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NONPARAMETRIC INFERENCE FOR REPAIR MODELS

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SUMMARY. We survey some recent results pertaining to nonparametric inference for repair models. Of particular interest is the development of estimators and simultaneous confidence bands for the distribution F of the time to first failure. Models considered include the minimal repair model, the Block, Borges and Savits (1985) model, and the general models of Dorado, Hollander and Sethuraman (1997), and Last and Szekli (1998).

1. Introduction

When an item fails in the field, time and cost considerations typically preclude replacement with a brand new item, though this would be preferable to maintain high reliability. Instead, the item is repaired and put back in use with a new effective age that depends on the nature of the repair. Thus it is of interest to estimate the life distribution F (and associated cumulative hazard function and failure rate) of time to first failure of a brand new item using data on repeated failures where, after each failure, some type of repair has been performed.

Repair models are probabilistic structures that specify the joint distribution of the failure times of the item (system) under consideration. These models vary in the manner in which the effective age of the item is determined as a consequence of the repair. Under minimal repair, for example, after a failure at age t the item is returned to the state of a working item of age t. In the Brown and Proschan (1983) model, when an item fails, with probability p a perfect repair is performed, so that the item is returned to

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the state of a new item, and with probability $1-p$, a minimal repair is performed. Block, Borges and Savits (1985) generalize the Brown-Proschan model so that p is age-dependent. That is when an item fails at age t, a perfect repair is performed with probability $p(t)$ and a minimal repair is performed with probability $1-p(t)$. Kijima (1989) introduced models that feature "degree of repair" random variables which allow for repairs that are between perfect and minimal, that is, repairs that could do better than minimal repair without necessarily restoring the item to a state equivalent to that of a new item. Dorado, Hollander, and Sethuraman (1997) introduced a general repair model that incorporates "better than minimal" repair as well as a boost to the residual life distribution of the repaired item. These notions are made precise in Section 2, where other repair models are also specified.

Section 2 presents examples of various repair models. Section 3 considers nonparametric inference for the general repair models of Dorado, Hollander and Sethuraman (1997) and Last and Szekli (1998). Section 4 considers nonparametric inference for the age-dependent Block, Borges and Savits (1985) model, a model that is not contained in the general DHS and LS models.

2. Examples of Repair Models

A system is the collection of failure times of an item that is instantly and repeatedly repaired at each failure according to some repair model. Let $\{S_j\}$ denote the failure times of a given system. To define the repair model, one needs to specify the joint distribution of the $\{S_j\}$. Let F be the distribution of the time to first failure of the system and let $T_j = S_j - S_{j-1}$ be the interfailure times. The repair models we consider postulate that the distributions of the interfailure times depend in some way on F and other evolving features relating to the nature of previous repairs. Some examples are as follows.

(i) Perfect Repair Model

In the perfect repair model, upon failure, a failed system is replaced by a brand new one identical to the original, so that $T_1, T_2, \ldots$ are independent and identically distributed (iid) according to F.

(ii) Minimal Repair Model

In the minimal repair model, the repair restores the system to its state just before failure. Under this model, $\{S_j\}$ is a Markov process with

$$P(S_j > x | S_{j-1} = y) = \frac{\bar{F}(x)}{\bar{F}(y)}, \qquad x > y. \tag{2.1}$$

(iii) Brown-Proschan (1983) Model

In the Brown and Proschan (1983) model, two types of repairs are possible at the time of each repair. With probability p, a perfect repair is performed, and with probability $1 - p$, a minimal repair is performed.

(iv) Block-Borges-Savits (1985) Model

Block, Borges and Savits (1985) generalize the Brown-Proschan model to allow age-dependent repairs. They allow the probability p to depend on the age of the failed system. The probability of a perfect repair is $p(t)$ where $p(\cdot)$ is a measurable function from $[0, \infty)$ to $[0, 1]$. Block, Borges and Savits showed that for continuous F, the condition

$$\int_{(0,\infty)} \frac{p(t)}{\bar{F}(t)} dF(t) = \infty \tag{2.2}$$

ensures that the waiting time between perfect repairs is almost surely finite with distribution

$$H(t) = 1 - \exp\left\{ -\int_0^t \frac{p(s)}{\bar{F}(s)} dF(s) \right\}, \qquad t \geq 0. \tag{2.3}$$

(Many a puzzled repairman has had to abandon the BBS model during repair because of lack of training for checking the measurability condition.)

(v) Kijima's (1989) Models

In Kijima's (1989) models I and II, the effective age A_j, $j = 1, 2, \ldots$, to which the system is restored after repair depends not only in its age just before failure but also on the degree of repair random variables D_j, $j = 1, 2, \ldots$. It will be assumed that the D's are independently distributed on $[0, 1]$ and independent of other processes.

The effective age A_j at the time of the $(j-1)^{\text{th}}$ failure will depend only on $T_1, \ldots, T_{j-1}, D_1, \ldots, D_{j-1}$. We will define $A_1 = 0$. The distribution of T_j, the j^{th} interfailure time given $T_1, \ldots, T_{j-1}, D_1, \ldots, D_{j-1}$, will depend only on the effective age A_j. In Kijima's model I, it is given by

$$P(T_j > x | T_1, \ldots, T_{j-1}, D_1, \ldots, D_{j-1}) = \frac{\bar{F}(x + A_j)}{\bar{F}(A_j)} \tag{2.4}$$

where

$$A_j = \sum_{i=1}^{j-1} D_i T_i, \qquad j > 1. \tag{2.5}$$

Note that with this specification

$$A_{j+1} = A_j + D_j T_j. \tag{2.6}$$

In Kijima's model II, $P(T_j > x | T_1, \ldots, T_{j-1}, D_1, \ldots, D_{j-1})$ is again given by (2.4) but with the specification

$$A_j = \sum_{k=1}^{j-1} \left(\prod_{i=k}^{j-1} D_i \right) T_k, \qquad j > 1. \tag{2.7}$$

Note that this choice,

$$A_{j+1} = D_j(A_j + T_j). \tag{2.8}$$

When $D_j = 1$ with probability p and $= 0$ with probability $1-p$, this reduces to the Brown-Proschan model.

(vi) Dorado-Hollander-Sethuraman (1997) Model

Dorado, Hollander and Sethuraman (1997) (hereafter DHS (1997)) introduced a general repair model that depends on two sequences, viz. the effective ages $\{A_j\}$ and the life supplements $\{\theta_j\}$. The idea is that not only can the repairman restore the item to an effective age that is less than the item's age just before failure, but he can also give the residual life distribution a beneficial supplement or boost.

For any distribution F, $\theta \in (0, 1]$ and $a \in [0, \infty)$, let the survival function $\bar{F}_{a,\theta}(x)$ be defined as

$$\bar{F}_{a,\theta}(x) = \frac{\bar{F}(a + \theta x)}{\bar{F}(a)}, \qquad x > 0. \tag{2.9}$$

The survival function $\bar{F}_{a,\theta}(x)$ gives the probability that an item that has lived a units of time will survive x more units of time when boosted by the supplement θ. Values of $\theta < 1$ correspond to longer lifetimes, $\theta = 1$ corresponds to no boost, and values of $\theta > 1$ correspond to deterioration.

Assume that the $\{A_j\}$, $\{\theta_j\}$ sequences satisfy

$$\begin{aligned} &A_1 = 0,\, \theta_1 = 1,\, A_j \geq 0, \quad \theta_j \in (0, 1] \\ &\text{and} \;\; A_j \leq A_{j-1} + \theta_{j-1} T_{j-1}, \qquad j \geq 2. \end{aligned} \tag{2.10}$$

The DHS (1997) model specifies the joint distribution of the $\{T_j\}$ as follows:

$$P\left(T_j \leq t | A_1, \ldots, A_j, \theta_1, \ldots, \theta_j, T_1, \ldots, T_{j-1}\right) = F_{A_j,\theta_j}(t), \tag{2.11}$$

for $t > 0$, $j \geq 2$. Under this model, when $\theta \leq 1$, we see that for $j \geq 1$, the effective age A_{j+1}, of the system after the j^{th} repair, is less than the effective age $X_j \stackrel{\text{def}}{=} A_j + \theta_j T_j$ just before the j^{th} failure which in turn is less than the actual age S_j.

The DHS (1997) model is quite general and contains many other models. For example, if we set $\theta_j = 1$, $A_j = 0$ for $j \geq 1$, we obtain the perfect repair model

$$P(T_j > t | T_1, \ldots, T_{j-1}) = \bar{F}(t).$$

If we set $\theta_j = 1$, $A_j = S_{j-1}$, $j \geq 1$, we obtain the minimal repair model

$$P(T_j > t | S_{j-1}) = \bar{F}_{S_{j-1},1}(t).$$

If we set $\theta_j = 1$ for each j and let A_j be defined by (2.5) so that $A_{j+1} = A_j + D_j T_j$, we obtain the Kijima I model,

$$P(T_j > t | T_i, D_i, 1 \leq i \leq j-1) = \bar{F}_{A_j,1}(t).$$

If we set $\theta_j = 1$ for each j and let A_j be defined by (2.7) so that $A_{j+1} = D_j(A_j + T_j)$, we obtain the Kijima II model,

$$P(T_j > t | T_i, D_i, 1 \leq i \leq j-1) = \bar{F}_{A_j,1}(t). \tag{2.12}$$

If we set $\theta_1 = 1$, $A_j = \sum_{i=1}^{j-1} \theta_i T_i$ and $0 < \theta_j < 1$ for $j > 1$ we obtain the supplemented life repair model. The term "supplemented life" is motivated as follows. If a minimal repair were performed at the time of the first failure, T_2 would have the distribution $F_{T_1,1}$. We can, however, provide a longer expected life for T_2 if we use the distribution F_{T_1,θ_2} for some θ_2 satisfying $0 < \theta_2 < 1$. Starting with the distribution F_{T_1,θ_2} for T_2 and using minimal repair after the second failure, T_3 would have the distribution $F_{A_3,1}$ where $A_3 = T_1 + \theta_2 T_2$. If we want a longer expected life for T_3 we can use the distribution F_{A_3,θ_3} for some $0 < \theta_3 < 1$. Continuing in this fashion yields the supplemented life model.

(vii) Last-Szekli (1998) Model

The restriction $\theta_j \in (0, 1]$ imposed by the DHS (1997) model does not incorporate deterioration due to repair. Last and Szekli (1998) propose model (2.12) with the larger range $[0,\infty)$ allowed for the factors $\{D_i\}$. Values of D_i larger than 1 correspond to a deterioration due to repair. Thus the DHS (1997) model almost contains the LS model as a special case, but does not since the LS model permits deterioration as a result of repair. Last and Szekli (1998) showed that their repair model contains many proposed repair models in the literature including those of Stadje and Zuckerman (1991) and Baxter, Kijima and Tortorella (1996).

3. Nonparametric Inference for the DHS (1997) Model

DHS (1997) fix a $T>0$ and define two processes $N(\cdot)$ and $Y(\cdot)$ by

$$N(t)=\sum_j I\{X_j\le t, S_j\le T\}$$

and

$$Y(t)=\sum_j I\{A_j<t\le (X_j\wedge[A_j+\theta_j(T-S_{j-1})])\}$$

where $X_j=A_j+\theta_j T_j$ is the effective age just before the j^{th} failure. Letting $\delta_j=I(X_j\le A_j+\theta_j(T-S_{j-1}))=I(S_j\le T)$ and

$$\widetilde{X}_j=X_j\wedge[A_j+\theta_j(T-S_{j-1})],$$

the random variables

$$\{(\widetilde{X}_1,\delta_1),(\widetilde{X}_2,\delta_2),\ldots\}$$

can be viewed as censored observations from a lifetime data model where the lifetime X_j of subject j is observed only if it is smaller than $A_j+\theta_j(T-S_{j-1})$. Thus one can think of the repair model as a survival model where the subject j enters the study at A_j and dies during the study at age X_j or leaves the study by age $A_j+\theta_j(T-S_{j-1})$. Hence the process $N(t)$ can be viewed as the number of observed (uncensored) deaths by time t and $Y(t)$ represents the number at risk at time t. DHS (1997) found this connection to survival studies helpful in developing estimators and simultaneous confidence bands for F.

Let Λ be the hazard function of F and define the process

$$M(t)=N(t)-\int_0^t Y(s)d\Lambda(s).$$

Though it cannot be claimed that $\{N(t)\}$ is a martingale or that $\left\{\int_0^t Y(s)d\Lambda(s)\right\}$ is its compensator, DHS (1997) (Theorem 3.1) proved that

$$E(M(t)) = 0$$

and

$$\operatorname{cov}(M(t), M(t')) = \int_0^{t\wedge t'} E(Y)(1-\Delta\Lambda)d\Lambda. \tag{3.1}$$

They assumed that n independent copies of the processes N and Y are observed on a finite interval. Let N_n and Y_n denote the sum of the n independent copies of N and Y, respectively. Let Λ denote the cumulative hazard function of F. The Nelson-Aalen estimator of Λ is

$$\widehat{\Lambda}_n(t) = \int_0^t \frac{J_n dN_n}{Y_n}$$

where $J_n(t) = I(Y_n(t) > 0)$ for $t \in (0, T]$. Since

$$F(t) = \int_0^t (1 - F(s-))d\Lambda(s)$$

it is natural to require an estimator $\widehat{F}_n$ to satisfy

$$\widehat{F}_n(t) = \int_0^t (1 - \widehat{F}_n(s-))d\widehat{\Lambda}_n(s).$$

The solution of this Volterra integral equation is

$$\hat{\hat{F}}_n(t) = \prod_{s\le t}(1 - d\widehat{\Lambda}_n(s)) \tag{3.2}$$

where $\prod_{s\le t}(1-d\widehat{\Lambda}_n(s))$ denotes the product integral (see Gill and Johansen (1990)).

Let $M_n = N_n - \int Y_n d\Lambda$. This is the sum of n iid processes in $D[0,T]$ with mean 0 and covariance function given in (3.1). Thus

$$\left\{W_n(t) = \frac{M_n(t)}{\sqrt{n}}, \qquad 0 \le t \le T\right\}$$

will converge to a Gaussian process, if tightness can be established. This is done in DHS (1997) (Theorem 5.1). It is easy to show that

$$\frac{\widehat{F}_n(t) - F(t)}{\bar{F}(t)} = \int \frac{\widehat{\bar{F}}_n(s-)J_n(s)}{\bar{F}(s)(Y_n(s)/n)} dM_n(s). \tag{3.3}$$

Let

$$C(t) = \int_0^t \frac{dF}{EY(1-F)}.$$

Assume that $F(T) < 1$ and F is an increasing failure rate distribution. From the continuity mapping principle, and a result on the uniform convergence of the integrand in (3.3), DHS (1997) (Corollary 5.1) show that

$$\sqrt{n}\left(\frac{\widehat{F}_n - F}{\bar{F}}\right) \Rightarrow B(C) \text{ on } D[0,T]$$

where B denotes the Brownian motion on $[0,\infty)$. Furthermore, they established

$$\sqrt{n}\frac{\bar{K}}{\bar{F}}(\widehat{F}_n - F) \Rightarrow B^0(K) \text{ on } D[0,T]$$

where B^0 denotes a Brownian bridge on $[0,1]$ and $K = C/(1+C)$.

DHS (1997) also derived a simultaneous confidence band for F. For $t \in [0,T]$, let $L_n = I(\widehat{F}_n(t) < 1)$ and set

$$\begin{aligned} \widehat{C}_n(t) &= \int_0^t \frac{J_n L_n d\widehat{F}_n}{(Y_n/n)(1-\widehat{F}_n)}, \\ \widehat{K}_n(t) &= \frac{\widehat{C}_n(t)}{1+\widehat{C}_n(t)}. \end{aligned}$$

For t such that $\widehat{F}_n(t) = 1$, set $\widehat{K}_n(t) = 1$. A nonparametric asymptotic simultaneous confidence band for F with confidence coefficient at least $100(1-\alpha)\%$ is

$$\widehat{F}_n \pm n^{-1/2}\lambda_\alpha \frac{\widehat{\bar{F}_n}}{\widehat{\bar{K}}_n} \tag{3.4}$$

where λ_α is such that

$$P\left(\sup_{t\in[0,1]}|B^0(t)| \le \lambda_\alpha\right) = 1-\alpha. \tag{3.5}$$

COMMENTS.

a. Gill (1981) considers the "testing with replacement" scenario where one observes $X_1, X_2, \ldots$ nonnegative iid random variables with distribution F. He derives a nonparametric product limit estimator of F based on the first n of an infinite sequence of independent realizations of $\widetilde{N}(t) = \#\{j \geq 1 : \sum_{i=1}^{j} X_i \leq t\}$, each observed over the fixed time interval $[0, T]$. If we set $A_j = 0$, $\theta_j = 1$ for all j in the DHS (1997) model, then the product limit estimator given by (3.2) reduces to Gill's product limit estimator. Furthermore, the nonparametric asymptotic simultaneous confidence band (3.4) provides a nonparametric asymptotic simultaneous confidence band for F in Gill's testing with replacement situation.

b. Let $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ be the distinct ordered values of the X's whose corresponding failure times are within $[0, T]$. Let δ_j denote the number of observations with value $X_{(j)}$. Then the estimator $\widehat{\widehat{F}}$ can be written as

$$\widehat{\widehat{F}}(t) = \prod_{X_{(j)} \leq t} \left(1 - \frac{\delta_j}{Y_n(X_j)}\right).$$

Furthermore

$$\widehat{C}_n(t) = n \sum_{X_{(j)} \leq t} \frac{\widehat{F}_n(X_{(j)}) - \widehat{F}_n(X_{(j-1)})}{Y_n(X_{(j)}) \widehat{\widehat{F}}_n(X_{(j)})}.$$

c. For some data sets, there may be a t_0, $0 < t_0 < T$ such that $\widehat{F}_n(t_0) = 1$. In such cases the confidence band is valid only on the interval $[0, \sigma)$ where $\sigma = \inf\{t \in [0, T] : \widehat{F}_n(t) = 1\}$.

d. Gäertner (2000) studied the repair models of Last and Szekli (1998) and obtained inferential results similar to those that DHS (1997) obtained for their general repair model. Gäertner's objective, however, is to estimate F and its cumulative hazard function from one observation of the repairable system.

e. Peña and Hollander (2002), in a recurrent event setting, introduced models that allow incorporation of covariates into the DHS (1997) and Last and Szekli (1998) models.

4. Nonparametric Inference for the BBS Model

Whitaker and Samaniego (1989) proposed an estimator for F, the distribution of the time to first failure, for the BBS model. Using counting process techniques, Hollander, Presnell and Sethuraman (1992) (hereafter HPS (1992)) extended the large sample theorems of Whitaker and Samaniego to the whole real line. HPS (1992) also derived a nonparametric asymptotic simultaneous confidence band for F.

Recall that in the BBS model, upon failure at age t, with probability $p(t)$ a perfect repair is performed whereas with probability $1-p(t)$ a minimal repair is performed. For $j=1,\ldots,n$, let $\{X_{j,0}\equiv 0, X_{j,1}, X_{j,2},\ldots\}$ be independent record value processes from F. These are Markov processes with $P(X_{j,k}>t|X_{j,0},\ldots,X_{j,k-1})=\bar{F}(t)/\bar{F}(X_{j,k-1})$, for $t>X_{j,k-1}$, $k\geq 1$. Let τ_F be the (possibly infinite) upper endpoint of the support of F. If $\Delta F(\tau_F)>0$, define $X_{j,\ell}=\infty$ for all ℓ larger than the first k for which $X_{j,k}=\tau_F$. Take $p(\tau_F)=1$ in all cases. These processes represent the failure ages of n systems under a "forever minimal repair" model. Perfect repair is introduced into the model by letting $\{U_{j,k}:1\leq j\leq n, k\geq 1\}$ be iid uniform random variables. Let

$$\delta_{j,k}=I(U_{j,k}\leq p(X_{j,k}))$$

and

$$\nu_j=\inf\{k:\delta_{j,k}=1\}.$$

Then observing $\{(X_{j,1},\ldots,X_{j,\nu_j}), j=1,\ldots,n\}$ is equivalent to observing n independent copies of the BBS process, each until the time of its first repair. HPS (1992) used this structure and martingale methods to establish their asymptotic results.

Let

$$N(t)=\#\{(j,k):X_{j,k}\leq t, k\leq\nu_j, 1\leq j\leq n\}$$

and

$$Y(t)=\#\{j:X_{j,\nu_j}\geq t, 1\leq j\leq n\}.$$

Assume F is continuous and the pair (F,p) satisfies

$$F(\tau_F)=1 \text{ and } \int_0^{\tau_F} p(t)\frac{dF(t)}{\bar{F}(t)}=\infty. \tag{4.1}$$

Let $X_{(k)}$ be the k^{th} ordered value of $\{X_{j,s}:s\leq\nu_j, 1\leq j\leq n\}$ and let

$$T=\min\{X_{(k)}:Y(X_{(k)})=1\}.$$

That is, the sampling scheme here is to observe the n BBS processes, each until the time of its first perfect repair but stop at the first failure age such that only one process remains at risk (that is, has not yet experienced a perfect repair). Then the Whitaker-Samingo estimator can be written as

$$\widehat{\bar{F}}(t) = \prod_{s \leq t} (1 - d\widehat{\Lambda}(s)) \tag{4.2}$$

where

$$\widehat{\Lambda}(t) = \int_0^t \frac{J(s)}{Y(s)} dN(s)$$

and $J(s) = I(s \leq T)$.

It is clear that $\{N(t)\}$ is an increasing counting process. By employing an appropriate filtration of σ-fields to account for the BBS model of minimal and perfect repairs through the function $p(t)$, HPS (1992) find the compensator of $\{N(t)\}$ and show that

$$M(t) = N(t) - \int_{(0,t]} Y(s) d\Lambda(s)$$

is a locally square-integrable martingale with a predictable variation process $\langle M \rangle$ given by

$$\langle M \rangle(t) = \int_{(0,t]} Y(s)(1 - \Delta\Lambda(s)) d\Lambda(s).$$

The estimator $\widehat{F}(t)$ given by (4.2) is related to the process $\{M(t)\}$ by the relation

$$\frac{\widehat{F}(t) - F(t)}{\bar{F}(t)} = \int_{(0,t]} \frac{\widehat{\bar{F}}(s)}{\bar{F}(s) Y(s)} dM(s). \tag{4.3}$$

Using Rebolledo's (1980) martingale central limit theorem and the methods of Gill (1983), HPS (1992) showed that, assuming F is continuous and (4.1) is satisfied, then (i), (ii) and (iii) below hold.

(i) As $n \to \infty$,

$$\sqrt{n}(\widehat{F} - F) \Rightarrow \bar{F} \cdot B(C) \text{ in } D[0, \infty],$$

where B is Brownian motion on $[0, \infty)$,

$$C(t) = \int_0^t \frac{dF(s)}{\bar{H}(s)\bar{F}(s)},$$

where H is given by (2.3).

(ii) As $n \to \infty$,

$$\sqrt{n}\frac{\bar{K}(\widehat{F} - F)}{\bar{F}} \Rightarrow B^0(K) \text{ in } D[0, \infty],$$

where B^0 is a Brownian bridge on $[0, 1]$ and $K = C/(1 + C)$.

(iii)

$$\sqrt{n}\frac{\hat{\bar{K}}}{\hat{\bar{F}}}(\widehat{F} - F) \Rightarrow B^0(K) \text{ in } D[0, \tau] \tag{4.4}$$

for any $\tau < \tau_F$, where $\widehat{K} = \widehat{C}/(1 + \widehat{C})$

$$\widehat{C}(t) = \int_0^t \frac{d\widehat{F}(s)}{\hat{\bar{H}}(s)\hat{\bar{F}}(s)}$$

and where $\widehat{H}$ is the empirical cdf of the X_{j,ν_j}. Result (4.4) justifies a nonparametric asymptotic simultaneous confidence band for F, on $[0, \tau]$, of the form

$$\widehat{F} \pm \sqrt{n}\lambda_\alpha \frac{\hat{\bar{F}}}{\hat{\bar{K}}}, \tag{4.5}$$

where λ_α is defined by (3.5).

Comments.

f. If there are tied observations, construct the confidence band by taking T to be the first age at which the number of items failing is equal to the number at risk. That is, set T to the first t such that $\Delta N(t) = Y(t)$, where $\Delta N(t) = N(t) - N(t-)$. Then set $\widehat{\bar{F}}$ equal to:

$$\widehat{\bar{F}}(t) = \prod_{s \le t} \left(1 - \frac{\Delta N(s)}{Y(s)}\right)$$

and $\hat{\bar{K}} = 1/(1 + \widehat{C})$, where

$$\widehat{C}(t) = \sum_{s \le t} \frac{n\Delta N(s)}{Y(s)(Y(s) - \Delta N(s))}.$$

g. The bands given by (4.5) have the feature that their width decreases as t increases, whereas the similarly constructed Hall and Wellner (1980) bands based on the Kaplan-Meier estimator in the randomly right-censored model increase in width. (The Kolmogorov-Smirnov bands for iid sampling have constant width.) The decreasing-width property of the bands given by (4.5) is due to the extra information about the tail of the distribution gained by the minimal repair sampling scheme whereas, by contrast, right-censored sampling yields less information about the tail.

h. HPS (1992) extended the two-sample Wilcoxon test to apply to BBS models. They assumed that for $i = 1, 2$, n_i BBS processed based on (p_i, F_i) are observed, each until its first perfect repair. They developed a nonparametric test of $H_0 : F_1 = F_2$ based on the statistic $W = \int_0^\infty \widehat{F}_1 d\widehat{F}_2$ where $\widehat{F}_i$ is the Whitaker-Samaniego estimator based on the n_i BBS processes, $i = 1, 2$.

i. Presnell, Hollander and Sethuraman (PHS (1994)) developed two nonparametric tests of the minimal repair assumption in the BBS model. Let $\widehat{\bar{F}}_e$ denote the empirical survival function based on the initial times of the systems under observation. PHS(1994) proposed a Kolmogorov-Smirnov-type test based on the maximum absolute difference between $\widehat{\bar{F}}_e$ and $\widehat{\bar{F}}$, the Whitaker-Samaniego estimator. They also developed a Wilcoxon-type test based on $\int_0^\infty \widehat{F}_e d\widehat{F}$.

j. Let $\lambda(t) = f(t)/\bar{F}(t)$ denote the failure rate of the distribution of the time to first failure. Agustin and Peña (2001) consider the BBS model and derive goodness-of-fit tests of $H_0 : \lambda(\cdot) = \lambda_0(\cdot)$ where $\lambda_0(\cdot)$ is a completely specified failure rate. The tests are based on reformulating Neyman's notion of smooth tests in terms of hazard functions.

k. In an imperfect repair model, Kvam, Singh and Whitaker (2002) obtain, using isotonic regression techniques, the nonparametric maximum likelihood estimator of the failure rate $\lambda(t) = f(t)/\bar{F}(t)$ when $F(\cdot)$ is absolutely continuous with an increasing failure rate function $\lambda(\cdot)$. For a BP model with $0 < p < 1$, they show that if λ is continuous and increasing on $[a, b]$, then their estimator $\widehat{\lambda}(t)$ converges a.s. to $\lambda(t)$, as $n \to \infty$ for $t \in [a, b]$.

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