $\alpha(X, p) \neq 0$ and X is not constant. Then $I(H, \alpha) < 1$ for “rate of loss.”*

PROOF. If at least one of the random variables is not constant, then it follows by elementary considerations of inf and sup that

$$\begin{aligned} & -\sup_s \sum_{(X,p) \in H} \alpha(X, p)[X(s) - p] \\ & < \sum_{(X,p) \in H} \max \left\{ 0, -\inf_s \alpha(X, p)[X(s) - p] \right\} = N(\alpha, H). \end{aligned}$$

Then the lemma is immediate from the assumption that $N(\alpha, H) > 0$. \square

PROOF OF THEOREM 4. Let H and H' be two sets of random variables and previsions that differ only in the previsions and let α be a strategy for the gambler that makes Dutch book against H' . Suppose that H dominates H' with respect to α . If α does not make Dutch book against H , then $I(H, \alpha) = 0 < I(H', \alpha)$, and we are done. For the remainder of the proof, assume that α makes Dutch book against both H and H' .

For every s ,

$$\begin{aligned} & \sum_{(X,p) \in H} \alpha(X, p)[X(s) - p] - \sum_{(X,p') \in H'} \alpha(X, p')[X(s) - p'] \\ & = \sum_{(X,p) \in H} \alpha(X, p)(p' - p) > 0, \end{aligned} \tag{6.1}$$

because H dominates H' with respect to α . Since none of the gambles is a guaranteed winner for the bookie,

$$N(\alpha, H') - N(\alpha, H)$$

$$\begin{aligned}
&= \sum_{(X,p') \in H'} -\inf_s \alpha(X,p')[X(s) - p'] - \sum_{(X,p) \in H} -\inf_s \alpha(X,p)[X(s) - p] \\
&= \sum_{(X,p) \in H} \alpha(X,p)(p' - p).
\end{aligned}$$

For convenience, let $G = \sup_s \sum_{(X,p) \in H} \alpha(X,p)[X(s) - p]$. Since α makes Dutch book against H' , we have

$$\begin{aligned}
I(H', \alpha) &= \frac{-\sup_s \sum_{(X,p') \in H'} \alpha(X,p')[X(s) - p']}{N(\alpha, H')} \\
&= \frac{-G + \sum_{(X,p) \in H} \alpha(X,p)(p' - p)}{N(\alpha, H) + N(\alpha, H') - N(\alpha, H)} \\
&= \frac{-G + \sum_{(X,p) \in H} \alpha(X,p)(p' - p)}{N(\alpha, H) + \sum_{(X,p) \in H} \alpha(X,p)(p' - p)}. \tag{6.2}
\end{aligned}$$

Lemma 3 tells us that

$$I(H, \alpha) = \frac{-G}{\sum_{(X,p) \in H} \alpha(X,p)(p' - p)} < 1. \tag{6.3}$$

If we add the positive number

$$\sum_{(X,p) \in H} \alpha(X,p)(p' - p)$$

to both the numerator and denominator of the expression for $I(H, \alpha)$ in (6.3), the ratio gets closer to 1 (increases) and we obtain the ratio in (6.2). Hence, $I(H', \alpha) > I(H, \alpha)$. \square

Next, we show by example that the “rate of loss” index does not reflect dominance when the normalization uses the maximum (rather than the sum) of the individual normalizations. That is, for the next example let

$$N(\alpha, H) = \max_{(X,p) \in H} \left\{ 0, -\inf_s \alpha(X,p)[X(s) - p] \right\}.$$

EXAMPLE 2. Let $S = \{s_1, s_2, s_3\}$ and let X_i ($i = 1, 2, 3$) be the indicator for s_i . Let $H = \{(X_1, 0.5), (X_2, 0.5), (X_3, 0.5)\}$ and let $H' = \{(X_1, 0.9), (X_2, 0.4), (X_3, 0.4)\}$. Let the gambler’s strategy be $\alpha = (1, 1, 1)$. Then $\sum_{(X,p) \in H} \alpha(X,p)[X(s_i) - p] = .5$, and $\sum_{(X,p') \in H'} \alpha(X,p')[X(s_i) - p'] = .7$ for each state s_i . Hence, H dominates H' with respect to α . Also, $N(\alpha, H) = .5$, whereas $N(\alpha, H') = .9$, so that $I(H, \alpha) = .5/.5 = 1 > .7/.9 = I(H', \alpha)$,

contrary to the ranking required to reflect dominance. In addition, based on Theorem 6 (part 2), we see that this example also fails the conclusion to Theorem 2 (part 1) even though H dominates H' as in the hypothesis of Theorem 2.

Last, we establish conditions under which the “rate of profit” index reflects dominance between sets of prevision.

THEOREM 5. *Let I be the “rate of profit” index of incoherence using for its normalizer $N(\alpha, H) = \sum_{(X,p) \in H} \max \{0, \sup_s \alpha(X, p)[X(s) - p]\}$. Assume that, for each $(X, p) \in H$, $\sup_s \alpha(X, p)[X(s) - p] > 0$. (That is, none of the gambles are guaranteed winners for the gambler.) If H dominates H' with respect to α and α makes Dutch book against H' and $N(\alpha, H) > 0$, then I reflects the dominance of H over H' with respect to α .*

PROOF. As in the proof of Theorem 4, let H and H' be two sets of random variables and previsions that differ only in the previsions, and let α be a strategy for the gambler that makes Dutch book against H' . As before, if α does not make Dutch book against H , then $I(H, \alpha) = 0 < I(H', \alpha)$. So assume for the rest of the proof that α makes Dutch book against both H and H' . Equation (6.1) still holds in this case. Since none of the gambles is a guaranteed winner for the gambler, we have that

$$\begin{aligned} &N(\alpha, H') - N(\alpha, H) \\ &= \sum_{(X,p') \in H'} \sup_s \alpha(X, p')[X(s) - p'] - \sum_{(X,p) \in H} \sup_s \alpha(X, p)[X(s) - p] \\ &= \sum_{(X,p) \in H} \alpha(X, p)(p - p'). \end{aligned}$$

As before, let $G = \sup_s \sum_{(X,p) \in H} \alpha(X, p)[X(s) - p]$. Now write

$$\begin{aligned} I(H', \alpha) &= \frac{-\sup_s \sum_{(X,p') \in H'} \alpha(X, p')[X(s) - p']}{N(\alpha, H) + N(\alpha, H') - N(\alpha, H)} \\ &= \frac{-G + \sum_{(X,p) \in H} \alpha(X, p)(p' - p)}{N(\alpha, H) + \sum_{(X,p) \in H} \alpha(X, P)(p - p')}. \end{aligned} \tag{6.4}$$

Since $\sum_{(X,p) \in H} \alpha(X, p)(p' - p) > 0$, the numerator of (6.4), is greater than $-G$ and the denominator is less than $N(\alpha, H)$, so the ratio is greater than $-G/N(\alpha, H)$. As $I(H, \alpha) = -G/N(\alpha, H)$, we have $I(H', \alpha) > I(H, \alpha)$. \square

7. Some Special Cases

In this section, we provide some theorems that calculate $\mathcal{I}(H)$ for some specific cases of interest. These cases concern three different normalizations e_1 , e_2 and e_3 and a single collection H consisting of the elements of a partition of S together with incoherent upper or lower previsions. For each normalization e , we shall use the same normalizing sequence $\{f_n\}_{n=1}^\infty$ defined by $f_n(x_1, \dots, x_n) = \sum_{i=1}^n x_i$ and define $N(\alpha, H)$ by (5.3). In each of the next three theorems $\{A_i\}_{i=1}^n$ is a partition of S into nonempty sets. We let q_1, \dots, q_n stand for upper previsions for A_1, \dots, A_n and we let p_1, \dots, p_n stand for lower previsions. Each function α is equivalent to an n -tuple $(\alpha_1, \dots, \alpha_n)$, which we shall denote α in these proofs. We also let $p_{(1)} \leq \dots \leq p_{(n)}$ be the ordered values of the lower previsions and define

$$s^+ = \sum_{i=1}^n q_i, \quad s^- = \sum_{i=1}^n p_i.$$

Assume that all p_i and q_i are nonnegative. Finally, in all of the proofs, we will let g stand for the function

$$g(\alpha) = \sup_s \sum_{i=1}^n (I_{A_i} - x_i),$$

where either $x_i = p_i$ for all i or $x_i = q_i$ for all i . This then makes

$$\mathcal{I}(H) = \max \left\{ 0, -\inf_{\alpha} g(\alpha_1, \dots, \alpha_n) \right\}.$$

So the proofs consist of minimizing g subject to $N(\alpha, H) \leq 1$.

THEOREM 6. *For the bookie's escrow e_1 ,*

(a) *Let $H = \{(I_{A_1}, q_1), \dots, (I_{A_n}, q_n)\}$, and assume that $s^+ < 1$. Then*

$$\mathcal{I}(H) = \frac{1 - s^+}{n - s^+},$$

and α_j all equal to each other achieves this value. If $0 < q_i < 1$ for all i , then the $\alpha_1, \dots, \alpha_n$ that achieve the value of $\mathcal{I}(H)$ are unique.

(b) *Let $H = \{(I_{A_1}, p_1), \dots, (I_{A_n}, p_n)\}$, and assume that $s^- > 1$. Then*

$$\mathcal{I}(H) = \frac{s^- - 1}{s^-},$$

and all α_j equal to each other achieves this value. If $0 < p_i < 1$ for all i , then the $\alpha_1, \dots, \alpha_n$ that achieve the value of $\mathcal{I}(H)$ are unique.

PROOF. For part (a), we can write $g(\alpha) = \max\{\alpha_1, \dots, \alpha_n\} - c$, where $c = \sum_{i=1}^n \alpha_i q_i$. Let $\alpha_j = \max\{\alpha_1, \dots, \alpha_n\}$, so that $g(\alpha) = \alpha_j - c$. Then

$$g(\alpha_j, \dots, \alpha_j) = \alpha_j - c + \sum_{i \neq j} (\alpha_i - \alpha_j) q_i \leq g(\alpha) \tag{7.1}$$

since $\alpha_i \leq \alpha_j$ for all $i \neq j$ and $q_i \geq 0$. Also,

$$N(\alpha_j, \dots, \alpha_j, H) = N(\alpha, H) + \sum_{i \neq j} (\alpha_i - \alpha_j)(1 - q_i) \leq N(\alpha_1, \dots, \alpha_n, H),$$

since $\alpha_i \leq \alpha_j$ for all $i \neq j$ and $1 - q_i \geq 0$. It follows that g is minimized (subject to $N(\alpha, H) \leq 1$) by setting all α_i equal to each other and rescaling them to make $h = 1$. Uniqueness in the case $0 < q_i$ for all $i = 1, \dots, m$ follows from the fact that the inequality in (7.1) is strict if all $q_i > 0$ and the α_i are not all equal.

For part (b), let $c = \sum_{i=1}^n \alpha_i p_i$. Then

$$g(\alpha) = \max\{\alpha_1, \dots, \alpha_n\} - c.$$

Clearly, having all $\alpha_i = 0$ does not achieve the value of $\mathcal{I}(H)$, so assume that at least one $\alpha_i > 0$. In such cases, $N(\alpha, H) = \sum_{i=1}^m \alpha_i p_i = c > 0$. Clearly, if $c < 1$, we can make $g(\alpha)$ smaller (strictly smaller if all $p_i > 0$) by scaling up the α_i to make $c = 1$. So assume that $c = 1$. Now, $g(\alpha) = \max\{\alpha_1, \dots, \alpha_m\} - 1$. We minimize g by making the largest α_i as small as possible subject to $\sum_{i=1}^m \alpha_i p_i = 1$. If the α_i are not all equal, then it is easy to see that we can lower the largest ones by raising the smallest ones while maintaining the constraint $\sum_{i=1}^m \alpha_i p_i = 1$. If $0 < p_i$ for all $i = 1, \dots, m$, then this maneuver strictly lowers g . This implies that g is minimized by choosing all of the α_i equal to the same value, which must, by the constraint, be the value $1/\sum_{i=1}^m p_i$. Plugging this value for all α_i into the formula for g yields the value of $\mathcal{I}(H)$ stated in the theorem. Uniqueness in the case $0 < p_i$ for all $i = 1, \dots, m$ follows from the series of strict decreases in g that occurred in the above discussion. \square

THEOREM 7. *For the gambler’s escrow e_2 ,*

(a) *Let $H = \{(I_{A_1}, q_1), \dots, (I_{A_n}, q_n)\}$, and assume that $s^+ < 1$. Then*

$$\mathcal{I}(H) = \frac{1 - s^+}{s^+},$$

and α_j all equal to each other achieves this value.

(b) Let $H = \{(I_{A_1}, p_1), \dots, (I_{A_n}, p_n)\}$, and assume that $s^- > 1$. Let k^* be the first k such that $p_{(n-k)} \leq (\sum_{i=n-k+1}^n p_{(i)} - 1)/(k-1)$ ($k^* = n$ if the inequality is never satisfied). Then

$$\mathcal{I}(H) = \frac{\sum_{i=n-k^*+1}^n p_{(i)} - 1}{\sum_{i=n-k^*+1}^n (1 - p_{(i)})}.$$

To achieve this value, set all α_i corresponding to $p_{(n-k^*+1)}, \dots, p_{(n)}$ equal to the same positive number and all other $\alpha_i = 0$.

PROOF. For part (a), we see that

$$g(\boldsymbol{\alpha}) = \max\{\alpha_1, \dots, \alpha_n\} - c,$$

where $c = \sum_{i=1}^n \alpha_i q_i$. Notice that $N(\boldsymbol{\alpha}, H) = -c$. It is clear that $N(\boldsymbol{\alpha}, H) = 0$ implies $g = 0$, which is not the smallest possible value. If $0 < N(\boldsymbol{\alpha}, H) < 1$ we can scale up all the α_i to make $N(\boldsymbol{\alpha}, H) = 1$ and make g smaller, so assume that $c = -1$. This makes g equal to the largest α_i plus 1. Set $\beta_i = -\alpha_i$ for each i and maximize $\min\{\beta_1, \dots, \beta_n\}$ subject to $\sum_{i=1}^n \beta_i p_i = 1$. If the β_i are not all equal, then we can raise the lowest ones and lower the highest ones while maintaining the constraint. It follows that g is minimized by setting all α_j equal to $-1/s^+$.

For part (b), write

$$g(\boldsymbol{\alpha}) = \max\{\alpha_1, \dots, \alpha_n\} - \sum_{i=1}^n \alpha_i p_i.$$

It is clear that $N(\boldsymbol{\alpha}, H) = \sum_{i=1}^n \alpha_i (1 - p_i)$. Note that $N(\boldsymbol{\alpha}, H) = 0$ implies $g = 0$, which is not the smallest possible value. Hence, we can assume that $\sum_{i=1}^n \alpha_i (1 - p_i) > 0$. This allows us to replace the constrained minimization problem by the minimization of $R(\boldsymbol{\alpha}) = g(\boldsymbol{\alpha})/N(\boldsymbol{\alpha}, H)$. For each i such that α_i is not the largest value, consider the effect on R of replacing α_i by $\alpha_i + \epsilon$. Let e_i be the unit vector with 1 in the i th coordinate and 0 elsewhere. Then

$$R(\boldsymbol{\alpha} + \epsilon e_i) = \frac{g(\boldsymbol{\alpha}) - \epsilon p_i}{N(\boldsymbol{\alpha}) + \epsilon(1 - p_i)}. \quad (7.2)$$

It is straightforward from (7.2) that $R(\boldsymbol{\alpha} + \epsilon e_i)$ is smaller than $R(\boldsymbol{\alpha})$ if $\epsilon > 0$ and $p_i/(1 - p_i) > -R(\boldsymbol{\alpha})$ or if $\epsilon < 0$ and $p_i/(1 - p_i) < -R(\boldsymbol{\alpha})$. So, we make R smaller by raising all α_i corresponding to large p_i and by lowering all α_i corresponding to small p_i . To see where the break between large and small p_i occurs, assume that $p_1 \leq \dots \leq p_n$. Clearly, the largest values of i should

have the largest values of α_i . Also, in order for $R(\alpha) < 0$, we need enough nonzero α_i so that the sum of the corresponding p_i is greater than 1. So, start with the first k such that $\sum_{i=n-k+1}^n p_i > 1$. Set $\alpha_n = \dots = \alpha_{n-k+1} = 1$ and $\alpha_1 = \dots = \alpha_{n-k} = 0$. Then $-R(\alpha) = (\sum_{i=n-k+1}^n p_i - 1) / \sum_{i=n-k+1}^n (1 - p_i)$. Notice that $p_i / (1 - p_i) > -R(\alpha)$ if and only if $p_i > (\sum_{i=n-k+1}^n p_i - 1) / (k - 1)$. If $p_{n-k} > (\sum_{i=n-k+1}^n p_i - 1) / (k - 1)$, we should set $\alpha_{n-k} = 1$, otherwise leave $\alpha_{n-k} = 0$. Also, notice that, if $p_{n-k} > (\sum_{i=n-k+1}^n p_i - 1) / (k - 1)$, then $(\sum_{i=n-k}^n p_i - 1) / k > (\sum_{i=n-k+1}^n p_i - 1) / (k - 1)$. So, we should let k^* be the first k such that $p_{n-k} \leq (\sum_{i=n-k+1}^n p_i - 1) / (k - 1)$ and then set $\alpha_i = 1 / \sum_{j=n-k^*+1}^n (1 - p_j)$ for $i \geq n - k^* + 1$ and $\alpha_i = 0$ for $i \leq n - k^*$. \square

THEOREM 8. *For the neutral escrow e_3 ,*

(a) *Let $H = \{(I_{A_1}, q_1), \dots, (I_{A_n}, q_n)\}$, and assume that $s^+ < 1$. Then*

$$\mathcal{I}(H) = \frac{1 - s^+}{n},$$

and α_j all equal to each other achieves this value.

(b) *Let $H = \{(I_{A_1}, p_1), \dots, (I_{A_n}, p_n)\}$, and assume that $s^- > 1$. let k^* be the first k such that $p_{(n-k)} \leq (\sum_{i=n-k+1}^n p(i) - 1) / k$ ($k^* = n$ if the inequality is never satisfied). Then*

$$\mathcal{I}(H) = \frac{\sum_{i=n-k^*+1}^n p(i) - 1}{k^*}.$$

To achieve this value, set all α_i corresponding to $p_{(n-k^+1)}, \dots, p_{(n)}$ equal to the same positive number and all other $\alpha_i = 0$.*

PROOF. For part (a), define $\beta_i = -\alpha_i$ for $i = 1, \dots, n$. Then

$$g(\alpha) = -\min\{\beta_1, \dots, \beta_n\} + \sum_{i=1}^n \beta_i q_i.$$

Here $N(\alpha, H) = \sum_{i=1}^n \beta_i$. We need to minimize g subject to $\sum_{i=1}^n \beta_i \leq 1$. If there is j such that $\beta_j > \min\{\beta_1, \dots, \beta_n\}$, we can replace β_j by $\min\{\beta_1, \dots, \beta_n\}$ which will lower g and lower $N(\alpha, H)$. Then scale up all the β_i so that $N(\alpha, H) = 1$ and this will lower g even more. Hence we need all β_i equal to a common value. Setting all $\alpha_i = -1/n$ makes $\mathcal{I}(H) = (1 - s^+) / n$.

For part (b), write

$$g(\alpha) = \max\{\alpha_1, \dots, \alpha_n\} - \sum_{i=1}^n \alpha_i p_i.$$

We need to minimize this subject to $N(\boldsymbol{\alpha}, H) = \sum_{i=1}^n \alpha_i = 1$ with all $\alpha_i \geq 0$. Since setting all $\alpha_i = 0$ clearly does not provide the minimum, we can assume that $\sum_{i=1}^n \alpha_i > 0$. This allows us to replace the constrained minimization problem by the minimization of $R(\boldsymbol{\alpha}) = g(\boldsymbol{\alpha})/N(\boldsymbol{\alpha}, H)$. For each i such that α_i is not the largest value, consider the effect on R of replacing α_i by $\alpha_i + \epsilon$. Let e_i be the unit vector with 1 in the i th coordinate and 0 elsewhere. Then

$$R(\boldsymbol{\alpha} + \epsilon e_i) = \frac{g(\boldsymbol{\alpha}) + \epsilon p_i}{N(\boldsymbol{\alpha}) + \epsilon}. \quad (7.3)$$

It is clear from (7.3) that $R(\boldsymbol{\alpha} + \epsilon e_i)$ is smaller than $R(\boldsymbol{\alpha})$ if $\epsilon > 0$ and $p_i > -R(\boldsymbol{\alpha})$ or if $\epsilon < 0$ and $p_i < -R(\boldsymbol{\alpha})$. So, we make R smaller by raising all α_i corresponding to large p_i and by lowering all α_i corresponding to small p_i . To see where the break between large and small p_i occurs, assume that $p_1 \leq \dots \leq p_n$. Clearly, the largest values of i should have the largest values of α_i . Also, in order for $R(\boldsymbol{\alpha}) < 0$, we need enough nonzero α_i so that the sum of the corresponding p_i is greater than 1. So, start with the first k such that $\sum_{i=n-k+1}^n p_i > 1$. Set $\alpha_n = \dots = \alpha_{n-k+1} = 1$ and $\alpha_1 = \dots = \alpha_{n-k} = 0$. Then $-R(\boldsymbol{\alpha}) = (\sum_{i=n-k+1}^n p_i - 1)/k$. If $p_{n-k} > (\sum_{i=n-k+1}^n p_i - 1)/k$ we should set $\alpha_{n-k} = 1$, otherwise leave $\alpha_{n-k} = 0$. Also, notice that, if $p_{n-k} > (\sum_{i=n-k+1}^n p_i - 1)/k$, then $(\sum_{i=n-k}^n p_i - 1)/(k+1) > (\sum_{i=n-k+1}^n p_i - 1)/k$. So, we should let k^* be the first k such that $p_{n-k} \leq (\sum_{i=n-k+1}^n p_i - 1)/k$ and then set $\alpha_i = 1/k^*$ for $i \geq n - k^* + 1$ and $\alpha_i = 0$ for $i \leq n - k^*$. \square

EXAMPLE 3. Consider three different incoherent lower previsions for the elements of a partition of S into three events: $(0.5, 0.5, 0.5)$, $(0.6, 0.7, 0.2)$, and $(0.6, 0.8, 0.1)$. In all three cases, $s^+ = 1.5$. For the first case, all three of our theorems say that $\mathcal{I}(H)$ is maximized by setting $\boldsymbol{\alpha} = (1, 1, 1)$. For the second case, both Theorems 6 and 8 say that $\mathcal{I}(H)$ is maximized at $\boldsymbol{\alpha} = (1, 1, 1)$, but Theorem 7 says that $\mathcal{I}(H)$ is maximized at $\boldsymbol{\alpha} = (1, 1, 0)$. For the third case, only Theorem 6 says that $\mathcal{I}(H)$ is maximized at $\boldsymbol{\alpha} = (1, 1, 1)$, while the other two say that $\mathcal{I}(H)$ is maximized at $\boldsymbol{\alpha} = (1, 1, 0)$.

For general finite sets of gambles defined on finite partitions, one can construct a linear programming problem whose solution provides both the rate of incoherence and the set of bets required to achieve that rate. Willis (1964) constructs such a linear programming problem to determine a gambler's strategy at a race track that posts incoherent pari-mutuel odds.

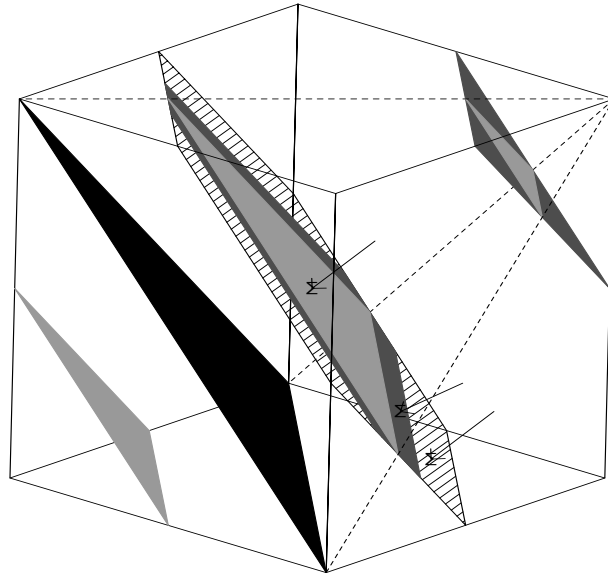


Fig. 1. Sets of bookie's previsions for which the gambler uses two or three events depending on the normalization. The black simplex is the set of coherent previsions. In the light gray shaded regions, the gambler bets on all three events, regardless of normalization. In the dark Gray shaded regions, the gambler bets on all three events for the neutral and bookie-escrow normalizations, but bets on only two events for the gambler-escrow normalization. In the line-shaded area, the gambler bets on only two events for the neutral and gambler-escrow normalizations. The three previsions in Example 3 are indicated in the plane where $s^+ = 1.5$. Also, the figure illustrates the simplex where three individually coherent previsions sum to 2.5.

8. Discussion

In this paper we have introduced a collection of three (families of) indices for gauging the incoherent sets of one-sided and two-sided previsions, using different (families of) normalizations to standardize the gains from a Dutch Book. Within each family, an index is based on a common escrow for an individual wager, though the normalizations within a family differ on the normalization for a set of wagers. Within each family, based on the 7 conditions of adequacy proposed in Section 4, the smallest normalization is the maximum of the individual escrows, and the largest normalization is the sum of the individual escrows.

One index family uses the bookie's escrow for a single wager (the bookie's maximum possible loss on that wager) as the basis for the normalization, and leads to what we call the "rate of loss" index. A second family uses the gambler's escrow as the basis for the normalization, and leads to what we call the "rate of profit" index. A third index family uses a neutral point of view, where in the case of simple bets, the normalization is based on the magnitude of the total stake of the bet.

In Section 5 we investigate those indices that are a continuous function of the gambler's previsions and the random variables for which these are previsions. We show that the index family based on the neutral point of view is continuous, as is the rate of loss index when the random variables are not constant. The index corresponding to the rate of profit is continuous when each (one or two-sided) prevision is coherent on its own.

In Section 6 we consider which of our indices reflects dominance among incoherent sets of (one and two-sided) previsions. We establish that the index-family based on the neutral point of view reflects dominance. The rate of loss index also does for non-constant previsions when the normalization is the largest from that family, using the sum of the bookie's escrows. The rate of profit likewise reflects dominance when individual previsions are coherent and using the same (sum-of-escrows) normalization.

In Section 7 we illustrate the gambler's strategies for achieving the respective indices. We use the simple case of incoherent previsions for events (indicator functions) that form a finite partition. The gambler's optimal strategies for these indices are distinct even for the case of incoherent (lower) previsions on a three-element partition, as illustrated in Figure 1.

This paper reports some of the basic findings for a theory of rates of incoherence. Other results, including rates of incoherence when some gambles are called-off, can be found in Schervish, Seidenfeld and Kadane (1997). We have already applied the rate of loss index to assess the well known incoherence of Classical Neyman-Pearson testing of a simple null hypothesis versus a simple alternative at a fixed level of type-1 error, regardless of the sample size (Schervish, Seidenfeld and Kadane, 2002). We do this by comparing the risk functions of different tests under different conditions. Such comparisons are generally accepted by classical statisticians concerned with admissibility. In future work, we hope to use the indices presented here to gauge the degree of incoherence of other, well known Classical Statistical procedures.

Acknowledgements. The authors would like to thank a referee for helpful suggestions. This research was supported by NSF grant DMS-9801401.

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