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AN ENTROPIC UNCERTAINTY PRINCIPLE FOR QUANTUM MEASUREMENTS

By M. KRISHNA
and
K.R. PARTHASARATHY
Institute of Mathematical Sciences, Chennai, India

SUMMARY. The Entropic uncertainty principle as outlined by Maassen and Uffink (1988) for a pair of non-degenerate observables in a finite level quantum system is generalized here to the case of a pair of arbitrary quantum measurements. In particular, our result includes not only the case of projective measurements (or equivalently, observables) exhibiting degeneracy but also an uncertainty principle for a single measurement.

1. Introduction

In the context of quantum computation and information, the notion of a measurement for a finite level quantum system has acquired great importance. See, for example, Nielsen and Chuang (2000). Suppose that a finite level quantum system is described by pure states which are unit vectors in a d-dimensional complex Hilbert Space $\mathcal{H}$ with scalar product $< \cdot, \cdot >$ which is linear in the second variable. By a measurement $\mathbf{X}$ we mean $\mathbf{X} = (X_1, X_2, \ldots, X_m)$, a finite sequence of positive operators satisfying the relation $\sum_{i=1}^{m} X_i = I$. If $\psi \in \mathcal{H}$ is a unit vector, then (in the Dirac notation) $p_i =< \psi|X_i|\psi >, i = 1, \ldots, m$ is a probability distribution on the set $\{1, 2, \ldots, m\}$ which is interpreted as a labelling of the possible elementary outcomes of the measurement. The corresponding uncertainty involved in such a measurement is measured by the entropy

$$H(\mathbf{X}, \psi) = -\sum_{i=1}^{m} p_i \ \log_2 \ p_i. \tag{1}$$

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Now consider two different measurements, $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_m)$ in the state ψ. We would then like to describe the Entropic uncertainty principle by a sharp lower bound for the sum $H(\mathbf{X}, \psi) + H(\mathbf{Y}, \psi)$ of the two entropies. Such an approach for observables was first initiated by Bialynicki-Birula and Mycielski (1975). Pursuing a conjecture of Kraus (1987), Maassen and Uffink (1988) obtained a sharp lower bound for the sum of entropies of two measurements $\mathbf{X}$ and $\mathbf{Y}$ when all the X_i's and the Y_j's are one dimensional projections, i.e., when $\mathbf{X}$ and $\mathbf{Y}$ reduce to observables without degeneracy. Following the arguments of Maassen and Uffink (1988) closely in using the Riesz-Thorin interpolation theorem and combining it with an application of Naimark's theorem (Helstrom, 1976) as outlined in (Parthasarathy, 1999) we obtain a lower bound in the case of a pair of arbitrary measurements of a finite level system. Our lower bound does coincide with the Maassen-Uffink lower bound in the case of observables without degeneracy.

2. The Main Result

We say that a measurement $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ is *projective* if each X_i is an orthogonal projection. In such a case one has

$$X_i X_j = \delta_{ij} X_j \quad \text{for all} \quad i, j \in \{1, 2, \ldots, m\}. \tag{2}$$

THEOREM 2.1. *Let* $\mathbf{P} = (P_1, P_2, \ldots, P_m)$, $\mathbf{Q} = (Q_1, Q_1, \ldots, Q_n)$ *be two projective measurements and let* ψ *be a pure state in* $\mathcal{H}$. *Then*

$$H(\mathbf{P}, \psi) + H(\mathbf{Q}, \psi) \geq -2 \ \log_2 \ \max_{i,j} \frac{| < \psi | P_i Q_j | \psi > |}{\| P_i \psi \| \| Q_j \psi \|}, \tag{3}$$

where, on the right hand side, the maximum is taken over all $1 \leq i \leq m$, $1 \leq j \leq n$ *satisfying the conditions* $P_i \psi \neq 0$, $Q_j \psi \neq 0$.

Before proceeding to the proof of this theorem, we shall present the well-known Riesz-Thorin interpolation theorem in a convenient form. Let $T = ((t_{ij}))$, $1 \leq i \leq m, 1 \leq j \leq n$ be any matrix of order $m \times n$ with entries from the field $\mathbb{C}$ of complex scalars. In any space $\mathbb{C}^k$ we define the norms

$$\|\mathbf{x}\|_p = \begin{cases} \left(\sum_{i=1}^k |x_i|^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max_{1 \leq i \leq k} |x_i| & \text{if } p = \infty, \end{cases} \tag{4}$$

where $\mathbf{x}^t = (x_1, x_2, \ldots, x_k)$.

Consider the operator $T : \mathbb{C}^n \to \mathbb{C}^m$ defined by

$$(T\mathbf{x})_i = \sum_{j=1}^{n} t_{ij} x_j \tag{5}$$

and define

$$\|T\|_{p,q} = \sup_{\mathbf{x}:\ \|\mathbf{x}\|_p = 1} \|T\mathbf{x}\|_q \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1. \tag{6}$$

With these notations we have the following theorem, which is a very special case of Theorem IX.17, pages 27-28 of Reed and Simon (1975).

THEOREM 2.2. *Suppose that* p_0, q_0, p_1, q_1 *are in the interval* $[1, \infty]$ *and* $\frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{p_1} + \frac{1}{q_1} = 1$ *and*

$$\|T\|_{p_0,q_0} \le m_0,\ \|T\|_{p_1,q_1} \le m_1. \tag{7}$$

Define p_t, q_t *for* $0 < t < 1$ *by*

$$\frac{1}{p_t} = t\frac{1}{p_1} + (1-t)\frac{1}{p_0},\ \frac{1}{q_t} = t\frac{1}{q_1} + (1-t)\frac{1}{q_0}. \tag{8}$$

Then

$$\|T\|_{p_t,q_t} \le m_t,\ \text{where } m_t = m_0^{1-t} m_1^t, \tag{9}$$

for every $0 < t < 1$.

PROOF OF THEOREM 2.1. Without loss of generality we can assume that $P_i\psi \neq 0$, $Q_j\psi \neq 0$ for every $1 \le i \le m, 1 \le j \le n$. Otherwise, we can restrict the following argument to the subset of indices which obey this condition. Define

$$\phi_i = \frac{P_i\psi}{\|P_i\psi\|},\ \psi_i = \frac{Q_i\psi}{\|Q_i\psi\|} \tag{10}$$

and observe that $\{\phi_i\}$ and $\{\psi_j\}$ are orthonormal sets. Put

$$t_{ij} = <\phi_i|\psi_j>,\ 1 \le i \le m, 1 \le j \le n. \tag{11}$$

For any $\mathbf{x} \in \mathbb{C}^n$ we have,

$$\sum_{i=1}^{m} \left| \sum_{j=1}^{n} t_{ij} x_j \right|^2 = \sum_{i=1}^{m} |<\phi_i| \sum_{j=1}^{n} x_j |\psi_j>|^2 \le \left\| \sum_{j=1}^{n} x_j |\psi_j> \right\|^2 = \sum_{j=1}^{n} |x_j|^2. \tag{12}$$

Thus the operator $T : \mathbb{C}^n \to \mathbb{C}^m$ defined by the matrix T satisfies the inequality

$$\|T\|_{2,2} \leq 1. \tag{13}$$

On the other hand,

$$\max_i \left| \sum_{j=1}^{n} t_{ij} x_j \right| \leq \max_{i,j} |t_{ij}| \sum_{j=1}^{n} |x_j|. \tag{14}$$

In other words,

$$\|T\|_{1,\infty} \leq R, \text{ where } R = \max_{i,j} |t_{ij}|. \tag{15}$$

Now apply Theorem 2.2 after putting

$$p_0 = q_0 = 2, p_1 = 1, q_1 = \infty, m_0 = 1, m_1 = R.$$

Then we have,

$$\|T\|_{p_t,q_t} \leq R^t, \ 0 < t < 1, \tag{16}$$

where a computation shows that $p_t = 2/(1+t)$ and $q_t = 2/(1-t)$. Define the vectors $\mathbf{a} \in \mathbb{C}^n, \mathbf{b} \in \mathbb{C}^m$ by

$$a_j =< \psi_j|\psi >, \ j = 1, 2, \ldots, n, \quad b_i =< \phi_i|\psi >, \ i = 1, 2, \ldots, m. \tag{17}$$

We have

$$\begin{aligned}
(T\mathbf{a})_i &= \sum_{j=1}^{n} t_{ij} a_j = \sum_{j=1}^{n} < \phi_i|\psi_j >< \psi_j|\psi > \\
&= \sum_{j=1}^{n} \frac{< \phi_i|Q_j\psi >< Q_j\psi|\psi >}{\|Q_j\psi\|^2} \\
&= \sum_{j=1}^{n} < \phi_i|Q_j\psi > = < \phi_i \left| \sum_{j=1}^{n} Q_j \right| \psi > = < \phi_i|\psi > = b_i.
\end{aligned} \tag{18}$$

By inequality (16) we now conclude that

$$\left(\sum_{i=1}^{m} | < \phi_i|\psi > |^{\frac{2}{1-t}} \right)^{\frac{1-t}{2}} \leq R^t \left(\sum_{j=1}^{n} | < \psi_j|\psi > |^{\frac{2}{1+t}} \right)^{\frac{1+t}{2}}, \tag{19}$$

for every $0 < t < 1$. Denoting

$$p_i =< \psi|P_i|\psi >= | < \phi_i|\psi > |^2, \ q_j =< \psi|Q_j|\psi >= | < \psi_j|\psi > |^2,$$

we see that the inequality (19) can be expressed as, after raising both sides to power $2/t$ and transferring the second factor on the right hand side to the left,

$$\left(\sum_{i=1}^{m} p_i p_i^{\frac{t}{1-t}}\right)^{\frac{1-t}{t}} \left(\sum_{j=1}^{n} q_j q_j^{-\frac{t}{1+t}}\right)^{-\frac{1+t}{t}} \leq R^2, \ 0 < t < 1. \tag{20}$$

Taking natural logarithms, letting $t \to 0$ and using L'Hospital's rule, we get

$$\sum_{i=1}^{m} p_i \ \log \ p_i + \sum_{j=1}^{n} q_j \ \log \ q_j \leq 2 \log R.$$

This completes the proof of the theorem. □

Corollary 2.3. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be projective measurements and let* ψ *be any pure state. Then*

$$H(\mathbf{P}, \psi) + H(\mathbf{Q}, \psi) \geq -2 \ \log \ \max_{i,j} \|P_i Q_j\|. \tag{21}$$

Proof. This is immediate from Theorem 2.1 when we note that

$$| < \psi | P_i Q_j | \psi > | = | < P_i \psi | P_i Q_j | Q_j \psi > | \leq \|P_i Q_j\| \ \|P_i \psi\| \ \|Q_j \psi\|. \qquad \square$$

Remark. Inequality (21) becomes trivial, in the sense that the right hand side vanishes, if and only if $\|P_i Q_j\| = 1$ for some i, j. This, in turn, is equivalent to finding a nonzero vector in the intersection of the ranges of P_i and Q_j for some i, j.

One can also consider a mixed state of the form

$$\rho = \sum_{i=1}^{r} \pi_i |\psi_i >< \psi_i|, \ \pi_i > 0, \ \sum_{i=1}^{r} \pi_i = 1,$$

where $\psi_i, \ i = 1, 2, \ldots, r$ are unit vectors. Then for any measurement $\mathbf{X} = (X_1, X_2, \ldots, X_m)$, one obtains a probability distribution

$$p_k = Tr(\rho X_k) = \sum_{i=1}^{r} \pi_i < \psi_i | X_k | \psi_i >, \ 1 \leq k \leq m.$$

We write

$$H(\mathbf{X}, \rho) = -\sum_{k=1}^{m} p_k \ \log_2 \ p_k.$$

Then we note that $(p_1, p_2, \ldots, p_m)$ is a convex combination of the probability distributions $(p_{i1}, p_{i2}, \ldots, p_{im}),\ 1 \leq i \leq r$, where

$$p_{ik} =< \psi_i|X_k|\psi_i >,\ 1 \leq k \leq m.$$

If now $\mathbf{P}$ and $\mathbf{Q}$ are two projective measurements it follows from the concavity property of entropy (see Nielsen and Chuang, 2000, Section 11.3.5, pp.516-518) that

$$H(\mathbf{P}, \rho) + H(\mathbf{Q}, \rho) \geq \sum_{i=1}^{r} \pi_i \left[H(\mathbf{P}, \psi_i) + H(\mathbf{Q}, \psi_i)\right] \geq -2\ \log\ \max_{i,j} \|P_i Q_j\|. \tag{22}$$

The importance of this inequality lies in the fact that the right hand side is independent of the state ρ.

THEOREM 2.4. *Suppose* $\mathbf{P} = (P_1, P_2, \ldots, P_m)$ *is a projective measurement and* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ *is an arbitrary measurement. Then for any pure state* ψ,

$$H(\mathbf{P}, \psi) + H(\mathbf{Y}, \psi) \geq -2\ \log\ \max_{i,j} \frac{| < \psi|P_i Y_j|\psi > |}{\|P_i\psi\|\ \|Y_j^{\frac{1}{2}}\psi\|}. \tag{23}$$

where the maximum is over all i, j *for which* $P_i\psi \neq 0, Y_j^{1/2}\psi \neq 0$.

PROOF. We look upon $\mathbf{Y}$ as a positive operator valued measure on the finite set $\{1, 2, \ldots, n\}$. In an orthonormal basis of $\mathcal{H}$, the operators $P_i, Y_j,\ 1 \leq i \leq m,\ 1 \leq j \leq n$ can all be viewed as positive semidefinite matrices. By Naimark's theorem (Helstrom, 1976) as interpreted in Parthasarathy (1999) for finite dimensional Hilbert spaces, we can construct matrices of the form

$$\tilde{Q}_j = \begin{bmatrix} Y_j & L_j \\ L_j^{\dagger} & Z_j \end{bmatrix},\ 1 \leq j \leq n \tag{24}$$

so that $\tilde{Q}_j$'s are projections in an enlarged Hilbert space $\mathcal{H} \oplus \mathcal{K}$ where $\mathcal{K}$ is also a finite dimensional Hilbert space and

$$\sum_{j=1}^{n} \tilde{Q}_j = I_{\mathcal{H}\oplus\mathcal{K}}.$$

Define

$$\tilde{P}_1 = \begin{bmatrix} P_1 & 0 \\ 0 & I_{\mathcal{K}} \end{bmatrix},\quad \tilde{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & 0 \end{bmatrix},\ 2 \leq i \leq m,\quad \tilde{\psi} = \begin{bmatrix} \psi \\ 0 \end{bmatrix}, \tag{25}$$

where the vectors in $\mathcal{H} \oplus \mathcal{K}$ are expressed as column vectors $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ with $u \in \mathcal{H}$ and $v \in \mathcal{K}$. Then $\tilde{\psi}$ is a pure state and $\tilde{\mathbf{P}} = (\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_m)$, $\tilde{\mathbf{Q}} = (\tilde{Q}_1, \tilde{Q}_2, \dots, \tilde{Q}_n)$ are projective measurements in an enlarged system. By Theorem 2.1 we have

$$H(\tilde{\mathbf{P}}, \tilde{\psi}) + H(\tilde{\mathbf{Q}}, \tilde{\psi}) \geq -2 \ \log \ \max_{i,j} \frac{|<\tilde{\psi}|\tilde{P}_i \tilde{Q}_j|\tilde{\psi}>|}{\|\tilde{P}_i \tilde{\psi}\| \ \|\tilde{Q}_j \tilde{\psi}\|}. \tag{26}$$

On the other hand, we have

$$\tilde{P}_i \tilde{\psi} = \begin{bmatrix} P_i \psi \\ 0 \end{bmatrix}, \ \tilde{Q}_j \tilde{\psi} = \begin{bmatrix} Y_j \psi \\ L_j^{\dagger} \psi \end{bmatrix}. \tag{27}$$

This implies

$$<\tilde{\psi}|\tilde{P}_i \tilde{Q}_j|\tilde{\psi}>=<\psi|P_i \ Y_j|\psi> \ \text{ and } \ <\tilde{\psi}|\tilde{P}_i|\tilde{\psi}>=<\psi|P_i|\psi> .$$

Since $\tilde{Q}_j$ is a projection we have

$$\|\tilde{Q}_j \tilde{\psi}\|^2 =<\tilde{\psi}|\tilde{Q}_j|\tilde{\psi}>=<\psi|Y_j|\psi>= \|Y_j^{\frac{1}{2}} \psi\|^2.$$

Thus (using the above two equations) inequality (26) reduces to inequality (23).

THEOREM 2.5. *Let* $\mathbf{X} = (X_1, X_2, \dots, X_m)$, $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$ *be two arbitrary measurements. Then for any pure state* ψ,

$$H(\mathbf{X}, \psi) + H(\mathbf{Y}, \psi) \geq -2 \ \log_2 \ \max_{i,j} \frac{|<\psi|X_i Y_j|\psi>|}{\|X_i^{\frac{1}{2}} \psi\| \ \|Y_j^{\frac{1}{2}} \psi\|}, \tag{28}$$

where the maximum is over all i, j *for which* $X_i^{1/2}\psi \neq 0, Y_j^{1/2}\psi \neq 0$.

PROOF. As in the proof of Theorem (2.4), use Naimark's theorem (Helstrom, 1976) and construct the projections $\tilde{Q}_j$ as in equation (24). Define

$$\begin{aligned} \tilde{X}_1 &= \begin{bmatrix} X_1 & 0 \\ 0 & I_{\mathcal{K}} \end{bmatrix}, \\ \tilde{X}_i &= \begin{bmatrix} X_i & 0 \\ 0 & 0 \end{bmatrix}, \ 2 \leq i \leq m, \end{aligned} \tag{29}$$

and consider the state $\tilde{\psi}$ as defined by equation (25). Then $\tilde{\mathbf{Q}} = (\tilde{Q}_1, \tilde{Q}_2, \dots, \tilde{Q}_n)$ is a projective measurement and $\tilde{\mathbf{X}} = (\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_m)$ is a measurement. Hence, by Theorem 2.4,

$$H(\tilde{\mathbf{Q}}, \tilde{\psi}) + H(\tilde{\mathbf{X}}, \tilde{\psi}) \geq -2 \ \log_2 \ \max_{i,j} \frac{|<\tilde{\psi}|\tilde{Q}_j \tilde{X}_i|\tilde{\psi}>|}{\|\tilde{X}_i^{\frac{1}{2}} \tilde{\psi}\| \ \|\tilde{Q}_j \tilde{\psi}\|}. \tag{30}$$

As in the proof of Theorem 2.4 we note that

$$< \tilde{\psi}|\tilde{Q}_j|\tilde{\psi} >= \|\tilde{Q}_j\tilde{\psi}\|^2 =< \psi|Y_j|\psi >= \|Y_j^{\frac{1}{2}}\psi\|^2.$$

Clearly, inequality (30) reduces to

$$H(\mathbf{X},\psi) + H(\mathbf{Y},\psi) \geq -2 \ \log_2 \ \max_{i,j} \frac{|< \psi|Y_j X_i|\psi >|}{\|Y_j{}^{\frac{1}{2}}\psi\| \ \|X_i{}^{\frac{1}{2}}\psi\|}, \tag{31}$$

which is the same as equation (28) owing to the self-adjointness of X_i and Y_j. □

COROLLARY 2.6. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_m)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ *be arbitrary measurements and let* ρ *be any state. Then*

$$H(\mathbf{X},\rho) + H(\mathbf{Y},\rho) \geq -2 \ \log_2 \ \max_{i,j} \|X_i{}^{\frac{1}{2}} Y_j{}^{\frac{1}{2}}\|. \tag{32}$$

PROOF. Owing to the concavity of Shannon entropy it is enough to prove the Corollary when ρ is a pure sate determined by a unit vector ψ. Now the required result is immediate from the theorem above if we observe that

$$|< \psi|X_i Y_j|\psi >| = |< X_i^{\frac{1}{2}}\psi|X_i^{\frac{1}{2}}Y_j^{\frac{1}{2}}|Y_j^{\frac{1}{2}}\psi >| \leq \|X_i^{\frac{1}{2}}Y_j^{\frac{1}{2}}\| \ \|X_i^{\frac{1}{2}}\psi\| \ \|Y_j^{\frac{1}{2}}\psi\|.$$

□

REMARK. Putting $\mathbf{X} = \mathbf{Y}$ in inequality (31) we get

$$H(\mathbf{X},\rho) \geq - \ \log_2 \ \max_{i,j} \|X_i{}^{\frac{1}{2}} X_j{}^{\frac{1}{2}}\|.$$

This yields a nontrivial uncertainty principle even for a single measurement since the right hand side need not vanish.

EXAMPLE. Let G be a finite group of cardinality N and let $\widehat{G}$ denote its dual space consisting of all the non-equivalent irreducible unitary representations of G. Denote by $L^2(G)$, the N-dimensional complex Hilbert space of all functions on G with the scalar product

$$< f|g >= \sum_{x \in G} \overline{f(x)} g(x), \ f,g \in L^2(G).$$

For any $\pi \in \widehat{G}$, let $d(\pi)$ denote the dimension of the representation space of π and let $\{\pi_{ij}(\cdot), \ 1 \leq i,j \leq d(\pi)\}$ denote the matrix elements of π in some orthonormal basis of its representation space. From the Peter-Weyl theory of representations we have two canonical orthonormal bases for $L^2(G)$:

1. $\{|x>=1_{\{x\}},\ x\in G\}$;

2. $\{\sqrt{\frac{d(\pi)}{N}}\pi_{ij}(\cdot),\ 1\le i,j\le d(\pi),\pi\in\widehat{G}\}$,

where $1_{\{x\}}$ denotes the indicator function of the singleton set $\{x\}$ in G. Consider the projective measurements

$$\mathbf{Q}=\{Q_x,x\in G\},\ Q_x=|x><x|,\ \mathbf{P}=\{P_{i,j,\pi},\ \pi\in\widehat{G},1\le i,j\le d(\pi)\},$$

where

$$P_{i,j,\pi}=\frac{d(\pi)}{N}|\pi_{ij}><\pi_{ij}|.$$

For any unit vector ψ in $L^2(G)$, we have

$$\begin{aligned}<\psi|Q_xP_{i,j,\pi}|\psi>&=\frac{d(\pi)}{N}<\psi|x><\pi_{ij}|\psi>\pi_{ij}(x),\\ \|Q_x\psi\|^2&=<\psi|Q_x|\psi>=|\psi(x)|^2,\\ \|P_{i,j,\pi}\psi\|^2&=\frac{d(\pi)}{N}|<\pi_{ij}|\psi>|^2.\end{aligned}\tag{33}$$

Thus our Entropic uncertainty principle assumes the form

$$\begin{aligned}&-\sum_{x\in G}|\psi(x)|^2\ \log_2\ |\psi(x)|^2-\sum_{\substack{1\le i,j\le d(\pi)\\ \pi\in\widehat{G}}}|\widehat{\psi}(i,j,\pi)|^2\ \log_2\ |\widehat{\psi}(i,j,\pi)|^2\\ &\ge -2\log_2\ \max_{i,j,\pi,x}\sqrt{\frac{d(\pi)}{N}}\ |\pi_{ij}(x)|,\end{aligned}\tag{34}$$

where

$$\widehat{\psi}(i,j,\pi)=\sqrt{\frac{d(\pi)}{N}}<\pi_{ij}|\psi>$$

is the (noncommutative) Fourier transform of ψ at the ijth entry of the irreducible representation π. Since $\pi_{ij}(x)$ is the ijth entry of the unitary matrix $\pi(x)$ and $\pi(e)=I_{d(\pi)}$ at the identity element e we have

$$\max_{i,j,\pi,x}|\pi_{ij}(x)|=1.$$

Thus the Entropic uncertainty principle reduces to

$$\begin{aligned}&-\sum_{x\in G}|\psi(x)|^2\ \log_2\ |\psi(x)|^2-\sum_{\substack{1\le i,j\le d(\pi)\\ \pi\in\widehat{G}}}|\widehat{\psi}(i,j,\pi)|^2\ \log_2\ |\widehat{\psi}(i,j,\pi)|^2\\ &\ge \log_2 N\ -\ \log_2\max_{\pi\in\widehat{G}}d(\pi),\end{aligned}\tag{35}$$

for every unit vector $\psi \in L^2(G)$. When G is Abelian, every π is one dimensional and the right hand side reduces to $\log_2 N$. In this case, when $\psi(x) \equiv 1/\sqrt{N}$, the inequality in (35) becomes an equality.

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M. Krishna and K.R. Parthasarathy
Institute of Mathematical Sciences
Taramani, Chennai 600 113, India
E-mails: krishna@imsc.ernet.in
krp@imsc.ernet.in