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# A CONTEMPORARY REVIEW AND BIBLIOGRAPHY OF INFINITELY DIVISIBLE DISTRIBUTIONS AND PROCESSES

*By* ARUP BOSE
*Indian Statistical Institute, Kolkata*
ANIRBAN DASGUPTA
and
HERMAN RUBIN
*Purdue University*

*SUMMARY.* This article provides a modern review of univariate and multivariate stable and infinitely divisible distributions and processes. Various characterizations and properties of stable and infinitely divisible distributions, including tail, moment and independence properties, and methods of simulation from an infinitely divisible distribution are discussed. Also discussed is the currently popular problem of estimating the index of a stable law and more generally, the heaviness of the tail of a distribution in the domain of attraction of a given stable law. A special feature of this article is its large collection of illustrative examples and a table of Lévy measures.

## 1. Introduction

Infinitely divisible distributions were introduced by de Finetti in 1929 and the most fundamental results were developed by Kolmogorov, Lévy and Khintchine in the thirties. The area has since continued to flourish and a huge body of deep and elegant results now exist in the literature. There have been many significant developments in the area in the last 20 to 25 years, especially in the areas of Lévy processes, uniform approximation of convolutions, and statistical inference, and a contemporary review seems to be needed. This article provides a review on both univariate and multivariate infinitely divisible distributions on $\mathbb{R}$ and $\mathbb{R}^d$ with a significant review

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of the recent developments. The intersection with Fisz (1962) and Steutel (1979a) is small. A special feature of our review is that we include the probabilistic, statistical as well as the simulation aspects. We also give numerous illustrative examples. The bibliography is far from being inclusive. This is particularly true of the bibliography on Lévy processes due to the huge literature in that area.

What are infinitely divisible (id) distributions? The following definition most fits the name, although other equivalent characterizations are available and are to be described later. See Chung (1976) and Feller (1966) for basic exposition and other basic examples.

Definition 1. A real valued random variable $X$ with cumulative distribution function (cdf) $F(\cdot)$ and characteristic function (cf) $\phi$ is said to be infinitely divisible (id), synonymously $F$ is an id law or $\phi$ is id, if for each $n > 1$, there exist iid random variables $X_1, \ldots, X_n$ with cdf say $F_n$ such that $X$ has the same distribution as $X_1 + \ldots + X_n$.

Remark 1. Since such a "division" of $X$ into "small" independent components is possible for each $n$, the name infinitely divisible seems appropriate. A degenerate random variable is by definition id. *This possibility is discounted in some of the subsequent discussion.*

Example 1. Let $X$ be $N(\mu, \sigma^2)$. For any $n > 1$, let $X_1, \ldots, X_n$ be iid $N(\mu/n, \sigma^2/n)$. Then $X$ has the same distribution as $X_1 + \ldots + X_n$. Thus $X$ is id.

Example 2. Let $X$ have a Poisson distribution with mean $\lambda$. For a given $n$, take $X_1, \ldots, X_n$ as iid Poisson variables with mean $\lambda/n$. Then $X$ has the same distribution as $X_1 + \ldots + X_n$. Thus it is id.

Example 3. Let $X$ have the continuous uniform $[0, 1]$ distribution. Then $X$ is *not* id. For if it is, then for any $n$, there exist iid random variables $X_1, \ldots X_n$ with some distribution $F_n$ such that $X$ has the same distribution as $X_1 + \ldots + X_n$. Since supremum of support of $X$ is 1, this forces the supremum of the support of $F_n$ to be $1/n$. This implies $V(X_1) \leq 1/n^2$ and hence $V(X) \leq 1/n$, an obvious contradiction.

Remark 2. Indeed, the above argument shows that no bounded non-degenerate random variable can be id. Hence, binomial, hypergeometric, and beta distributions are not id.

This raises the natural question:

Question 1. Which real valued random variables with unbounded support are id?

This question can be completely answered via several equivalent characterizations available for id laws. Some of these are given below. Interestingly most common univariate random variables with unbounded support *are* id. But there are a few common univariate random variables with unbounded support that are *not* id. Here are two lists which cover univariate distributions in common use:

LIST 1. *Those that are id:* Includes the discrete distributions Poisson, geometric and negative binomial and the continuous distributions normal, lognormal, noncentral chi-square, $t$, exponential, Gamma, double exponential, Pareto, Cauchy, half Cauchy, squared Cauchy, extreme value distributions, logarithm of a gamma, logarithm of a beta, and product of standard normals.

See section 2.4 for many more examples.

LIST 2. *Those that are not id:* Includes finite mixtures of normals, integer part of a normal, (certain) products of Poissons and products of geometrics, discrete normal, half normal, maximum of independent normals, maximum and minimum of independent Poissons, inverse normal and inverse $t$.

REMARK 3. The proof that a distribution is or is not id is often nontrivial. Instances of this are the proofs that half Cauchy, lognormal, and $t$ distributions are id. Similarly, the proofs that inverse normal and inverse $t$ distributions are not id require tricks that are not well known. It is a peculiarity of the subject that hard special techniques may be needed for particular special problems. See Bondesson (1978, 1981, 1987, 1992), Steutel (1970, 1974) and Thorin (1977a,b).

## 2. Characterizations

Now let us return to Question 1. A number of equivalent characterizations will be given. We shall also discuss results known for subclasses such as the class of all nonnegative random variables and all nonnegative random variables which have density. Feller (1966) and Chung (1976) contain most of the results below.

2.1 *General characterizations.* First, let us see another familiar but motivating example.

EXAMPLE 4. Fix $n > 1$, and take $X_{n1}, \ldots, X_{nn}$ to be iid Bernoulli $(p_n)$ random variables. Then $S_n = X_{n1} + X_{n2} + \ldots + X_{nn}$ has a binomial$(n, p_n)$

distribution, and if $p_n \to 0$ as $n \to \infty$ in such a way that $np_n \to \lambda$, for some $0 < \lambda < \infty$, then $S_n$ converges in distribution to a Poisson random variable with mean $\lambda$ which is id.

Note that the distribution of $X_{in}$ does depend on $n$, and that the limit distribution is Poisson, which is id. Hence we may ask the following question:

QUESTION 2. Fix $n \geq 1$. Take $X_{n1}, X_{n2}, \ldots, X_{nn}$ to be iid with some common distribution say $H_n$. Let $S_n = X_{n1} + X_{n2} + \ldots + X_{nn}$. If $S_n$ has a limit distribution, say $F$, can we assert anything interesting about the nature of $F$?
The answer provides our first characterization:

*Characterization # 1:*

Such an $F$ is id and conversely, every id $F$ arises in this fashion.

REMARK 4. In Question 2, the random variables at the $n$th stage have a distribution $H_n$ that depends on $n$. Suppose instead that we have one sequence of summands which are iid but we allow appropriate centering and normalization. This raises the following question:

QUESTION 3. Suppose $X_1, X_2, \ldots$ is an iid sequence with a common distribution $H$. Take $S_n = X_1 + X_2 + \ldots + X_n$; suppose, for some sequences of numbers $a_n$ and $b_n, (S_n - a_n)/b_n$ has a limit distribution, say $F$. Can we assert anything interesting about the nature of $F$?

Question 3 is a special case of Question 2 by taking $X_{in} = X_i/b_n - a_n/(nb_n)$. So certainly our limit law $F$ in Question 3 is id. The collection of all such $F's$ is thus a subclass of the class of all id laws. This subclass is the class of all *stable laws* and can also be defined as follows:

DEFINITION 2. A cdf $F$ on the real line is said to be stable if for every $n \geq 1$, there exist constants $b_n$ and $a_n$ such that $S_n = X_1 + X_2 + \ldots + X_n$ and $b_n X_1 + a_n$ have the same law. Here $X_1, X_2, \ldots, X_n$ are iid with distribution $F$.

REMARK 5. Two random variables $X$ and $Y$ are said to be of same type if $Y \stackrel{\mathcal{D}}{=} aX + b$ for some constants $a$ and $b$. Thus $F$ is stable if $S_n$ and $X_1$ are of same type. It turns out that $b_n$ has to be $n^{1/\alpha}$ for some $0 < \alpha \leq 2$. The constant $\alpha$ is said to be the *index* of the stable distribution $F$.

EXAMPLE 5. All stable laws have infinitely differentiable densities with bounded derivatives. However, the density functions are known in simple closed forms in only three cases: (i) the normal distribution (index 2), (ii) the Cauchy distribution (index 1) and (iii) the *Lévy distribution* (index 1/2).

Let us return to the issue of characterizing id laws at large. There is a very elegant characterization of id laws as *compound Poisson* distributions.

*Characterization # 2.*

Let $N_i$ be a sequence of Poisson random variables. and let $\{X_{ij}\}$ be iid for every fixed $i$. Let $Z$ be a single normal random variable. Suppose all the random variables are mutually independent, and $\{c_i\}$ is a sequence of constants such that the series below converges. Then $Z + \sum_{i=1}^{\infty}(\sum_{j=1}^{N_i} X_{ij} - c_i)$ is infinitely divisible, and conversely every infinitely divisible law has such a representation, provided the random variables are allowed to be degenerate.

A nice use of this characterization is the following:

EXAMPLE 6. Take $X$ to have a noncentral chi–square distribution with say one degree of freedom and some noncentrality parameter. It is well known (see Feller, 1966, for example) that $X$ may be written as $Y_1 + Y_2 + \ldots + Y_{2N+1}$, where the $Y_i$ are iid central chi-squares with one degree of freedom and $N$ is an independent Poisson random variable. Write $X_1 = Y_2 + Y_3, X_2 = Y_4 + Y_5$, etc. Then $X = Y_1 + (X_1 + X_2 + \ldots + X_N)$, where $Y_1$ is id. The quantity in parentheses is also id by characterization # 2, and so is their sum. That is, $X$ is id.

The most common means of characterization of id laws is by their characteristic functions. Several forms are available– some are easier to describe, while others are easier to apply. We give two of these forms, Form A for the finite variance case, and Form B for the general case. For a history on representation formulae, see Gnedenko and Kolmogorov (1954, page 68).

*Characterization # 3.*

Form A (Kolmogorov, 1932). Let $F$ be an id law with mean $b$ and finite variance and let $\phi(t)$ be its characteristic function. Then

$$\log \phi(t) = ibt + \int_{-\infty}^{\infty} (e^{itx} - 1 - itx) \frac{d\mu(x)}{x^2} \tag{1}$$

where $\mu$ is a finite measure on the real line. Further, $\mu(\mathbb{R}) = Var(X)$.

EXAMPLE 7. Suppose $F$ is the normal distribution with mean 0 and variance $\sigma^2$. Then the measure $\mu$ is degenerate at 0 with point mass $\sigma^2$ there and $b$ is 0.

EXAMPLE 8. Suppose $Y$ has a Poisson distribution with mean $\lambda$ and let $F$ be the distribution of $X = c(Y - \lambda)$. Then $\mu$ is degenerate with mass $c^2$ at $c$ and $b = 0$.

REMARK 6. Form A can be roughly interpreted as follows. The measure $\mu$ is degenerate when $F$ is normal or Poisson. A general $\mu$ can be approximated by linear combinations of such degenerate measures. In other words, a general id law may be written as limit of finite convolutions of normal and Poisson type random variables. This is in fact true without the finite variance assumption made in Form A above.

Form B (Lévy, 1937, Khintchine, 1937). Let $F$ be an id law and let $\phi(t)$ denote its characteristic function. Then $w(t) = \log \phi(t)$ admits the representation

$$w(t) = ibt - t^2\sigma^2/2 + \int_{-\infty}^{\infty} (e^{itx} - 1 - \frac{itx}{1+x^2})\frac{1+x^2}{x^2} d\lambda(x) \tag{2}$$

where $b$ is a real number, and $\lambda$ is a finite measure on the real line giving mass 0 to the value 0, i.e., $\lambda\{0\} = 0$. The integrand is defined to be $-t^2/2$ at the origin, by continuity.

REMARK 7. This is the original canonical representation given by Lévy. For certain applications and special cases, Form A is more useful. The measures $\mu$ and $\lambda$ in the two forms are both termed as *Lévy measure* of the cdf $F$.

Since normal laws are also limits of convolutions of Poisson type variables, we have the following characterization.

*Characterization # 4.* $F$ is id if and only if it is the limit in distribution of $S_n = X_1 + X_2 + \ldots + X_n$, where $X_i$ are independent *Poisson type* random variables.

*Lévy Measures for some distributions*

We now give the explicit identification of Lévy measures for some specific distributions. Most of the details may be found in Gnedenko and Kolmogorov (1954), Feller (1966) and Bondesson (1982). Below, Form A will be used for cases of finite variance.

1. Normal distribution with mean 0 and variance $\sigma^2$. Here $w(t) = -\sigma^2 t^2/2$. Thus, $b = 0$ and $\mu$ is concentrated at 0 with mass $\sigma^2$.

2. Poisson distribution with mean $\lambda$. Here $w(t) = \lambda \exp\{it - 1\}, b = \lambda$ and $\mu$ is concentrated at 1 with mass $\lambda$.

3. Geometric distribution with parameter $0 < p < 1$. Here $w(t) = \sum_{x=1}^{\infty}(e^{ixt} - 1)q^x/x$. It is easy to check that $b = \frac{q}{p}$ and $d\mu(x) = xq^x, \ x = 1, 2, \ldots$

4. For the (number of failures) negative binomial distribution with parameter $n$ and $p$, using the above, $b = \frac{nq}{p}$ and $d\mu(x) = nxq^x$, $x = 1, 2, \ldots$

5. For the exponential distribution with mean 1, $\phi(t) = (1 - it)^{-1}, b = 1$ and $d\mu(x) = x \exp\{-x\}dx, x > 0$.

6. For the gamma density $f(x) = \frac{\alpha^p}{\Gamma(p)} e^{-\alpha x} x^{p-1}$, $\phi(t) = (1 - \frac{it}{\alpha})^{-p}$ and

$$w(t) = i\frac{p}{\alpha} + p\int_0^\infty \{e^{itx} - 1 - itx\}\frac{e^{-\alpha x}}{x}dx$$

Hence $b = \frac{p}{\alpha}$ and $d\mu(x) = px \exp\{-\alpha x\}dx, \ \ x > 0$.

7. The double exponential distribution has density $f(x) = \exp(-|x|)/2, \ -\infty < x < \infty$ with cf $\phi(t) = (1-it)^{-1}(1+it)^{-1}$. Using the representation for the exponential, in this case, $b = 0$ and $d\mu(x) = |x| \exp\{-|x|\}dx, -\infty < x < \infty$.

8. Logistic distribution. Suppose $Y$ has the logistic density $\exp(-x)/(1 + \exp(-x))$, $-\infty < x < \infty$. Then its Lévy measure is

$$\lambda(dx) = \frac{x^2}{1+x^2} \cdot \frac{\exp(-|x|)}{|x|(1-\exp(-|x|))}dx, \quad -\infty < x < \infty.$$

9. Non-Gaussian Stable distributions with index $\alpha$, $0 < \alpha < 2$. In this case,

$$w(t) = ibt + \beta_1 \int_{-\infty}^0 \frac{A(t,x)}{|x|^{1+\alpha}}dx + \beta_2 \int_0^\infty \frac{A(t,x)}{x^{1+\alpha}}dx$$

where $A(t,x) = e^{itx} - 1 - \frac{itx}{1+x^2}$ and $\beta_1$ and $\beta_2$ are nonnegative. Hence $\sigma^2 = 0$, and $d\lambda(x) = \frac{x^2}{1+x^2}|x|^{-(1+\alpha)}dx, \ \ -\infty < x < \infty$.

10. The hyperbolic cosine density is $f(x) = (\pi \cosh x)^{-1}, \ \ -\infty < x < \infty$ Its characteristic function is $\phi(t) = [\cosh(\pi t/2)]^{-1}$. See Feller (1966, p.476). Here $b = 0$ and $d\mu(x) = \frac{x}{exp\{x\} - exp\{-x\}}dx$.

11. Bessel distribution. The Bessel function of order $p \geq -1$ is defined as $I_p(x) = \sum_{k=0}^\infty \frac{1}{k!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k+p}$. Then for every $r > 0$, the Bessel density $f_r(x) = e^{-x}\frac{r}{x}I_r(x), \ \ (x > 0)$ has cf $\phi(t) = \left[1 - it - \sqrt{(1-it)^2 - 1}\right]^r$. Here $d\mu(x) = rx \exp\{-x\}I_0(x)$ and $b = r$.

12. Log beta distribution. Suppose $Y$ is a beta random variable with density $f_Y(x) = [Beta(\alpha,\beta)]^{-1}x^{\alpha-1}(1-x)^{\beta-1}, \ \ 0 < x < 1$. Let $X = -\log Y$. Then $X$ is id and its Lévy measure concentrates on the positive real line with measure $\lambda(dx) = \frac{x^2}{1+x^2}x^{-1}e^{-\alpha x}(1-e^{-\beta x})/(1-e^{-x})dx$.

13. Log gamma distribution. Suppose $Y$ has the Gamma $(1,p)$ distribution. Let $X = \log Y$. Then its Lévy measure is $\lambda(dx) = \frac{x^2}{1+x^2}\exp(px)/(1-\exp(x))dx$, $x < 0$.

2.2 *Characterization of nonnegative discrete id laws.* There is an elegant characterization of all distributions supported on the nonnegative integers that are id. The characterization says the following. See Dharmadhikari and Joag-Dev (1988), and Katti (1967, 1977).

Let $X$ take values $0, 1, 2, \ldots$, with $P(X = k) = p_k$. Then $X$ is id if and only if

$$\eta_i = \frac{ip_i}{p_0} - \sum_{j=1}^{i-1} \eta_{i-j}\frac{p_j}{p_0} \geq 0 \ \ \forall i \geq 2, \text{ where } \eta_1 = p_1/p_0. \qquad (3)$$

Remark 8. It is difficult to verify (3) to establish infinite divisibility. The relation is more useful to exclude a given distribution from being id.

Example 9. Consider the discrete standard normal distribution with mass function $p_k = 2[\theta(2\pi)^{-1}) + 1]^{-1}\exp(-k^2/2)$ where $\theta(\cdot)$ is the Jacobi theta function. Then $\eta_1 = .6065 > 0$, but $\eta_2 = -.0972 < 0$, and it follows that the discrete normals are not id. (3) can also be used to prove that the product of two independent Poissons with small mean is not id.

However, simple and verifiable sufficient conditions that imply the above characterization are available. One such sufficient condition is the following:

*Sufficient Condition.* Let $X$ take values $0, 1, 2, \ldots$, with $P(X = k) = p_k$. Then $X$ is id if $\log p_k$ is convex in $k$.

Remark 9. The support of a discrete id law on nonegative integers cannot have any gaps if $P(X = 1)$ is strictly positive. We will see below that a similar result holds for positive id random variables with a density.

2.3 *Nonnegative id laws with densities.* The details of the results described in this section may be found in Steutel (1969, 1979) and Goldie (1967).

One of the most important and useful results on id laws is the Goldie-Steutel law for positive random variables. The proof that a certain positive random variable having a density is id may be accomplished in one of the following ways: (a) verify that the Goldie-Steutel law applies; (b) try to verify a known characterization result parallel to the discrete case; (c) use a known sufficient condition; (d) use a special technique for that particular problem.

The problems that are solved by using technique (d) generally turn out to be the difficult ones; the lognormal and the half Cauchy are two instances.

*The Goldie-Steutel Law.* Let a positive random variable $X$ have a density $f(x)$ which is *completely monotone*. Then $X$ is id.

REMARK 10. *Complete monotonicity* means that the function is continuous, decreasing, and derivatives of successive orders have opposite signs. It is well known that such functions may be written as exponential mixtures. So the Goldie-Steutel Law says that a positive random variable $X$ with density $f$ is id if it can be written as $X = YZ$, where $Z$ is exponential with mean 1, and $Y$ is nonnegative and independent of $Z$. Many positive random variables are known to be of this variety, and a fortiori, they are id. In fact, the same proof can be used to remove the restriction that $Y$ is nonnegative; see DasGupta (2002a) for applications of this extended version.

EXAMPLE 10. Let $X$ have the Pareto density $f(x) = \frac{\alpha}{\mu}(\frac{\mu}{x+\mu})^{\alpha+1}$. Then, easily, $f$ is completely monotone, and so $X$ is id. It can be verified that in the representation $X = YZ$ as above, $Y$ has a Gamma density.

The following extension of the Goldie-Steutel law is sometimes useful:

*Extension of the Goldie-Steutel Law.* Let $X$ have a density $f(x)$ of the form

$$f(x) = \int_0^\infty \exp(-xt)g(t)dt,$$

where $g(\cdot)$ changes sign once. Then $X$ is id.

Now let us see the continuous analogs of some of the results we saw in the discrete case. First, a characterization of id laws in terms of the density function; see Steutel (1979) (to our knowledge, the result first appears in McCloskey (1965), where it is credited to H. Rubin).

*Characterization of nonnegative id laws with densities.* $X > 0$ with density $f(x)$ is id *if and only if* there is a nondecreasing function $\tau(u)$ on $[0, \infty]$ with $\int_1^\infty u^{-1} d\tau(u) < \infty$, such that $f(x) = x^{-1} \int_0^x f(x-u) d\tau(u)$.

This integral equation has an interpretation in terms of size-biased sampling and decomposability of the size-biased version of an id law. See Perman, Pitman and Yor (1992) and Steutel (1995). For applications of this formula, see Pitman and Yor (1997).

Verifying whether a given $f$ may be written in the above form corresponds to solving an integral equation with difficult constraints and hence is often difficult to implement. But fortunately, there are certain verifiable sufficient conditions and necessary conditions for applications. We give one pair below.

*Sufficient condition.* Let $X$ be a positive random variable with a strictly positive decreasing and twice continuously differentiable density $f(x)$. Then

$X$ is id if

$$\frac{f'(y)}{f(y)} \leq \frac{1}{x} + \frac{f''(x)}{f'(x)}, \quad \forall\, 0 < y \leq x. \tag{4}$$

*Necessary condition.* Let $X$ be a positive id random variable with a density $f(x)$. Under regularity conditions, if $f > 0$ in some neighbourhood of 0, then it cannot have any zeroes. See Sharpe (1995) for precise statements.

REMARK 11. Note that this parallels the result for the discrete case that the support of $X$ cannot have any gaps.

2.4 *Basu's theorem and infinite divisibility.* DasGupta (2002a) shows by using Basu's theorem (Basu, 1955) that a large class of functions of random variables $X_1, ..., X_n$, two of which are independent $N(0,1)$, is infinitely divisible. Precisely, one has the following theorem.

Let $X_1, X_2$ be iid $N(0,1)$ and $X_3, ..., X_n$ arbitrary random variables such that $(X_1, X_2)$ is independent of $(X_3, ..., X_n)$. Let $f(X_1, X_2)$ be any homogeneous function of degree 2 and $g(X_3, ..., X_n)$ any arbitrary function. Then $f(X_1, X_2)g(X_3, ..., X_n)$ is id.

The following examples are some consequences of the above result. See DasGupta (2002a) for more examples.

EXAMPLE 11. If $Z_i \overset{iid}{\sim} N(0,1), 1 \leq i \leq n$, then $Z_1 Z_2 ... Z_n$ is id.

EXAMPLE 12. If $X_i \overset{iid}{\sim}$ Cauchy $(0,1), 1 \leq i \leq n$, then $X_1 X_2 ... X_n$ is id.

EXAMPLE 13. Let $X_1, X_2, X_3$ be iid $N(0,1)$ then $X_1 X_2 \pm X_2 X_3$ is id.

EXAMPLE 14. If $Z_i \overset{iid}{\sim} N(0,1), 1 \leq i \leq n$ where $n \geq 3$, then $\frac{Z_1\ Z_2\ ... Z_k}{Z_{k+1} ... Z_n}$ is id for $2 \leq k \leq n-1$.

EXAMPLE 15. If $X_i \overset{iid}{\sim}$ Cauchy $(0,1), 1 \leq i \leq n$, where $n \geq 3$, then $\frac{X_1\ X_2 ... X_k}{X_{k+1} ... X_n}$ is id for $2 \leq k \leq n-1$.

## 3. Properties of id and Stable Laws

Id laws have very interesting properties in terms of their characteristic function, moments and tails. Moreover, subclasses of id laws such as those that are unimodal, totally positive, etc, turn out to be quite interesting. We shall discuss some of these below. The important subclass of *stable laws* is treated separately in subsection 3.3.

3.1 *Properties of the characteristic function.* Characteristic functions of id laws satisfy some interesting properties. Such properties are useful to exclude particular distributions from being id and to establish further properties of id laws as well. They generally do not provide much probabilistic insight, but are quite valuable as analytical tools in studying id laws. A collection of properties is listed below. See Chung (1976) for proofs of most of these results.

1. Let $\phi(t)$ be the characteristic function (cf) of an id distribution. Then $\phi$ has no real zeroes. The converse is false.

2. Let $\phi(t)$ be the characteristic function (cf) of an id distribution. Then for all $\lambda > 0$, $\phi^\lambda(t)$ is also a cf. Here $\phi^\lambda(t)$ is to be defined as $\exp(\lambda Log[\phi(t)])$, where $Log[.]$ denotes the *distinguished logarithm.*

3. Let $\phi_1(t), \phi_2(t)$ be two id cfs; then $\phi_1(t)\phi_2(t)$ is also an id cf.

4. Let $\phi(t)$ be the characteristic function (cf) of an id distribution. Then $\overline{\phi}$, the complex conjugate of $\phi$, and $|\phi|^2$ are also id cfs.

5. Let $\phi_n(t)$ be a sequence of id cfs, converging pointwise to another cf $\phi(t)$. Then $\phi(t)$ is also an id cf.

6. Let $\phi(t)$ be the characteristic function (cf) of an id distribution. Then there exist real constants $a, b$, such that $|\log \phi(t)| \leq a + bt^2$ for all $t$.

REMARK 12. Property 3 just says that the convolution of id laws is id. Property 4 says that the negative of an id random variable $X$ is id, and therefore if $X_1$ and $X_2$ are iid and id, then $(X_1 - X_2)$ must also be id. Property 5 is essentially a restatement of Characterization # 1. Let us see a quick example that the converse of Property 1 is false.

EXAMPLE 16. The function $\phi(t) = (\cos t + 2)/3$ is the cf of a non-id symmetric distribution on $\{-1, 0, 1\}$ and obviously has no real roots.

3.2 *Moments and tails of id laws.* An id random variable may have all moments, some moments, or even no moments: the normal has all moments, the Cauchy has no moments, and intermediate $t$ distributions have some moments. But one can say some definite things about the tails of id laws. For example, roughly speaking, no id law can have tails thinner than that of a normal. We state these and connections to the canonical measures of id laws below. See Steutel (1979).

Let $X$ be an id random variable with cdf $F(x)$ and corresponding canonical measure $\lambda$ as in Form B of its characteristic function. Then,

(1) $-\log(1 - F(x) + F(-x)) = O(x \log x)$ as $x \to \infty$ unless $F$ is a normal cdf;

(2) There cannot exist any reals $a > 0,\ b > 1$ such that $1 - F(x) + F(-x) = O(\exp(-ax^{1+b}))$ as $x \to \infty$ unless $F$ is degenerate;

(3) There cannot exist any reals $a > 0, 0 < b \leq 1$ such that $1 - F(x) + F(-x) = O(\exp(-ax^{1+b}))$ as $x \to \infty$ unless $F$ is normal;

(4) If $\lim_{x \to -\infty} F(x)/\Phi(x) = 1$, then $F$ must be $\Phi$ itself;

(5) For a given $p > 0, 1 - F(x) = O(x^{-p})$ as $x \to \infty$ if and only if $\int_x^\infty d\lambda(u) = O(x^{-p})$ as $x \to \infty$; and $F(-x) = O(x^{-p})$ as $x \to \infty$ if and only if $\int_{-\infty}^{-x} d\lambda(u) = O(x^{-p})$ as $x \to \infty$.

(6) For a given $p > 0, E(|X|^p) < \infty$ if and only if $\int_{-\infty}^{\infty} |u|^p d\lambda(u) < \infty$.

Remark 13. The connections of $F$ to the canonical measure $\lambda$ via their respective tails as in (5) are nice; so is the equivalence between existence of absolute moments. It is also interesting that the assertion of (4) is false if the cdf in the denominator is an id cdf $G(x)$ other than $\Phi(x)$.

Example 17. Suppose $X$ has the density function

$$f(x) = \exp[-x^4]/(2\Gamma(5/4)).$$

Then, from (1) or (2), it follows that $X$ cannot be id. The tail of $f(x)$ is too thin. Similarly if $X$ has a mixture normal distribution $pN(0, \sigma_1^2) + (1 - p)N(0, \sigma_2^2)$, for unequal $\sigma_1^2, \sigma_2^2$, then, from (1), it follows that $X$ cannot be id.

3.3 *The stable laws.* The subclass of stable laws occupies a special position in the class of id laws. Their probabilistic properties have been studied extensively. They have also found numerous applications in statistics. In this subsection we discuss some of the probabilistic properties of stable laws. In section 6 we shall look at the statistical importance of this class. Basic properties of stable laws can be seen in Feller (1966), Zolotarev (1986), and Dharmadhikari and Joag-Dev (1988) and Samorodnitsky and Taqqu (1994).

*Characteristic function of stable laws*: Starting from Form B of the characteristic function of id laws, it is possible to derive the following characterization:

$\phi(t)$ is the $cf$ of a stable law $F$:

with index $\alpha \neq 1$ if and only if it has the representation

$$\log \phi(t) = ibt - \sigma^\alpha |t|^\alpha (1 - i\beta \text{sign}(t) \tan(\pi\alpha/2)); \tag{5}$$

with index $\alpha = 1$, if and only if it has the representation

$$\log \phi(t) = ibt - \sigma|t| \left(1 + i\beta \text{sign}(t) \frac{2}{\pi} \log|t|\right). \tag{6}$$

The *scale parameter* $\sigma > 0$, the *location parameter* $b$ and the *skewness* parameter $\beta$ of $F$ above are *unique* (except that if $\alpha = 2$, the value of $\beta$ is irrelevant). The possible value of $\beta$ ranges in the closed interval $[-1, 1]$. The possible values of $b$ are the entire real line. It follows trivially from the characteristic function that $F$ is *symmetric* (about $b$), if and only if $\beta = 0$. If $\alpha = 2$ then $F$ is normal. If $\alpha = 1$ and $\beta = 0$, then $F$ is a Cauchy law with scale $\sigma$ and location $b$.

*Moments and tails of a stable law.* The stable laws also have some very nice moment and tail properties. But first an easy fact: The first moment of any random variable, if it exists, is equal to the first derivative of the cf at zero.

By using the above characterization, it is easy to see that if $X$ is stable with $\alpha > 1$, then $E(X) = b$. What can be said about other moments? Of course, if $\alpha = 2$, then all moments exist. If $X$ is stable with $0 < \alpha < 2$, then for any $p > 0$,

$$E|X|^p < \infty \text{ if and only if } 0 < p < \alpha.$$

This property of the moments *suggests* that the tails of a stable law behave as $x^{-\alpha}$. This is essentially correct: If $X$ is stable with index $0 < \alpha < 2$, then there exists a non-zero constant $C_\alpha \neq 0$, such that,

$$\lim_{x\to\infty} x^\alpha P\{X > x\} = C_\alpha (1+\beta)\sigma^\alpha/2 \tag{7}$$

$$\lim_{x\to\infty} x^\alpha P\{X \leq -x\} = C_\alpha (1-\beta)\sigma^\alpha/2. \tag{8}$$

Thus a stable law of index $0 < \alpha < 2$ has *at least* one of the tails of *exact* asymptotic order $x^{-\alpha}$. If $\beta \neq 1, -1$, then *both* tails are of this order.

3.4 *Unimodality, total positivity and the class L.* The standard examples of id laws are all unimodal. Even the discrete ones are discrete unimodal. See Wolfe (1971), Yamazato (1982), Ibragimov and Cernin (1959) and Dharmadhikari and Joag-Dev (1988) for general theory. But it is not difficult to construct simple continuous id random variables which do not have unimodal densities. Let us see an example. See Sato (1994) for additional information.

EXAMPLE 18. Suppose $X_1$ has the $N(0, \sigma^2)$ distribution, and $X_2$, independent of $X_1$, has a Poisson ($\lambda$) distribution. Consider the convolution $X = X_1 + X_2$. Evidently, $X$ is id. However, for given $\lambda$, the density of $X$ will not be unimodal for sufficiently small $\sigma$. Figure 1 gives the density of $X$ when $\lambda = 1$ and $\sigma = 1/4$. An $X$ of this form, in general, has a density with finitely many distinct local maxima.

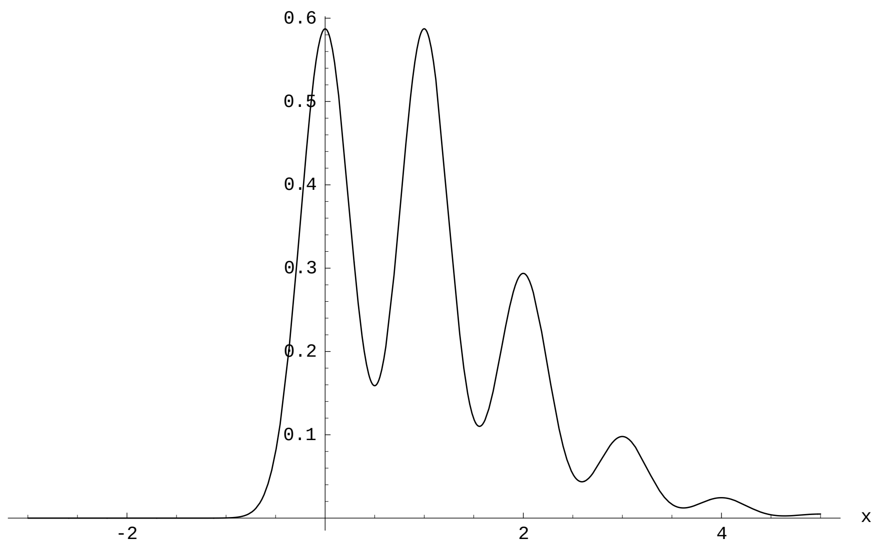


Figure 1. Density of $X$ in Example 18 ($\lambda = 1$, $\sigma = 1/4$)

In view of this, it is interesting to ask what can be said about unimodality of id laws. It turns out that a large class of id laws having a certain property known as *self decomposability* are indeed unimodal. We first give the definition.

DEFINITION 3. Let $X$ be a random variable with characteristic function $\phi(t)$. $X$ is said to be *self decomposable* if for every $0 < c < 1, \phi(t)$ can be factorized as $\phi(t) = \phi(ct)\psi(t)$, where $\psi$ is another characteristic function. In other words, $X$ can be written as a convolution $X \stackrel{\mathcal{D}}{=} cX + Y$ for every $0 < c < 1$. The class of all such laws is called *the class L*.

A recent paper with further references for self decomposable laws is Jeanblanc et al. (2002).

REMARK 14. Clearly, normal and Cauchy distributions are self decomposable.

The following three results give the interconnections between id/stable laws, class $L$ and unimodality. For details see Yamazato (1975).

1. Every id random variable that is in the class $L$ is unimodal.

2. All stable distributions belong to the class $L$.

3. All stable distributions, which are a fortiori id, are unimodal.

REMARK 15. Thus, we have a nice subclass of id laws, namely all stable laws, that are unimodal. The proof that every density in the class $L$ is unimodal is nontrivial. Note that the result stated above indicates how to construct other large subclasses of id laws that would also be unimodal. For instance, take a convolution of a normal random variable with a stable random variable. This will be id, and will also be unimodal because a normal random variable is *strongly unimodal* and a stable one, as we just stated, is unimodal.

For one sided, i.e., either positive or negative, stable laws, it is sometimes possible to assert a very strong kind of unimodality. See Karlin (1968). It is the following :

Let $X$ be a positive stable random variable with index $\alpha = 1/k$ for some natural number $k$ and $|\beta| = 1$ in the canonical representation of its characteristic function. Then the density of $X$ is *totally positive.*

3.5. *Approximation of sums in total variation by id laws.* Take iid random variables $X_1, X_2, \ldots$ having some common distribution $H$. Under well known conditions, $S_n = X_1 + X_2 + \ldots + X_n$, when centered and normalized, will converge to a normal distribution. However, the convergence is not necessarily in total variation. A simple example is that of iid Bernoulli random variables $X_n$. In this case, for all $n$, the total variation distance remains equal to 1. However, if $H$ is continuous with a unimodal density, then the convergence will also be in total variation.

So it is interesting that if the approximating class is enlarged to the class of id laws, then it is possible to say definite things about convergence in total variation. See Fisz (1962). A few results are as follows:

Let $X_1, X_2, \ldots$ be an iid sequence with some common distribution $H$. Let $\mathcal{I}$ denote the class of all id laws and let $H_n(x) = P(S_n \leq x)$. Then

1. $\lim_{n\to\infty} \sup_H \inf_{F\epsilon\mathcal{I}} d_{TV}(H_n, F) = 0$;
2. There exist finite constants $c_1, c_2$, such that
$$c_1(n\log n)^{-1} < \sup_H \inf_{F\epsilon\mathcal{I}} d_{TV}(H_n, F) < c_2 n^{-1/3}(\log n)^2;$$
3. If $X_j$ are iid Bernoulli $(p)$, then
$$\sup_{0<p<1} \inf_{F\epsilon\mathcal{I}} d_{TV}(H_n, F) = O(n^{-2/3}).$$

3.6. *Kolmogorov's Uniform limit theorems.* For two distribution functions $F$ and $G$ on $\mathbb{R}$, let $\rho(F,G)$ denote their Kolmogorov distance $\sup_x |F(x)-G(x)|$. Prohorov (1955) proved that $\inf_{G\varepsilon\mathcal{D}} \rho(F^n, G) \to 0$ for any $F$, where $F^n$ is the $n$-fold convolution of $F$ and $\mathcal{D}$ is the class of id laws. Kolmogorov (1956) addressed the question whether the convergence is uniform over $F$ and showed that $\psi(n) = \sup_F \inf_{G\varepsilon\mathcal{D}} \rho(F^n, G) = O(n^{-\frac{1}{5}})$. Over the next twenty five years, numerous researchers in the former Soviet Union obtained progressively improved results on the asymptotic behaviour of $\psi(n)$, culminating in the brilliant resolution of the problem in Arak (1981, 1982). The table below sketches the evolution of this problem.

| *Author* | *Result* |
|---|---|
| Prohorov (1960) | $\psi(n) = O(n^{-\frac{1}{3}}(\log\ n)^2)$ |
| Mesalkin (1961) | $\psi(n) \geq c\ n^{-\frac{2}{3}}(\log\ n)^{-\frac{7}{2}}$ |
| Kolmogorov (1963), Le Cam (1965) | $\psi(n) = O(n^{-\frac{1}{3}})$ |
| Arak (1981,1982) | $c_1 n^{-\frac{2}{3}} \leq \psi(n) \leq c_2 n^{-\frac{2}{3}}$ with the $n^{-\frac{2}{3}}$ rate attained at suitable Bernoulli distributions. |

Arak also shows that if $F$ is restricted to distributions whose $cf$'s do not change sign, then the rate of $\psi(n)$ is $n^{-1}$. Thus, by allowing all id distributions to approximate $F^n$, the $n^{-\frac{1}{2}}$ rate of the Berry-Esseen theorem is improved. Discussion of other metrics including the Lévy metric may be seen in Zaitsev and Arak (1983).

3.7. *Slepian inequalities.* If $\boldsymbol{X}, \boldsymbol{Y}$ are zero mean normal vectors and $\mathrm{corr}(X_i, X_j) \geq \mathrm{corr}(Y_i, Y_j)$, then
$$P(\boldsymbol{X} \leq \boldsymbol{c}) \geq P(\boldsymbol{Y} \leq \boldsymbol{c}).$$

Thus the quadrant probabilities (either lower or upper) are increasing functions of the correlations. This is the original Slepian inequality (Slepian, 1962). It has many applications.

A version of the Slepian inequality for stable distributions with $\alpha > 1$ was obtained in Marcus and Pisier (1984) and for some id distributions in Samorodnitsky and Taqqu (1993). But the conditions are difficult to verify. Brown and Rinott (1988) provide Slepian inequalities for a subfamily of multivariate id distributions under simple conditions.

Samorodnitsky and Taqqu (1994) revisit this problem and relate Slepian inequalities for id distributions to those for the corresponding Lévy measures.

3.8. *Operator Stable measures and C-decomposibility.* Two concepts that generalize infinite divisibility are the concepts of operator stable and C-decomposable measures. Primary references are Sharpe (1969b) and Bunge (1996,1997). First we introduce operator stability.

3.8.1 *Operator stable measures.* Let $V$ be a finite dimensional inner product space. For a probability measure $\lambda$ on $V$, and a nonsingular linear transformation $B$, let $\lambda^n$ denote the $n$-fold convolution $\lambda * \lambda * \cdots * \lambda$ and $B\lambda$ the measure $\lambda B^{-1}$. Let also $\delta(a)$ denote the point mass at the point 'a'.

A probability measure $\mu$ on $V$, not supported on any proper hyperplane, is called operator stable (os) if there exist a sequence of nonsingular transformations $\{A_n\}$, a sequence of points $\{a_n\}$, and a probability measure $\lambda$ on $V$ so that $A_n\lambda^n * \delta(a_n)$ converges weakly to $\mu$.

Sharpe (1969b) shows two key facts:

1) If $\mu$ is os, then $\mu$ is id;

2) Every os $\mu$ admits the following decomposition: there exist independent subspaces $V_1, V_2$ and measures $\mu_1$, $\mu_2$ *os* on $V_1, V_2$ such that $\mu_1$ is Gaussian, $\mu_2$ is Gaussian-free, $V = V_1 \oplus V_2$, and $\mu = \mu_1 * \mu_2$.

Sharpe (1969b) also characterizes the os measures in terms of the family of measures $\{\mu^t\}, t > 0$, where $\mu^t$ is the measure with $cf$ $(\hat{\mu})^t$, $\hat{\mu}$ being the $cf$ of $\mu$. Further refinements of some of these results may be seen in Hudson and Mason (1981a,b) and Kucharczak (1975). Characterization of laws in the domain of attraction of an os law and a series representation akin to ordinary stable laws are given in Hahn et al. (1989).

3.8.2 *C and N decomposibility.* Bunge (1996,1997) describes two generalizations of infinite divisibility that have various applications.

A random variable $X$ is called $N$-divisible if there exist iid random variables $X_1, X_2, \ldots$ such that $X$ has the same distribution as $X_1 + X_2 + \ldots + X_N$, where $N$ takes positive integral values and is independent of the $\{X_i\}$. If the $\{X_i\}$ can be chosen to be scaled copies of $X$, then $X$ is called $N$-stable. Bunge (1996) discusses representations of $N$-divisible measures and applications to quantities modelled as random sums.

Bunge (1997) calls $X$ to be $C$-decomposable for suitable subgroups $C$ of [0,1] if for every $c$ in $C$, $X$ has the same distribution as $cX + Y_c$, where $Y_c$ is independent of $X$. The relation to infinite divisibility is that the intersection of a nested family of subsets of $C$-decomposable laws gives the id distributions.

3.9. *Infinitely divisible laws on group structures.* Extension and representation of id laws on group structures more general than $\mathbb{R}^d$ are discussed in various works of Parthasarathy, Zolotarev, Heyer, Port and Stone.

In a pioneering article, Parthasarathy and Sazanov (1964) give a formula for the characteristic function of an id law on locally compact abelian groups. They show that every id law can be broken down into three components, a Gaussian, a Poisson and a Haar component. Heyer (1972) discusses existence and various definitions of Gaussian and Poisson measures on locally compact abelian groups. He also gives a generalization of the characterization of id laws as limits of triangular arrays in the Euclidean case for compact lie groups. Zolotarev (1975) gives explicit conditions on the spectral measure of an id law $P$ on a locally bicompact abelian group for $P$ to be discrete, continuous or absolutely continuous. He also gives a series of interesting examples. See Fisz and Varadarajan (1963) and Tucker (1965) also for similar results. Id processes on locally compact abelian groups are discussed in Port and Stone (1969), where they show extensions of various results on hitting times of transient random walks to the continuous time case. They also discuss potential theory for the continuous time case.

3.10. *Infinitely divisible random sets.* Analogous to the theory of id random variables, there is an elegant set of results on id random sets. A random compact convex set (in the sense of Matheron, 1975) $K$ is said to be id if for every $n \geq 1$, the law of $K$ is the same as the law of the Minkowski sum $K_1 + ... + K_n$ of iid sets $K_1, ..., K_n$. Mase (1979) is a pioneering article. For the special case of $\mathbb{R}^1$, it is shown that an id set containing the origin is of the form $[-X, Y]$ for $X, Y \geq 0$ where $(X, Y)$ have a joint id distribution.

The restriction that the sets contain the origin is removed in some special cases in Vitale (1983) and Lyashenko (1983). A Lévy type representation for the general case is given in Gine and Hahn (1985), where support functions of convex sets are used to reduce the problems to id laws on certain function spaces. Tools of probability theory on Banach spaces are then heavily used.

In a later paper, Gine and Hahn (1985), a lucid review of the topic and some discussion of the infinite dimensional case are presented. Some practical applications can be seen in Artstein and Hart (1981).

## 4. Multivariate id and Stable Laws

Let $X$ be a $k$-dimensional random vector. Then the definition of infinite divisibility of $X$ is the same as in one dimension. Many of the results are similar too, for instance inclusion of the multivariate stable laws, canonical representations of the characteristic functions, etc. The basic theorems about id random variables were generalized to id vectors as early as 1954. Early references in this area are Rvaĉeva (1954), Takano (1954) and Dwass and Teicher (1957). See also Stoyanov (1987).

Of course if a random vector is id, then all the lower dimensional components are also so. However, interesting things happen when we consider other aspects. In the subsections below, we discuss some aspects of multivariate id laws, such as, independence, Gaussianity, existence of moments etc.

4.1. *Some interesting examples.* This section is a collection of results and examples exploring the connection between infinite divisibility of the full vector and of lower dimensional transformations.

Example 19. It is possible that a random vector $X$ is not id, but all linear combinations of the coordinates of $X$ are id. Let $Z$ be a standard bivariate normal vector. Define a new bivariate random vector $X = (c'Z, Z'AZ)$, where $c$ is a 2–tuple and $A$ is a $2 \times 2$ symmetric matrix. If $c$ is not in the null space of $A$, the vector $X$ is not id. However, every linear combination of the two coordinates of $X$ is infinitely divisible.

Example 20. Even if $X$ is not id, every lower dimensional projection *may* be id. Take $Z_1, Z_2$ to be iid $N(0,1)$, and define a new trivariate vector $X$ as $X = (Z_1^2, Z_1 Z_2, Z_2^2)$. Then, it is easily verified that each two dimensional projection is id, but $X$ itself is not id.

Example 21. For iid univariate normal variables, the sample variance is a scaled chi–square and hence id. However, for iid $k$–dimensional normal vectors with nonsingular dispersion matrix, the Wishart matrix of sample variances and covariances is not id.

Example 22. Let $X$ be an id random vector. Then it is possible that although $X$ is not multivariate normal, certain linear combinations $c'X$ of

$X$ are univariate normal. Indeed, $c'X$ is univariate normal if and only if the Lévy measure corresponding to the distribution of $X$ is supported on the manifold $\{x : c'x = 0\}$. Of course, such examples of normal projections of non-normal vectors are well known; but now the full vector itself is id.

EXAMPLE 23. Nonlinear functions of the coordinates of a non-id vector may be id. Let $Z$ be $N(0,1)$ and write $Z = X_1 X_2$, where $X_1, X_2$ are iid. This is possible. Let $X = (X_1, X_2)$. Then $X_1 X_2$ is id by construction, but $X$ is not id. To see the latter, note that $P(|X_1| > x) \leq P(|Z| > x^2)^{1/2} = O(x^{-1}\exp(-x^4/2))$, and from fact (1) listed under moments and tails of id laws (section 3.2), we see that $X$ cannot be id.

4.2 *When are components of an id vector independent?* Recall the universally known fact about Gaussian random vectors: if $(X_1, \ldots, X_n)$ is Gaussian then the components are mutually independent if and only if they are pairwise independent which in turn happens if and only if $\text{Cov}(X_i, X_j) = 0$ $\forall\, i \neq j$.

Now assume that $X = (X_1, \ldots, X_n)$ is id. A natural question is when are the components independent? Are there any necessary and sufficient conditions available as in the normal case? This is discussed in Pierre (1971) and Veeh (1982).

It turns out that if the id vector has finite fourth moment, then pairwise independence is still equivalent to total independence.

Since an id vector can, in general, have Poisson components, it is clear that the covariance condition which is necessary and sufficient for pairwise/total independence when $X$ is normal does not remain so when $X$ is merely id. But interestingly, the addition of one extra condition leads to a satisfactory solution. Assume that $X$ is id and has a finite fourth moment. Since total independence is equivalent to pairwise independence, it is enough to concentrate on the case where $X$ is a 2-vector, $X = (X_1, X_2)$. To simplify expressions, assume that $E(X_i) = 0$ for $i = 1,\ 2$.

Let

$$\beta = (2,2) \text{ cumulant of } (X_1, X_2) = \text{Cov}(X_1^2,\ X_2^2) - 2(\text{Cov}(X_1, X_2))^2.$$

From the results of Pierre (1971) (see also Sclove, 1981), it is known that $\beta \geq 0$.

In general, the two components $X_1$ and $X_2$ are independent if and only if $\text{Cov}(X_1,\ X_2) = 0$ and $\text{Cov}(X_1^2,\ X_2^2) = 0$.

In several special cases, $\beta$ carries the information on independence of the components. For example, if $(X_1, X_2)$ has *no* Gaussian component, then $X_1$ and $X_2$ are independent if and only if $\beta = 0$. In particular, if $X$ is discrete then $X_1$ and $X_2$ are independent if and only if $\beta = 0$. Hence Poisson components are independent if and only if $\beta = 0$.

4.3. *When are id vectors Gaussian?* Suppose $X = (X_1, \ldots, X_n)$ is id. As we have discussed in Example 22, it is possible that *certain* linear combinations are normal but $X$ is not normal. What happens if sufficiently many linear combinations are normal? Indeed, if each $X_k$ is Gaussian, then $X$ is Gaussian. One can say more. If there is at least one component $k$ such that the 4th cumulant of $X_k$ is zero then also $X$ is Gaussian.

Recall that the regression functions of Gaussian vectors are linear. Further, all conditional distributions are homoscedastic. That is, the dispersion matrix of any sub-vector given any other is free of the conditioning sub-vector. For characterization of normal vectors using such ideas, see Kagan et al. (1973).

However, for id $X$, homoscedasticity and the linearity for vectors up to a pair guarantees Gaussianity of $X$:

Suppose $X$ is id and square integrable, linearly independent (to avoid trivialities) and with pairwise nonzero correlations. Suppose for some $i$, $j, k$,

$$\begin{aligned}
E(X_i|X_j) &= a_{ij}X_j + \mu_{ij}, \\
Var(X_i|X_j) &= b_{ij}, \\
E(X_i|X_j, X_k) &= a_{j,(i,j,k)}X_j + a_{k,(i,j,k)}X_k + \alpha_{i,(j,k)}, \\
Var(X_i|X_j, X_k) &= b_{i,(j,k)}.
\end{aligned}$$

Then $X$ is Gaussian. For more information on such characterizations, see Wesolowski (1993) and Arnold and Wesolowski (1997).

4.4 *Multivariate stable distributions.* Multivariate stable laws forms a subclass of multivariate id laws. While they have not found much applications in statistical modelling yet, it is believed that this situation will change in the near future. In particular, they are anticipated to be of much use in economic data modelling. There are different ways of extending the univariate notion of stability, giving rise to different classes of multivariate stable laws. A general reference is Press (1972); also see Horn and Steutel (1978). We will take the following as our definition:

Definition 4. A random vector $X = (X_1, \ldots, X_k)$ with distribution $F$ is said to be stable, equivalently $F$ is said to be stable if for independent copies $X^{(1)}$ and $X^{(2)}$ of $X$, and for any positive numbers $a$ and $b$, there exists a positive number $c$ and a vector $D$ such that $aX^{(1)} + bX^{(2)} \stackrel{\mathcal{D}}{=} cX + D$. If $D = 0$, then $X$ is said to be *strictly stable.*

As in the univariate case, if $X$ is stable, there is an $\alpha$, $0 < \alpha \leq 2$, called the index of $X$ or $F$, such that for any $n \geq 2$, there is a vector $D_n$ such that $X^{(1)} + X^{(2)} + \ldots + X^{(n)} \stackrel{\mathcal{D}}{=} n^{1/\alpha}X + D_n$ where $X^{(1)}, \ldots, X^{(n)}$ are iid copies of $X$. Moreover, this can be taken as the definition of multivariate stability, equivalent to the one given above.

EXAMPLE 24. Of course, as in the univariate case, if $\alpha = 2$, then $X$ is multivariate normal.

EXAMPLE 25. It is not hard to verify the following: If $X$ is $\alpha$ stable (respc. strictly stable) then *all* linear combinations are $\alpha$ stable (respc. strictly stable).

It turns out that the converse is *partially* true and we have the following facts:

(1) If *all* linear combinations of the coordinates of $X$ are stable with $\alpha \geq 1$, then $X$ is stable.

(2) If *all* linear combinations of the coordinates of $X$ are *strictly stable*, then $X$ is strictly stable.

(3) If *all* linear combinations of the coordinates of $X$ are *symmetric stable*, then $X$ is symmetric stable. (Here symmetry is defined as $X \stackrel{\mathcal{D}}{=} -X$).

EXAMPLE 26. The conclusion in (1) is *false* in general if $0 < \alpha < 1$. To see this take $\Psi(t_1, t_2) = \exp\{-r^\alpha + i\rho r \cos(3\phi)\}$ where $t_1 = r\cos(\phi)$, $t_2 = r\sin(\phi)$. Then for sufficiently small $\rho > 0$, $\Psi$ is a characteristic function of a vector $X$ which is *not* stable. However, it is rather easy to check via the characteristic function that *any* linear combination of the two coordinates of $X$ is stable.

REMARK 16. Actually, in the above example, $X$ is not even id. In general, if we assume that $X$ is id and all linear combinations are stable then $X$ is also stable.

*The spectral measure of a stable law.* If $X$ is stable with $0 < \alpha < 2$, then its characteristic function has the following representation. This representation can be arrived at starting from the representation of id laws.

Let $S$ denote the unit sphere in $k$ dimensions and $\Gamma$ a finite measure on $S$. Then, with $<,>$ denoting inner product, the cf of a stable law has the representation:

if $\alpha \neq 1$,

$$\Psi(t) = \exp\left\{i < t,\ \mu > - \int_S | < t, s > |^\alpha \times \left[1 - i\,\mathrm{sign}\left(< t,\ s > \tan\frac{\pi\alpha}{2}\right)\right]\Gamma(ds)\right\}, \tag{9}$$

if $\alpha = 1$, then

$$\begin{aligned}\Psi(t) = \exp\Big\{ & i < t,\ \mu > \\ & - \int_S | < t, s > | \times \left[1 + i\frac{2}{\pi}\mathrm{sign}(< t,\ s > \log | < t,\ s > |)\right]\Gamma(ds)\Big\}. \end{aligned} \tag{10}$$

The pair $(\Gamma, \mu)$ is unique. The above representation is called the *spectral representation.* $\Gamma$ is called the *spectral measure.*

EXAMPLE 27. The characteristic function of the *multivariate Cauchy* random variable $X$ is given by $\Psi(t) = \exp\{-(t'\Sigma t)^{1/2} + i < t, \mu >\}$. If $\Sigma$ is the identity matrix and $\mu = 0$, then $X$ is spherically symmetric stable with $\Gamma$ being the *uniform measure*. For the bivariate case, its density is given by $f(x) = (2\pi)^{-1}(1 + x_1^2 + x_2^2)^{-3/2} \quad -\infty < x_1,\ x_2 < \infty$.

EXAMPLE 28. From the above representation, we can derive a criterion for the independence of the components of a stable vector. If $X$ is stable, then its components are independent if and only if the spectral measure $\Gamma$ is discrete and is concentrated on the intersection of the axes with the sphere $S$.

4.5. *Joint moments and linearity of conditional expectations.* Recall that if $X$ is a one dimensional stable variable with index $\alpha$, then $E|X|^p < \infty$ for all $0 < p < \alpha$. The moment of order equal to $\alpha$ need not be finite as the Cauchy law where $\alpha = 1$ shows. Thus when we deal with stable vectors, we must at least assume that $\alpha > 1$ for the expectation to exist in general. By using Holder's inequality, this not only assures the finiteness of the first moment of *every* component, it also implies every product moment of combined order $p < \alpha$ is finite. To be precise, if $X$ is stable with index $\alpha$, then

$$\sum_{i=1}^n p_i < \alpha \ \Rightarrow E\prod_{i=1}^n |X_i|^{p_i} < \infty.$$

The converse is false in general. However if $X = (X_1,\ X_2)$ is a stable vector with only *two* coordinates, and with index $\alpha < 2$ then, the converse is indeed true.

What happens to regression functions for stable vectors? In particular, if $X = (X_1, \ldots, X_n)$ is stable, is $E(X_1|X_2, \ldots, X_n)$ linear in $X_2, ...X_n$?

Again, the answer is yes, if we have a two vector: if $X = (X_1,\ X_2)$ is stable with index $1 < \alpha < 2$, then $E(X_2|X_1) = cX_1$ for some constant $c$. If $X = (X_1, \ldots, X_n)$, $n > 2$, then in general it is *not* true that $E(X_1|X_2, \ldots, X_n)$ is linear in $(X_2, \ldots, X_n)$.

There are several conditions under which this linearity can be claimed. We give one which is related to the spectral measure $\Gamma$.

If $X$ is strictly stable with index $1 < \alpha < 2$, and spectral measure $\Gamma$, then $E(X_n|X_1, \ldots, X_{n-1}) = \sum_{i=1}^{n-1} a_i X_i$ if and only if

$$\forall\ \boldsymbol{r},\quad \int_S \left(x_n - \sum_{i=1}^{n-1} a_i x_i\right) \left(\sum_{i=1}^{n-1} r_i x_i\right)^{\alpha-1} d\Gamma\ (x) = 0. \tag{11}$$

4.6. *Point processes and infinite divisibility.* Point processes have received enormous attention for their numerous applications and there are several good books in this area: Daley and Vere-Jones (DVJ, 1988), Kallenberg (1983) and Karr (1986) are standard references. Here we will give a very brief exposition of infinitely divisible point processes.

Any probability mechanism by which a random number of points $N(A)$ is allocated to a Borel subset $A$ of the real line defines a point process on the real line. We assume that $N(A)$ is finite for every bounded set $A$. So $N$ is a random counting measure on $R$. In general, the real line may be replaced by any complete separable metric space (c.s.m.s.).

The Poisson process is the simplest example, where for all $a,\ b$, $N((a,\ b])$ has a Poisson distribution. If $E[N((a,\ b])] = \lambda(b-a)$ then it is a *stationary Poisson process*. There are several extensions of the Poisson process such as the compound Poisson process, the mixed Poisson process etc.

A point process $N$ is said to be id if for every $k$, it is a superposition of $k$ iid point processes. The stationary Poisson process is clearly id.

A typical *finite dimensional distribution* (fidi) of a point process is the distribution of $N(A_i),\ i = 1, 2, \ldots k$ where $k$ is any integer and the $A_i$'s are any bounded Borel sets. It is an important fact that:

*Fact*: A point process is id if and only if all its fidi are id.

A widely used class of point processes which has close connections to the study of structure of id point processes is the cluster process. A point process $N$ is called a *cluster process* on the c.s.m.s. $Y$ with center point process $N_c$ on the c.s.m.s. $X$ and component point processes $\{N(\cdot|x), x \in X\}$ if for every bounded set $A$,

$$N(A) = \int_X N(A|x)N_c(dx) \ = \sum_i N(A|x_i)$$

is finite almost surely.

REMARK 17. Note that there is no requirement that individual clusters are almost surely boundedly finite ($N(Y|x) < \infty$). However, this is indeed the case in most common examples.

The cluster process includes all natural extensions of the basic Poisson process. Two other general examples of cluster processes are: the Poisson cluster process (when the cluster center process is Poisson) and the Neyman Scott process (when the individual cluster members are iid). See DVJ for many other examples.

The notion of the probability generating function, so useful for integer valued random variables, has a natural extension to point processes. For any suitable function $h$ define

$$G(h) = E \exp \int \log h(x) N(dx).$$

$G$ is called the *probability generating functional* (p.g.fl.) of the point process $N$.

The following is a representation result for the p.g.fl of id point processes:

RESULT. Suppose that the point process $N$ is a.s. finite and id. Then there exists a uniquely defined a.s. finite point process $N^*$ such that $P(N^*(X) = 0) = 0$ and a finite positive number $\alpha$ such that

$$G(h) = \exp(\alpha(G_{N^*}(h) - 1))$$

for all real valued Borel measurable functions $h$ on $X$ which satisfy $0 \leq h \leq 1$ everywhere and $1 - h$ vanishes outside some bounded set. Conversely, any functional of the above form represents the p.g.fl. of an a.s. finite id point process.

Thus $N$ may be regarded as a Poisson randomization of some other point process $N^*$. The reader may draw a parallel with the Lévy representation for discrete id variables. There is another interpretation. It can be shown that $G$ is indeed the p.g.fl. of a Poisson cluster process. See DVJ for details.

When the almost sure finiteness condition made above is relaxed, the situation gets a little more complicated. However, note that any id point process will remain id and will become finite a.s. if it is restricted to any given bounded set. Thus, locally the above representation holds. This is the basic idea behind a general representation for id point processes. To describe this, we need the concept of the KLM measure, so called due to the basic contributions of Kerstan and Matthes (1964) and Lee (1964, 1967).

Let $\widehat{N}_X$ denote the class of boundedly finite counting measures on $X$ and $\widehat{N}_0(X) = \widehat{N}_X \backslash \{N(X) = 0\}$. A boundedly finite measure $\widetilde{Q}$ on the Borel sets of $\widehat{N}_0(X)$ such that $\widetilde{Q}\{N : N(A) > 0\} < \infty$ for any bounded $A$ is a KLM measure.

Characterizations of id point processes through representability of its p.g.fl. by means of KLM measures may be seen in DVJ. Additional understanding of the structure of id point processes may also be gained through the KLM measure. We first define the following:

Definition 5. An id point process is *regular* if its KLM measure is carried by the set

$$V_r = \{N^* : N^*(X) < \infty\}.$$

Definition 6. An id point process is *singular* if its KLM measure is carried by the complementary set

$$V_s = \{N^* : N^*(X) = \infty\}.$$

Then the following holds:

Result. Every id point process admits the decomposition

$$N = N_R + N_S$$

where $N_R$ and $N_S$ are independent id point processes which are respectively regular and singular.

Finally there are some connections between various classes of KLM measures and id point processes:

Result. Any id process

(i) can be represented as a Poisson randomization if and only if its KLM measure is totally finite.

(ii) is a.s. finite if and only it is regular and its KLM measure is finite.

(iii) is a Poisson cluster process with a.s. finite clusters if and only if it is regular.

For a wealth of further information in this area, we refer the reader to DVJ and Kallenberg (1983). In particular, stationarity of id point processes and the connections to the stationarity of random measures is very thoroughly discussed in DVJ.

## 5. Lévy processes

Lévy processes provide a conceptually natural generalization of Poisson processes and Brownian motion and were introduced by Paul Lévy. The theory of Lévy processes has seen great advances in the last two decades and numerous serious applications are being made. We recommend Bertoin (1996), Sato (2001) for a wealth of information, applications, and numerous references to the literature. The richest single source for applications of Lévy processes to numerous areas of science and economics is Barndorff-Nielsen et al. (2001).

5.1. *Definitions.* Lévy processes can be thought of as strictly stationary processes with independent increments. $X_t$ is a Lévy process on a probability space $(\Omega, B, P)$ if for all $s, t \geq 0$, $X_{s+t} - X_t$ has the same distribution as $X_s$ and is independent of $\sigma\{X_u : 0 \leq u \leq t\}$. In particular, $X_0 = 0$ a.s. If for some $\alpha$ $(0 < \alpha \leq 2)$, $X_t$ and $t^{\frac{1}{\alpha}} X_1$ have the same law for all $t > 0$, then the Lévy process $X_t$ is called a stable process with index $\alpha$.

Lévy processes include Brownian motion and compound Poisson process as special cases. Stable processes with $\alpha = 2$ are proportional to Brownian motion and with $\alpha = 1$ correspond to the centered Cauchy processes.

5.2. *Characteristic Exponents.* The function $\psi : \mathbb{R}^d \to \mathbb{C}$ defined by $Ee^{i<X_t,\lambda>} = e^{-t\psi(\lambda)}$ is called the characteristic exponent of the $d$-dimensional process $X$. Lévy processes with the same characteristic exponent are equal in distribution.

An explicit construction of a Lévy process $X$ with a given characteristic exponent $\psi(.)$ is possible. Towards this end, let $a$ be an element of $\mathbb{R}^d$, and $\Pi$ a measure on $\mathbb{R}^d - \{0\}$ such that

$$\int \min(1, |x|^2)\Pi(dx) < \infty. \tag{12}$$

Let also $Q(.)$ be a positive semi-definite quadratic form. Define

$$\psi(\lambda) = i < a, \lambda > + \frac{Q(\lambda)}{2} + \int_{\mathbb{R}^d - \{0\}} (1 - e^{i<x,\lambda>} + i < x, \lambda >) I\{|x| < 1\}\Pi(dx); \tag{13}$$

then there exists a unique Lévy process $X$ with characteristic exponent $\psi$ and with $\Pi$ as the characteristic measure of the jump process of $X$. See Ito (1942).

5.3. *Basic properties.* Some basic properties of Lévy and Stable processes are listed below.

a. If $E(X_1) = 0$ then the Strong Law holds, i.e., $\frac{X_t}{t} \to 0$ a.s.

b. Every Lévy process satisfies the Strong Markov property.

c. Appropriate versions of the LIL hold. Thus, define $\Phi : (0, \infty) \to (0, \infty)$ by $E(e^{-\lambda X_t}) = e^{-t\Phi(\lambda)}$ and let $\phi$ denote the inverse function $\Phi^{-1}$. Let

$$f(t) = \frac{\log \log |t|}{\Phi(t^{-1} \log \log |t|)}.$$

Suppose $\Phi$ is of regular variation with index $\rho$, $0 < \rho < 1$. Then

$$\liminf_{t \to 0+} \frac{X_t}{f(t)} = \rho(1-\rho)^{\frac{1-\rho}{\rho}} \, a.s.$$

For stable processes with $\alpha \neq 2$, and for any increasing function $g : (0, \infty) \to (0, \infty)$,

$$\limsup_{t \to 0+} \frac{|X_t|}{g(t)} = 0 \text{ or } \infty \quad a.s.$$

according as $\int_0^\infty (g(t))^{-\alpha} dt < \infty$ or $= \infty$.

The same results hold with $t \to 0+$ replaced by $t \to \infty$. See Bertoin (1996) for additional information and Feller (1966), Resnick (1987) for construction and properties of regularly varying functions. For many other sample path properties, see Zolotarev (1964) and Marcus and Rosen (1992,1993, 1994).

5.4. *Recurrence and transience.* There is an elegant dichotomy and an equally elegant integral test to decide recurrence vs. transience of a Lévy process. Let $X$ be a Lévy process not supported on any proper subgroup of $\mathbb{R}^d$. Then, either neighbourhoods of every point of $\mathbb{R}^d$ are visited at arbitrarily large times, a.s., or else $|X_t| \to \infty$ a.s. as $t \to \infty$.

The integral test to determine recurrence vs. transience says that a Lévy process with characteristic exponent $\psi(.)$ is transient if and only if $\exists r > 0$ such that

$$\limsup_{q \to 0} \int_{B(0,r)} Re\left(\frac{1}{q + \psi(\lambda)}\right) d\lambda < \infty; \tag{14}$$

see Bertoin (1996). For the special case of $\mathbb{R}^1$, a centered Lévy process is recurrent; if, on the other hand, $E(X_1) \neq 0$ and $E(|X_1|) < \infty$, then $X_t$ is transient.

For Stable processes, one has recurrence for $1 \leq \alpha \leq 2$ and transience for $0 < \alpha < 1$ on $\mathbb{R}^1$, and recurrence for $\alpha = 2$ and transience otherwise on $\mathbb{R}^2$. On $\mathbb{R}^d$ for $d > 2$, one always has transience.

5.5. *Local times and Fluctuation theory.* Given a Lévy process $X_t$, let $L = L(t,x) = \lim_{\varepsilon \downarrow 0} \frac{Leb(s<t:\ x-\varepsilon<X(s)<x+\varepsilon)}{2\varepsilon}$ denote the local time. Biane and Yor (1987) give many results and pretty identities for Brownian local times. Yor (1982) discusses Hilbert transforms of the local time; recall that the Hilbert transform $\mathcal{H}(\cdot)$ is defined as $\mathcal{H}(f) = \lim_{\varepsilon \to 0+} \int_{|t|>\varepsilon} f(x-t)\frac{1}{\pi t}dt$. Time spent over a given level and fluctuation theory are discussed in Bertoin (1996) and Doney (2001).

Dvoretzky, Erdos and Kakutani (DEK, 1961) introduced the concept of an increase time. $t$ is said to be an increase time of $X$ if for some $\epsilon > 0$, $X(t-s) \leq X(t) \leq X(t+s)$ for all $s < \epsilon$.

DEK (1961) proved the fundamental result that Brownian motion never increases. Generalization of this result to certain types of Lévy processes can be seen in Bertoin (1991,1995,1996).

5.6 *Range and Level Sets.* $(H) = \{X_s(w) : \quad s > 0\}$ is called the Range of $X$. It was proved by Lévy (1953) that the Hausdorff dimension of $d$-dimensional Brownian motion equals 2 a.s. for $d \geq 2$ (for $d = 1$, the range equals all of $\mathbb{R}^1$ a.s.). For a stable process with index $\alpha$, the Hausdorff dimension equals min $(\alpha, d)$ a.s.; this result was first proved by McKean (1955) and subsequently by many others. As regards level sets, $\{t : X_t = x\}$ is called the level set of $x$. For stable processes with $\alpha > 1$, the Hausdorff dimension of the level set is $1 - \frac{1}{\alpha}$ for any $x$. In particular, it is $\frac{1}{2}$ for the Brownian motion. See Sato (1999) for a thorough exposition on this topic.

5.7. *Moments of functionals and the Riemann Zeta Function.* A number of functionals of the Brownian motion and the Brownian Bridge are related to the Riemann Zeta function through their Mellin transforms. Some properties of the Zeta function and their roots follow from these relations. See Pitman and Yor (2001), DasGupta and Lalley (2001) and Williams (1990) for derivations of some of these relations. See also Biane et al. (2001) for a nice review of this area.

If $X(t)$ denotes a Brownian Bridge on $[0,1]$ and $W$ equals its range

$\sup_t X(t) - \inf_t X(t)$, then $M(s)$, the Mellin transform of $W$ equals

$$M(s) = \frac{s(s-1)\Gamma(\frac{s}{2})\zeta(s)}{2^{s/2}}, \quad s \geq 1; \tag{15}$$

at $s = 1$, $\zeta(s)$ has a simple pole and the formula remains valid. In particular, the first four moments of $W$ are $\sqrt{\frac{\pi}{2}}$, $\zeta(2) = \frac{\pi^2}{6}$, $\frac{3}{2}\sqrt{\frac{\pi}{2}}\zeta(3)$ and $3\zeta(4) = \frac{\pi^4}{30}$. Various inequalities on the values of the Zeta function at integer arguments follow from the Mellin transform formula (15); see DasGupta (2002b).

Formula (15) also leads to some properties of the roots of the Zeta function. Let $\rho_k = \frac{1}{2} + it_k$, $t_k > 0$, be the roots on the critical line with a positive imaginary part and consider the real function $f(s) = \sum_{k=1}^{\infty} \log\left(1 + \frac{s(s-1)}{\frac{1}{4}+t_k^2}\right)$, $s > 1$. This series converges and it follows from (15) that $f$ is convex. Moreover, if the Riemann hypothesis is true, then (15) and a result in Bach and Shallit (1996) imply that $f'(s) > 2 + \gamma - \log(4\pi)$ for all $s > 1$. A proof of this is given in DasGupta and Lalley (2001); here $\gamma$ is the Euler constant.

If $Z = \sup_t |X(t)|$, then its Mellin transform equals

$$M(s) = \frac{s(2^{s-1}-1)\Gamma(\frac{s}{2})\zeta(s)}{2^{\frac{3}{2}s-1}}, \quad s \geq 1; \tag{16}$$

in particular, the first four moments are $\sqrt{\frac{\pi}{2}} \log\, 2$, $\frac{\pi^2}{12}$, $\frac{9\sqrt{\pi}}{16\sqrt{2}}\zeta(3)$ and $\frac{7\pi^4}{720}$. Analogous expressions for the Mellin transforms of $Z = \sup_t |X(t)|$ where $X(t)$ is either the Brownian motion or the Brownian meander are also available. For the Brownian motion, the first two moments of $Z$ are $\frac{\pi}{4}$ and $\frac{8}{3\sqrt{2\pi}}G$, where $G$ is the Catalan constant. For the Brownian meander, these are $\sqrt{2\pi} \log\, 2$ and $\frac{\pi^2}{3}$ respectively (see DasGupta, 2002b).

## 6. Simulation of id Laws

To understand the behaviour of different statistical procedures where id laws are involved, it is important to be able to simulate id and stable laws. We concentrate here on simulation of id laws in general. The simulation of stable laws is a significantly more specialized task and will not be discussed here. The interested reader may consult Adler et al. (1998) for material on that. We sketch two approaches.

*Approach 1: via Poisson processes.* Bondesson (1982) noticed an interesting connection between Poisson processes and id laws as follows.

Let $Z(u)$, $u > 0$ be a family of non-negative independent random variables. Let $T_i$, $i = 1, 2 \ldots$ be the points (in increasing order) in an independent Poisson point process of rate $\lambda$ on $(0,\ \infty)$. Let $X = \sum_{i=1}^{\infty} Z(T_i)$ and let $X_T = \sum_{T_i \le T} Z(T_i)$ where $T$ is a truncation point. The Laplace transform (LT) of $X_T$ is given by

$$E[\exp\{-sX_T\}] = \exp\left\{\lambda \int_{(0,\ T)} (\psi(s,u) - 1) du\right\}$$

where $\psi(s,u) = E[\exp\{-sZ(u)\}]$. It follows that $X$ has the LT

$$\phi(s) = \exp\left\{\lambda \int_{(0,\ \infty)} (\psi(s,u) - 1) du\right\}. \tag{17}$$

We assume that $X < \infty$ almost surely. (Otherwise $X = \infty$ almost surely: this follows from the zero one law and the fact that the process $X_T$, $T > 0$ has independent increments.) Clearly $X$ is id.

Now let $Z(u)$ have the distribution function $H(y,u)$. Under suitable regularity conditions on the measure $H(dy,u)$, changing the order of integration, we may rewrite (17) as

$$\phi(s) = \exp\left\{\lambda \int_{[0,\ \infty)} (e^{-sy} - 1) \left(\int_{(0,\ \infty)} H(dy,u) du\right)\right\}. \tag{18}$$

Now we consider simulation from an id distribution $F$ with Lévy measure $\Lambda(dy)$. Suppose we can find a simple family of distribution functions $H(y,u)$ on $[0,\ \infty)$ and a $\lambda$ such that on $(0,\ \infty)$,

$$\lambda \int_{(0,\ \infty)} H(dy,u) du = \Lambda(dy)$$

or equivalently, for $x > 0$,

$$\lambda \int_{(0,\ \infty)} \bar{H}(x,u) du = \int_{(x,\ \infty)} \Lambda(dy) = \bar{N}(x)$$

where $\bar{H} = 1 - H$. Then simulate points $T_i$ in a Poisson $(\lambda)$ process by for example adding independent exponential random numbers and after that, values $Z(T_i)$ from the distribution functions $H(x,\ T_i)$ and set $X = \sum_{i=1}^{\infty} Z(T_i)$. Then $X$ has the desired distribution. If the sum converges rapidly, only a few terms are needed to get a good approximate value of $X$.

Bondesson (1982) showed how different classes of $H$ lead to different id distributions such as, the generalized convolutions of mixtures of exponentials (class $\mathcal{T}_2$ of Bondesson, 1981), generalized gamma convolutions (Thorin, 1978a,b) and the generalized negative binomial convolutions (Bondesson, 1979a).

*Approach 2: via structural theorem(characterization #4).* Recall characterization #4 given earlier for id laws as limit of sums of independent Poisson type random variables. We state this fact again here in the form of a theorem, commonly known as the *structural theorem.* A proof may be found in Loeve (1960, p.298).

THEOREM. *A characteristic function $\psi$ is id if and only if it is the limit of sequences of products of Poisson types. That is there exists $a_{nk}$ and $b_{nk}$ such that*

$$\psi(t) = \lim_{n\to\infty} \exp\left[\sum_{k=1}^{n} ita_{nk} + \lambda_{nk}\{\exp(-itb_{nk}) - 1\}\right] \tag{19}$$

The algorithm of Damien et al. (1995) to generate an observation from a given id law with characteristic function $\psi$ proceeds as follows:

Let $\Lambda$ be the appropriately defined (finite) Lévy-Khintchine measure associated with $\psi$.

Let $\Lambda_1,\ldots,\Lambda_n$ be i.i.d. from the distribution $\frac{1}{k}d\Lambda(x)$ where $k = \int_{-\infty}^{\infty} d\Lambda(x)$.

Let $Y_i \sim \text{Poi}\left(\frac{k(1+\Lambda_i^2)}{n\Lambda_i^2}\right)$, $i = 1,\ldots,n$.

Let $X_n = \sum_{i=1}^{n}\left(\Lambda_i Y_i - \frac{k}{n\Lambda_i}\right)$. Then $\psi_{X_n}(t) \longrightarrow \psi(t)\ \forall\ t$, as $n\to\infty$.

In particular, they use this algorithm to generate observations from several stable distributions and study the accuracy via the Kolmogorov-Smirnov metric.

This has interesting applications in Bayesian nonparametrics. Consider the problem of estimating an unknown cdf $F$ on $[0,\infty)$ based on $n$ iid observations (possibly censored) from $F$. This requires putting a prior distribution on the space of distribution functions $\mathcal{F}$. Viewing $F$ as a stochastic process, let $F(t) = 1 - \exp(-Y_t)$ where $\{Y_t\}$ is a Lévy process. The posterior distribution is also a Lévy process. See Ferguson and Phadia (1979) for details. The increments of this process, when the jumps are removed, are id. Using the above approach, these continuous increments can be simulated. The jump components are independent and hence simulating the increments corresponding to these jumps is standard. Combining these two simulations, the total increments of the process are simulated. This implies that a complete Bayesian analysis of the posterior distribution is possible. In particular, the authors show how to implement the idea in estimating the survival function using the three priors, gamma process, Dirichlet process and the simple homogeneous process.

Apparently, no results are known regarding the rate of convergence of the generated samples, but the simulation results of the authors are quite promising.

Related papers in Bayesian nonparametrics where particular Lévy processes have been used are (i) Hjort (1990) who uses beta processes, and (ii) Ramgopal and Smith (1993) who use extended gamma processes.

## 7. Stable Laws in Inference

As we have seen earlier, no id law can have tails thinner than the normal tail. However, the tails of an id law can be quite heavy. As data from a steadily increasing number of fields have exhibited heavy tailed behaviour, the importance of id laws in statistical modelling and inference has grown. See Adler et al. (1998) and DuMouchel (1973b,1975).

We have very briefly mentioned the use of id processes in Bayesian inference in the previous section. However, since the class of id laws consists of the weak limits of triangular sums, it is a huge class and is not convenient for most statistical modelling and inference problems.

On the other hand, any stable law is obtained as the weak limit of sums of iid random variables. Thus it serves as a very natural model in situations where aggregation is involved. This explains the importance of the normal distribution when the observations have finite second moments. But this leaves out the distributions with heavy tails.

As we have seen in section 3.3, at least one of the tails of a stable law decreases as the $\alpha$th power. This offers flexibility in modelling heavy tailed phenomena by stable laws with an appropriate choice of $\alpha$, $0 < \alpha < 2$. Instances where the stable model holds exactly are not very frequent.

EXAMPLE 29. As early as 1919, before the concept of stable laws was introduced by Paul Lévy, Holtsmark found that under certain natural assumptions, the random fluctuation of the gravitational field of stars in space has a probability density whose cf is given by $\exp\{-\lambda|t|^{3/2}\}$, $t \in \mathbb{R}^3$ where $\lambda$ is a positive constant determined by certain physical characteristics. This is a three dimensional spherically symmetric stable law with $\alpha = 3/2$ and is known as the *Holtsmark distribution.*

Since we cannot hope to have exact stability of the observations we must look for approximate stability. This leads to the concept of domain of attraction.

7.1. *Domain of attraction*

DEFINITION 7. A distribution $F$ is said to belong to the *domain of attraction* of a stable law with index $\alpha$ if there exists real sequences $\{a_n > 0\}$ and $\{b_n\}$ such that if $X_1, \ldots X_n \ldots$ are iid with distribution $F$ then $b_n^{-1}(X_1 + \ldots + X_n - a_n)$ converges in distribution to this stable law. We write $F \in \mathcal{D}(\alpha)$.

EXAMPLE 30. Any stable distribution is in its own domain of attraction. All distributions with finite second moments are in the domain of attraction of the normal law.

Plenty of distributions with infinite second moments are also in the domain of the normal law. It will be easier to provide such examples after we give the criteria for checking whether a distribution belongs to $\mathcal{D}(\alpha)$.

There are two such simple but powerful criteria. To state these, recall that a function $L(\cdot)$ is said to be *slowly varying* if for every $x > 0$, $L(tx)/L(t) \to 1$ as $t \to \infty$. Below, $L$ is any such slowly varying function.

*Criterion 1.* A distribution $F$ belongs to the domain of attraction of a stable law
(i) with index $0 < \alpha < 2$, if and only if there exists $0 \leq p \leq 1$ such that,

$$\lim_{x \to \infty} \frac{1 - F(x)}{1 - F(x) + F(-x)} = p \tag{20}$$

and

$$1 - F(x) + F(-x) \sim \frac{2-\alpha}{\alpha} x^{-\alpha} L(x) \text{ as } x \to \infty \tag{21}$$

(ii) with index $\alpha = 2$ (normal law), if and only if

$$\int_{-x}^{x} y^2 dF(y) \sim L(x) \text{ as } x \to \infty. \tag{22}$$

EXAMPLE 31. Consider the Pareto law discussed in Example 10 which has the density $f(x) = \frac{\alpha}{\mu}(\frac{\mu}{x+\mu})^{\alpha+1}$, $x > 0$. It is easy to see that it belongs to $\mathcal{D}(\alpha)$.

EXAMPLE 32. By using the criterion above, it is easy to construct examples of distributions $F$ whose second moments are infinite but which belong to the domain of attraction of the normal law. For instance, the distribution $F$ with density $f(x) = 2|x|^{-3}\log x$ for $|x| \geq 1$ has infinite second moment and belongs to the domain of attraction of the normal law. The $t$ distribution with one degree of freedom is the Cauchy law and so is stable. The $t$ distribution with degrees of freedom three or more has finite second moment and hence is in the the domain of attraction of the normal law. The $t$ distribution with *two* degrees of freedom has the density $f(x) = c(1+x^2)^{-3/2}$ where $c$ is a constant. So it does not have finite second moment. However, it is easy to check that Criterion 1 (ii) is satisfied with $L(x) = \log x$. Hence the $t$ distribution with two degrees of freedom belongs to the domain of attraction of the normal law.

EXAMPLE 33. In the definition of domain of attraction, we used *sums* of variables. If we use other composition operations, we obtain other notions of stability. We discuss one such alternate notion of stability, obtained by taking *maximums.* Suppose that $X_1, \ldots, X_n$ are iid $F$. Let $M_n = \max\{X_1, \ldots, X_n\}$. Suppose that $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ are sequences such that $a_n^{-1}(M_n - b_n)$ converges in distribution to some distribution $G$. We write $F \in \max\mathcal{D}(G)$. The class of limit distributions obtained in this way is called the class of *extreme value distributions* or *max stable laws.* See Resnick (1987) and Reiss (1989). One subclass of this class consists of the *Frechet distributions* defined as:

$$\Phi_\alpha(x) = \exp\{-x^{-\alpha}\},\ x > 0,\ \text{where } \alpha > 0.$$

From the extreme value theory, $F \in \max\mathcal{D}(\Phi_\alpha)$ if and only if $1 - F(x) = x^{-\alpha}L(x)$. Note that this condition is a *part* of the condition for $F$ to belong to $\mathcal{D}(\alpha)$.

We now present the second domain of attraction criterion. An application to the problem of estimation of $\alpha$ may be found in section 7.3.

*Criterion 2.* A distribution $F$ belongs to the domain of attraction of a stable law with index $0 < \alpha < 2$, if and only if

$$\frac{x^2[1 - F(x) - F(-x)]}{\int_{-x}^{x} y^2 dF(y)} \sim \frac{2-\alpha}{\alpha} \text{ as } x \to \infty. \tag{23}$$

7.2. *Estimation of $\alpha$, preliminaries.* The normal law has a rapidly decreasing tail and corresponds to $\alpha = 2$. For modelling heavy tailed phenomena, we restrict our discussion to the class of stable laws with index $0 < \alpha < 2$. This leads to the following basic question:

*Question:* Suppose we have iid observations from a distribution $F \in \mathcal{D}(\alpha)$. How does one estimate the parameter $\alpha$?

Note that even if we assume that $F$ itself is stable, the problem is still not easy. As mentioned earlier, except for the three special distributions normal, Cauchy and Lévy, no closed form expressions are known for the density of stable laws. This makes the problem of estimating $\alpha$ quite difficult. Possible approaches to the estimation problem are already offered indirectly in the discussion of section 7.1:

(i) Example 33 suggests that the extreme order statistics have a role to play.

(ii) Criterion 1 suggests how the sample versions of $F$, the *empirical distribution* $F_n$ may be used to obtain estimates of $\alpha$. Likewise, Criterion 2 also suggests estimates for $\alpha$.

(iii) The cf of stable laws is available in a closed form. Thus the use of the *empirical characteristic function* offers another possible approach, at least when $F$ is exactly stable. For general theory on estimation of $\alpha$, see Csorgo et al. (1985) DuMouchel (1983), Hill (1975), McCullough (1997), Resnick and Starica (1997b), and de Haan and Peng (1998).

EXAMPLE 34. Consider the *one* parameter Pareto distribution with parameter $\alpha$ whose cdf is given by

$$1 - F(x) = x^{-\alpha}, \quad x > 1. \tag{24}$$

Assume that $X_1, \ldots, X_n$ are iid observations from this Pareto law. Since the distribution and the density in this case are explicitly known, we can use the *method of maximum likelihood* to estimate $\alpha$. By writing down the joint density of $X_1, \ldots, X_n$, it is easily seen that the *maximum likelihood estimator* of $\gamma = \alpha^{-1}$ is given by

$$\hat{\gamma}_n = n^{-1} \sum_{i=1}^{n} \log X_i \ = n^{-1} \sum_{i=1}^{n} \log X_{(i)}.$$

Above, $X_{(1)} < \ldots < X_{(n)}$ are the *order statistics* of $X_1 \ldots X_n$. We shall use this notation in our subsequent discussion also.

The nice thing about this estimate is that it involves the random variables through their logarithms which have finite second moments. Indeed,

$$E(\log X_1) = \gamma \quad \text{and } Var(\log X_1) = \gamma^2. \tag{25}$$

By using the central limit theorem, we thus have

$$n^{1/2}(\hat{\gamma}_n - \gamma) \overset{\mathcal{D}}{\Rightarrow} N(0,\ \gamma^2). \tag{26}$$

Suppose now that $F \in \mathcal{D}(\alpha)$. Suppose that the right tail is nontrivial so that equation (20) is satisfied with $p > 0$. Then Criterion 1 implies that the right tail behaves like the Pareto tail (24) in Example 34, except for a slowly changing function. This feature is the basis of many estimators of $\alpha$ in the literature. In the next few subsections we shall describe some of the estimators of $\alpha$. For some comparisons of these estimators based on simulations, see Pictet et al. (1998). The general recommendation is that Hill's estimator, discussed in section 7.3., is the best to use.

7.3. *The Hill estimator* Assume that $F \in \mathcal{D}(\alpha)$ is such that

$$1 - F(x) = x^{-\alpha} L(x), \quad \text{as} \quad x \to \infty \tag{27}$$

where $L(\cdot)$ is a *slowly varying function*. Note that this implies that *if* $X_{(n-k)}$ *is large*, then the following approximate relation holds:

$$\frac{1 - F(x\ X_{(n-k)})}{1 - F(X_{(n-k)})} \approx x^{-\alpha}. \tag{28}$$

Conditional on $X_{(n-k)}$, $\left(\frac{X_{(n)}}{X_{(n-k)}}, \ldots, \frac{X_{(n-k+1)}}{X_{(n-k)}}\right)$ is distributed as the order statistics from a sample of size $k$ from the distribution with tail

$$\frac{1 - F(x\ X_{(n-k)})}{1 - F(X_{(n-k)})}, \quad x \geq 1$$

which, as (27) holds, is approximately the Pareto tail. Thus going back to the estimate introduced in the special case of the Pareto, it appears intuitively justified to use the above ratios, for some large value of $k$, in the same way as all the observations were used in defining (24) in the exact Pareto case. This leads to the famous Hill's estimator (Hill, 1975): choose $k < n$ large in some appropriate way. Then the *Hill estimate* of $\gamma = \alpha^{-1}$ on the basis of $n$ iid observations from the distribution $F$ satisfying (27) is defined as

$$\hat{\gamma}_{k,n} = k^{-1} \sum_{i=n-k+1}^{n} \log \frac{X_{(i)}}{X_{(n-k)}}. \tag{29}$$

The Hill estimate uses only the *upper* $(k+1)$ ordered statistics of the sample and ignores the rest of the sample. The uneasy aspect is the dependence on the choice of $k$. We shall address this issue below. But first let us see a result which guarantees that this method works, at least asymptotically.

*Consistency of the Hill estimator*. Suppose that $n \to \infty$ so that we have a sample size which increases indefinitely. Let $k = k_n$ be such that $k \to \infty$

but $k/n \to 0$. This means that we use a very large proportion of the ordered statistics. It turns out that this guarantees (Mason, 1982)

$$\hat{\gamma}_{k,n} \xrightarrow{P} \gamma. \tag{30}$$

Note that no additional assumptions on $F$ are required for the above result. So the Hill estimator is *consistent* under minimal assumptions.

In practice, sometimes one has to deal with data which are not iid. Extensions of the above consistency to situations where $\{X_i\}$ is a dependent sequence may be found in Rootzen et al. (1990), Hsing (1991), and Resnick and Stărică (1998).

*Asymptotic distribution and confidence interval.* In applications, one is not satisfied with a *point* estimate and a consistent *interval* estimate is more comforting. This requires establishing a non-degenerate (asymptotic) distribution of the estimator with an appropriate norming and centering. Unfortunately, the class of all $F$ which are in the domain of a stable law with index $\alpha$ is still too large and such a result is *not* available. However, under suitable restrictions on $F$ the same limit law (26) as in the exact Pareto case holds. This result is actually true under several different sets of sufficient conditions. The reader may consult de Haan and Resnick (1998) and the references contained there for more details. *Under suitable conditions on* $k$ and $F$,

$$k^{1/2}(\hat{\gamma}_{k,n} - \gamma) \xrightarrow{\mathcal{D}} N(0,\ \gamma^2). \tag{31}$$

It is assuring that the limiting variance involves $F$ only through $\gamma$. This makes setting up an approximate confidence interval for $\gamma$ easy. Fix a confidence coefficient $1 - \beta$ and let $\Phi^{-1}(\beta/2)$ be the *upper* $\beta/2$ percentile of the standard normal distribution. Then a $100(1-\beta)\%$ asymptotically correct confidence interval for $\gamma$ is given by:

$$I_n \;=\; [\hat{\gamma}_{k,n}\{1 - k^{-1/2}\Phi^{-1}(\beta/2)\},\ \hat{\gamma}_{k,n}\{1 + k^{-1/2}\Phi^{-1}(\beta/2)\}]. \tag{32}$$

The consistency result (30) and the asymptotic normality result (31) together imply that $P\{\hat{\gamma}_{k,n} \in I_n\} \to 1 - \beta$ under the conditions alluded to. The equivalent statement for the estimate of $\alpha$ is of course obtained by taking the interval with the end points as the inverses of the end points of $I_n$.

*Choice of $k$: the Hill plot.* The consistency and asymptotic normality property of the Hill estimator depends on $k = k_n$ going to infinity at an appropriate rate. In practice, given a sample of size $n$, one has to decide on the value of $k$ to use. One approach is to use the *Hill plot.* This is simply a plot of the estimator $(\hat{\gamma}_{k,n})^{-1}$ against $k$. On this plot we look for a range of values of $k$ where the plot is flat. This gives a range of possible values of $k$ which can be used to calculate the estimate. Empirically, it has been seen that the estimator is quite insensitive to the eventual choice of $k$ in the chosen range. For more information, see Drees et al. (2000). This article also carries information on various refinements of the Hill plot.

*Bias of the Hill estimator in small samples.* Since the Hill estimator is based on an approximation of the tail of $F$, it is natural for it to have some *bias* in finite samples. The amount of bias is determined by the finer behaviour of the tail of $F$. One possibility in studying the bias is to work with specified subclasses of $F$. Here is one such result. Consider the class of $F \in \mathcal{D}(\alpha)$ which satisfy for some $\alpha > 0$ and $\beta > 0$,

$$1 - F(x) = ax^{-\alpha}[1 + bx^{-\beta} + o(x^{-\beta})]. \tag{33}$$

Then if $k = k_n \to \infty$, $k/n \to 0$, the asymptotic bias $B$ of the Hill estimator is given by:

$$B = -\frac{\beta b}{\alpha(\alpha+\beta)} a^{-\frac{\beta}{\alpha}} \left(\frac{k}{n}\right)^{\frac{\beta}{\alpha}} \{1 + o(1)\}. \tag{34}$$

An asymptotic expression for the variance can also be derived:

$$Var(\hat{\gamma}_{k,n}) = \left[\frac{\beta^2 b^2}{\alpha^2(\alpha+\beta)^2} a^{-\frac{2\beta}{\alpha}} \left(\frac{k}{n}\right)^{\frac{2\beta}{\alpha}} + \frac{1}{\alpha^2 k}\right] + o(1). \tag{35}$$

See Goldie and Smith (1987), Hall and Welsh (1985) and Pictet et al. (1998) for bias expressions in various situations and recommendations for the choice of $k$.

*Unsatisfactory behaviour near $\alpha = 2$.* While the Hill estimator is one of the best and popular methods, its unsatisfactory performance is documented in the literature when $\alpha$ is close to 2. A possible explanation is that while the tail of a stable law with index $\alpha < 2$ is like $x^{-\alpha}$, the tail of the normal law ($\alpha = 2$) is exponentially decreasing. Further, a few upper order statistics cannot be expected to yield good estimators for "near normal" laws.

7.4. *de Haan and Pereira's estimator.* de Haan and Pereira (1999) focussed on the situation where $\alpha$ may be close to 2. Let $\beta = \frac{2-\alpha}{\alpha}$. Note that if $\alpha$ is close to 2 then $\beta$ is close to zero. In this situation, it appears to be reasonable to consider Criterion 2 and start with (23) to build an estimator. Consideration of the sample analogue of Criterion 2 leads to their estimator.

So suppose we have iid observations on $F \in \mathcal{D}(\alpha)$. Let the order statistics of $|X_i|$, $1 \le i \le n$ be denoted by $|X|_{(1)} \le \ldots \le |X|_{(n)}$. Let $\mathcal{G}_n$ be the empirical distribution of $\{|X_i|,\ 1 \le i \le n\}$. Motivated by (23), we may choose a $k = k_n \to \infty$ and define the estimator $\hat{\beta}_n$ of $\beta = \frac{2-\alpha}{\alpha}$ as

$$\hat{\beta}_n = \frac{k|X|^2_{(n-k)}}{\sum_{i=1}^{n-k} |X|^2_{(i)}}. \tag{36}$$

It may be noted that this estimate uses the $(n-k)$ *lower* order statistics of the absolute values. The estimate of $\beta$ is easily transformed into an estimate $\hat{\alpha}_n$ of $\alpha$ as $\hat{\alpha}_n = 2(1+\hat{\beta}_n)^{-1}$.

Under various assumptions on $\{k_n\}$, and $F$, the consistency and asymptotic normality of $\hat{\beta}_n$ hold. However, the norming is not as simple as the one in the Hill estimator. As with the Hill estimator, in our statement, we shall leave out the exact assumptions required. For details of the conditions required, see de Haan and Pereira (1999). To state the asymptotic normality of $\hat{\beta}_n$, let

$$\begin{aligned} N_n &= \frac{1}{2}\sum_{i=0}^{k-1}\{\log|X|^2_{(n-i)} - \log|X|^2_{(n-k)}\} \\ \beta_n &= \frac{k}{n}\{\mathcal{G}^{-1}(1-k/n)\}^2 \Big/ \int_0^{1-\frac{k}{n}} (\mathcal{G}^{-1}(t))^2 dt \end{aligned}$$

where $\mathcal{G}^{-1}$ is the inverse of $\mathcal{G}(x) = F(x) - F(-x)$. Then *under appropriate conditions*,

$$\frac{k^{1/2}}{N_n}\left(\frac{\hat{\beta}_n}{\beta_n} - 1\right) \xrightarrow{\mathcal{D}} N(0,\ (2\beta+1)^{-1}). \tag{37}$$

7.5. *A moment estimator.* In Example 33 we have seen a parametric subclass of the max-stable laws. Indeed the entire collection of extreme value distributions can also be parametrized. The approach to estimating this parameter leads to an estimate for the stability index $\alpha$ as well.

A distribution is an *extreme value distribution* if and only if up to a scale and location shift, it is of the form:

$$G_\gamma(x) = \exp(-(1+\gamma x)^{-\gamma^{-1}}), \quad \gamma \in \mathbb{R},\ x > 0. \tag{38}$$

The case of $\gamma = 0$ is interpreted as

$$G_0(x) = \exp(-e^{-x}). \tag{39}$$

The parameter $\gamma$ may be called the *extreme value index* of the distribution. For $\gamma > 0$, let $\alpha = \gamma^{-1}$. Then this parametrization is consistent with the parametrization of the stable class. That is, $F \in \mathcal{D}(\alpha)$ for some $0 < \alpha < 2$, if and only if $F \in \max \mathcal{D}(G_\gamma)$. Now consider the problem of estimating $\gamma$ when it is known that $F \in \max \mathcal{D}(G_\gamma)$. Suppose that $\gamma > 0$. Dekkers et al. (1989) considered the problem of estimation of $\gamma$ and one of the estimators they consider is obtained by a moment approach. For $r = 1, 2$, let

$$H_{k,n}^{(r)} = k^{-1} \sum_{i=n-k+1}^{n} \left( \log \frac{X_{(i)}}{X_{(n-k)}} \right)^r. \tag{40}$$

Hence, $H_{k,n}^{(1)}$ is Hill's estimator. Define the estimator $\hat{\gamma}_n$ of $\gamma$ as

$$\hat{\gamma}_n = H_{k,n}^{(1)} + 1 - \frac{1/2}{1 - \left(H_{k,n}^{(1)}\right)^2 / H_{k,n}^{(2)}}. \tag{41}$$

Note that the estimator $\hat{\gamma}_n$ is an estimator of the extreme value index $\gamma$ and is defined even if $F$ does not belong to $\mathcal{D}(\alpha)$. That is, it is an estimator of $\gamma$ irrespective of whether $\alpha = 1/\gamma$ is in the interval $(0,\ 2)$. This is an important aspect: suppose we do not know whether $F$ has heavy tails. Then we will be wary of using the Hill estimator since it is specially geared towards the heavy tailed situation. We can then consider using the current estimator. The estimator is consistent for all values of $\gamma$: if $F \in \max \mathcal{D}(\gamma)$, $k \to \infty$ and $k/n \to 0$, then

$$\hat{\gamma}_n \xrightarrow{P} \gamma. \tag{42}$$

So then why use Hill's estimator at all? This is reflected in the asymptotic distribution of the estimator. As before we skip the precise conditions, which can be seen in Dekkers et al. (1990). Under suitable conditions,

$$k^{1/2}(\hat{\gamma}_n - \gamma) \xrightarrow{D} N(0, 1 + \gamma^2). \tag{43}$$

Recall that the asymptotic variance of the Hill estimator is $\gamma^2$ and so, the current estimator has a larger asymptotic variance than the Hill estimator.

7.6. *Other estimators.* There are many other estimators that are available in the literature. We will not go into a detailed description of these. Here are some of the more well known ones:

1. *Pickands estimator*: This is a very quick and easy estimator proposed in Pickands (1975). It involves calculating the 25%, 50% and 75% quantiles.

See Dekkers and de Haan (1989) for its strong consistency and asymptotic normality under appropriate conditions. The estimator is defined as

$$\hat{\gamma}_n^P = (\log 2)^{-1} \log \left[ \left( X_{(k)} - X_{(2k)} \right) / \left( X_{(2k)} - X_{(4k)} \right) \right]. \tag{44}$$

*2. de Haan-Resnick estimator*: This estimator is given in de Haan and Resnick (1980) and involves only the maximum and *one* other extreme order statistics. It is thus a simplified version of the Hill estimator.

$$\hat{\gamma}_n^R = \left( \log X_{(1)} - \log X_{(k)} \right) / \log k. \tag{45}$$

*3. The CD plot estimator.* The log-log *complementary distribution* (CD) plot estimator also has its genesis in the Pareto expression

$$1 - F(x) \sim x^{-\alpha} \text{ as } x \to \infty. \tag{46}$$

This implies that $\log(1 - F(x))$ and $x$ are linearly related *for large* $x$ with slope $-\alpha$. In practice, we plot $\log(1 - F_n(x))$ against $x$ and choose a *large* $x_0$ beyond which the plot looks linear. Estimate the (negative) slope by fitting a straight line (with equally spaced chosen points on the $X$-axis) and the negative of the slope is the estimate for $\alpha$.

REMARK 18. Even though in practice observations can rarely be assumed to be exactly stable, it is illuminating to consider such a situation and investigate how the different parameters ($\alpha$, $\beta$ and $b$) in the corresponding cf representation can be estimated. These estimators can also serve as preliminary estimators in more complicated procedures which involve observations which are not exactly stable. The *McCulloch estimator* is the simplest among these and is designed for the situation when the observations are from a stable law with $\alpha \in [0.6,\ 2]$. The main virtue of the estimator is its simplicity of calculation. It may be termed as the *method of five quantiles* and is known to perform remarkably well in practice.

Suppose that $F$ is stable with the cf given in section 3.3. Let $F_p$ denote the $p$th quantile of $F$. Let

$$\Phi_1(\alpha,\ \beta) = \frac{F_{0.95} - F_{0.05}}{F_{0.75} - F_{0.25}} \text{ and } \Phi_2(\alpha,\ \beta) = \frac{F_{0.95} + F_{0.05} - 2F_{0.50}}{F_{0.95} - F_{0.05}}. \tag{47}$$

It turns out that $\Phi_1$ is monotonic in $\alpha$ and $\Phi_2$ is monotonic in $\beta$ (for fixed $\alpha$) and so we can invert these functions to get

$$\alpha = \Psi_1(\Phi_1,\ \Phi_2) \text{ and } \beta = \Psi_2(\Phi_1,\ \Phi_2). \tag{48}$$

McCulloch (1986) tabulated these values for various values of $\Phi_1$ and $\Phi_2$.

To form the estimators of $\alpha$ and $\beta$, first estimate the five quantiles above by the respective sample quantiles. Use these to obtain estimates of $\Phi_1$ and $\Phi_2$. Then use McCulloch's tables to obtain the estimates of the two parameters.

Another common approach is to use the representation of the characteristic functions of stable distributions. The corresponding sample cf is used to construct these estimators. The reader may consult Kogan and Williams (1998) and the references contained in that paper for material on this topic.

## 8. Applications

Stable and id distributions have found the greatest applications in finance and economics. There have been other applications as well in problems involving heavy tails; see the recent book by Uchaikin and Zolotarev (1999). Here we will mention a few applications in the areas of finance and economics.

Benoit Mandelbrot made the first attempt to use stable distributions for modelling stock returns by questioning the use of normal distributions for that purpose; see Mandelbrot (1963). Use of stable laws for analysing stock returns is also made in Officer (1972).

Applications in capital asset pricing are discussed in Gamrowski and Rachev (1994), and in a very nice review article by McCulloch (1996). Stable laws have also been used in option pricing and for modelling foreign exchange rates; see McCulloch (1996) for a comprehensive review of the models.

The finance and economics literature also contain methods for estimation of stable law parameters, and this development has been partially independent of the probability and statistics literature. Methods of parameter estimation are discussed in Arad (1980) in the context of stock returns, and in Liu and Brorsen (1995) in the context of modelling foreign exchange rates, in particular. We enthusiastically recommend these references and Adler et al. (1998) to the readers for all sorts of applications.

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ARUP BOSE
STATISTICS AND MATHEMATICS UNIT
INDIAN STATISTICAL INSTITUTE
KOLKATA, INDIA
E-mail: abose@isical.ac.in

ANIRBAN DASGUPTA AND HERMAN RUBIN
DEPARTMENT OF STATISTICS
PURDUE UNIVERSITY
WEST LAFAYETTE, IN 47907, USA
E-mail: dasgupta@stat.purdue.edu
hrubin@stat.purdue.edu