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THE BALAYAGE ORDER DEFINED BY A GAMBLING HOUSE

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SUMMARY. We describe the balayage order defined by an analytic gambling house on a Borel subset of a Polish space in terms of our construction of the saturation of the gambling house.

1. Introduction

A relation between probability measures, known as the balayage order, was introduced independently in two different contexts during the 1950's.

G. Choquet (1956) studied the purely mathematical problem of how to represent a point x belonging to a metrizable, compact, convex subset K of a locally convex linear topological space V by a probability measure μ_x on the extreme points of K in the sense that

$$f(x) = \int_K f \, d\mu_x \,,$$

for all continuous linear functionals $f : V \mapsto \mathbb{R}$. To obtain μ_x, Choquet defined the balayage relation between measures μ, ν by the rule $\mu \vdash \nu$ if and only if $\int g \, d\mu \leq \int g \, d\nu$ for all continuous, concave $g : K \mapsto \mathbb{R}$. Then μ_x is a minimal element of the set $\Gamma_C(x) = \{\mu : \mu \text{ represents } x\}$. Cartier (cf. Dellacherie and Meyer, 1983, X 40) showed that $\mu \vdash \nu$ if and only if there is a Markov kernel Q on K such that, for each x, $Q(x, \cdot)$ represents x, and $\mu = \nu Q$.

The balayage order also plays a crucial role in D. Blackwell's (1951) work on the comparison of experiments in statistics. An *experiment* $\mathcal{E}$ is

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a collection $\{P_\theta, \theta \in \Theta\}$ of probability measures on a measurable space $(X, \mathcal{A})$. Given another experiment $\mathcal{F}$ corresponding to probability measures $\{Q_\theta, \theta \in \Theta\}$ on $(Y, \mathcal{B})$, we prefer $\mathcal{F}$ to $\mathcal{E}$ and write $\mathcal{E} \le \mathcal{F}$ if, roughly speaking, a statistician can, for every action set and loss function, obtain an expected loss based on $\mathcal{F}$ that is no larger than that obtained from using $\mathcal{E}$. Suppose that the parameter space Θ is a finite set with n elements. Then Blackwell associates to every experiment $\mathcal{E}$ a canonical measure $\mu_{\mathcal{E}}$ on the unit simplex S^{n-1}. Furthermore, $\mathcal{E} \le \mathcal{F}$ if and only if $\int g\, d\mu_{\mathcal{E}} \ge \int g\, d\mu_{\mathcal{F}}$ for every continuous, concave $g : S^{n-1} \mapsto \mathbb{R}$. Thus $\mathcal{E} \le \mathcal{F}$ if and only if $\mu_{\mathcal{F}} \vdash \mu_{\mathcal{E}}$. A recent exposition of the Blackwell theory is in Le Cam (1996).

A gambling house Γ on a Borel set X is a correspondence that assigns to each $x \in X$ a set $\Gamma(x)$ of probability measures on X. In gambling theory, the set of Γ-excessive functions defined in the next section plays the role of the concave functions in Choquet theory. So, for probability measures μ, ν on X, we define $\mu \vdash \nu$ if and only if $\int f\, d\mu \le \int f\, d\nu$ for all Γ-excessive f. If $X = K$ and $\Gamma = \Gamma_C$ as defined above, we recover the balayage order of Choquet. There is also an interpretation akin to that of the Blackwell theory; namely, for any bounded utility function u, a gambler who begins play with a fortune having distribution μ can do no better than a gambler who begins with distribution ν. (Precise definitions are in the next section.)

A natural approach to the study of the balayage order in gambling theory is through the introduction of the *saturated house* Γ^s, which is defined as the largest gambling house on X having the same excessive functions as Γ. For an analytic gambling house Γ, Dellacherie and Meyer (1983, XI 38) gave a construction of Γ^s from Γ, which showed, inter alia, that Γ^s is also analytic. They then established various properties of the balayage order by using their construction (see Dellacherie and Meyer, 1983, XI 39-40). We gave a different, more gambling theoretic, construction of Γ^s in Maitra and Sudderth (2000a). The aim of this article is to study the balayage order in terms of our construction. This approach yields somewhat more precise results and avoids the use of the theory of capacitary operators on which the Dellacherie - Meyer method is heavily reliant. Our description of the balayage order is also more constructive because we bypass Strassen's theorem (Strassen, 1965, Theorem 1) on the disintegration of continuous linear functionals on separable Banach spaces. The present article is a sequel to Maitra and Sudderth (2000a), which will hereafter be cited as MSa.

The article is organized as follows. Section 2 contains definitions, notation, and a quick summary of results from MSa. The main results of the paper are given in section 3. Section 4 contains examples which serve to illustrate some of our results.

2. Definitions, Notation and Preliminaries

Let X be a Borel subset of a Polish space. Denote by $\mathbb{P}(X)$ the set of probability measures on the Borel σ-field of X. We equip $\mathbb{P}(X)$ with the weak topology so that $\mathbb{P}(X)$ is again a Borel subset of a Polish space (Kechris, 1995, 17E). A *gambling house* Γ is a subset of $X \times \mathbb{P}(X)$ such that each vertical section $\Gamma(x)$ of Γ at x is nonempty. A gambling house Γ is *analytic* if Γ is an analytic subset of $X \times \mathbb{P}(X)$. If Γ is a gambling house on X, a Γ-*excessive* function f is a bounded, upper analytic function on X such that $\gamma(f) \leq f(x)$ for every $\gamma \in \Gamma(x)$ and $x \in X$, where we write $\gamma(g)$ for $\int g \, d\gamma$. Recall that a function g is *upper analytic* if the set $\{g > a\}$ is analytic for every real a.

A gambling house Γ induces a relation $\vdash_\Gamma$ (or simply $\vdash$, when Γ is understood from the context), called the *balayage order* on $\mathbb{P}(X)$ as follows:

$$\mu \vdash_\Gamma \lambda \qquad \text{if} \qquad \mu(f) \leq \lambda(f)$$

for every Γ-excessive f. In case $\mu \vdash_\Gamma \lambda$, we say that μ is a *balayée* of λ. The *saturation* Γ^s of a gambling house Γ is the largest gambling house Γ' such that every Γ-excessive function is Γ'-excessive. It is then easy to see that

$$\Gamma^s(x) = \{\gamma \in \mathbb{P}(X) \, : \, \gamma \vdash_\Gamma \delta(x)\} \tag{2.1}$$

for every $x \in X$, where $\delta(x)$ is point mass at x. A gambling house Γ is said to be *saturated* if $\Gamma = \Gamma^s$.

We will now describe our construction of the saturation Γ^s. First we need some definitions. Let, then, Γ be a gambling house on X. We say that $\sigma = (\sigma_o, \sigma_1, \ldots)$ is a *strategy available in* Γ *at* x if $\sigma_o \in \Gamma(x)$ and, for $n \geq 1$, σ_n is a universally measurable function on X^n into $\mathbb{P}(X)$ such that $\sigma_n(x_1, x_2, \ldots, x_n) \in \Gamma(x_n)$ for every $x_1, x_2, \ldots, x_n \in X$. Plainly, a strategy σ will induce, via the Ionescu-Tulcea theorem (Bertsekas and Shreve, 1978, 7.45), a unique probability measure on the Borel sigma-field of the history space $H = X^N$, where N is the set of positive integers and H is given the product topology. This probability measure will also be denoted by σ. A *stop rule* t is a universally measurable function on H into $N \cup \{0\}$ such that $t(h) = k$ and $h \equiv_k h'$ imply $t(h') = k$, where $h \equiv_k h'$ means that h and h' agree through the first k coordinates. In particular, if $t(h) = 0$ for some h, then t is identically zero. A *policy available in* Γ *at* x is a pair $\pi = (\sigma, t)$ such that σ is a strategy available in Γ at x and t is a stop rule. If $h \in H$, and t a stop rule ≥ 1, h_t will abbreviate the $t(h)$-th coordinate of h. A policy $\pi = (\sigma, t)$ available in Γ at x defines the *terminal gamble* $\gamma(\pi)$ *available in*

Γ *at* x as follows:

$$\begin{aligned}\gamma(\pi) &= \delta(x), \text{ if } t \equiv 0, \\ \text{and } \gamma(\pi)(g) &= \int g(h_t)\,\sigma(dh), \text{ if } t \geq 1,\end{aligned}$$

for any bounded, Borel measurable function g on X. With Γ we associate two operators G_Γ and R_Γ as follows: if u is a bounded, universally measurable function on X, define, for $x \in X$,

$$\begin{aligned}G_\Gamma u(x) &= \sup\{\gamma(u) \,:\, \gamma \in \Gamma(x)\} \\ \text{and } R_\Gamma u(x) &= \sup_\pi \int u(h_t)\,\sigma(dh),\end{aligned}$$

where the second sup is over all policies $\pi = (\sigma, t)$ available in Γ at x and $h_t = x$ for $t \equiv 0$. Then G_Γ is the optimal reward operator when the gambler plays for one day and R_Γ is the optimal reward operator for the leavable gambling problem (Γ, u) in the sense of Dubins and Savage (1976). If Γ is an analytic gambling house and u is a bounded, upper analytic function on X, then both $G_\Gamma u$ and $R_\Gamma u$ are upper analytic (MSa, Lemma 4.1).

The first step in our construction of the saturation Γ^s is to define the house Γ^c by taking $\Gamma^c(x)$ to be the collection of all terminal gambles $\gamma(\pi)$ such that $\pi = (\sigma, t)$ is a policy available at x in Γ and t is a bounded stop rule. The main properties of Γ^c, established in MSa (Theorem 3.3, Lemma 4.1), are given below.

THEOREM 2.1. *Let* Γ *be an analytic gambling house on* X. *Then* Γ^c *is an analytic gambling house containing* Γ *such that* $\delta(x) \in \Gamma^c(x)$ *for every* $x \in X$. *Moreover, the operators* R_Γ *and* G_{Γ^c} *agree on bounded, upper analytic functions on* X.

To complete the construction, we recall that if $M \subseteq \mathbb{P}(X)$, then the *strong convex hull* of M, written $\operatorname{sco} M$, is the set of all probability measures $\gamma \in \mathbb{P}(X)$ such that for some probability measure $\eta \in \mathbb{P}(\mathbb{P}(X))$ with $\eta^*(M)=1$

$$\gamma(g) = \int_{\mathbb{P}(X)} \mu(g)\,\eta(d\mu)$$

for all bounded, Borel measurable functions g on X, where η^* is outer measure. Say that $M \subseteq \mathbb{P}(X)$ is *strongly convex* if $M = \operatorname{sco} M$.

THEOREM 2.2. *(MSa, Theorem 1.2) Let* Γ *be an analytic gambling house on* X. *Then*

$$\Gamma^s(x) = \textit{norm-closure}\,(\operatorname{sco}\Gamma^c(x)), \qquad x \in X,$$

where the closure is in the total variation norm.

Throughout the paper, the operations of forming the strong convex hull and the (variation) norm closure will be performed (vertical) sectionwise on subsets of $X \times \mathbb{P}(X)$ or $\mathbb{P}(X) \times \mathbb{P}(X)$. Thus if Γ is a subset of $X \times \mathbb{P}(X)$, then $\operatorname{sco}\Gamma$ is the subset of $X \times \mathbb{P}(X)$ whose x-section is the strong convex hull of $\Gamma(x)$ and $\overline{\operatorname{sco}}\,\Gamma$ is the subset of $X \times \mathbb{P}(X)$ whose x-section is the norm-closure of the strong convex hull of $\Gamma(x)$.

Also throughout the paper, for each $\mu \in \mathbb{P}(X \times X)$, we fix a version $\mu(x)$ of the regular μ-conditional distribution of the second coordinate given that the first is x such that the map $(\mu, x) \mapsto \mu(x)$ is jointly Borel measurable in μ and x. Such a version is available thanks to Lemma 2.2 in Maitra et al. (1990).

A *selector* for a gambling house Γ is a function $\phi : X \mapsto \mathbb{P}(X)$ such that $\phi(x) \in \Gamma(x)$ for every $x \in X$. We will write a selector ϕ as a transition function Q thus:

$$Q(x, B) = \phi(x)(B)$$

for $x \in X$, B a Borel subset of X. We will say that Q is universally measurable or Borel measurable according as ϕ is universally measurable or Borel measurable. If Q is universally measurable and $\nu \in \mathbb{P}(X)$, we write νQ for the probability measure defined by

$$\nu Q(B) = \int Q(x, B)\, \nu(dx)$$

for Borel subsets B of X.

Following Dubins and Savage (1976, p.124), we say that a gambling house is *closed under composition* if for $\gamma \in \Gamma(x)$ and any universally measurable selector Q for Γ, the gamble $\gamma Q \in \Gamma(x)$. We remark for use later that, if Γ is closed under composition, then $\Gamma^c \subseteq \Gamma$. Indeed, if $\pi = (\sigma, t)$ is a policy available in Γ at x, then it is easily verified by induction on the ordinal-valued function $j(t)$ that $\gamma(\pi) \in \Gamma(x)$. The function $j(t)$ is defined as follows:

$$j(t) = \begin{cases} 0, & \text{if } \ t \equiv 0, \\ \sup\{j(t[x]) + 1 \ : \ x \in X\}, & \text{if } \ t \geq 1, \end{cases}$$

where $t[x]$ is the stop rule defined by:

$$t[x](h) = t(xh) - 1\,,\ h \in H,$$

xh being the catenation of x followed by h (cf. Maitra and Sudderth, 1996, section 2.6).

3. The Balayage Order

In this section we take up the study of the balayage order. We will characterize the order in terms of gambles in the house $\operatorname{sco}\Gamma^c$. We will assume throughout that Γ is an analytic gambling house and that the balayage order is defined by Γ.

THEOREM 3.1. *For μ, $\lambda \in \mathbb{P}(X)$, $\mu \vdash \lambda$ if and only if for every Borel measurable function $g : X \mapsto [0,1]$,*

$$\mu(g) \leq \sup\{\lambda Q(g)\ :\ Q \textit{ is a Borel measurable selector for } \Gamma^c\}. \tag{3.1}$$

PROOF. For the "only if" part, first recall that $R_\Gamma u = G_{\Gamma^c} u$ for any bounded, upper analytic function u on X (Theorem 2.1). Fix $\epsilon > 0$ and let $g : X \mapsto [0,1]$ be Borel measurable. By a known selection theorem (Maitra et al, 1991, Lemma 2.1), there is a universally measurable selector Q^* for Γ^c such that

$$\int g(y)\, Q^*(x, dy) > R_\Gamma g(x) - \epsilon, \qquad x \in X. \tag{3.2}$$

Now $Q^*(x,\cdot)$ is clearly Borel measurable outside a λ-null Borel set. Since $\delta(x) \in \Gamma^c(x)$ for every $x \in X$, we can set $Q^*(x,\cdot) = \delta(x)$ on the null set so that we may assume Q^* is a Borel measurable selector for Γ^c and (3.2) holds a.s. (λ). Hence

$$\begin{aligned}\mu(g) \leq \mu(R_\Gamma g)\ &\leq\ \lambda(R_\Gamma g)\ <\ \int\int g(y)\, Q^*(x,dy)\lambda(dx) + \epsilon \\ &\leq \sup\{\lambda Q(g)\ :\ Q \text{ is a Borel measurable selector for } \Gamma^c\} + \epsilon,\end{aligned}$$

where the first inequality is because $g \leq R_\Gamma g$, the second by virtue of the facts that $R_\Gamma g$ is Γ-excessive (MSa, Theorem 1.1) and $\mu \vdash \lambda$, and the third by virtue of (3.2). Since $\epsilon > 0$ is arbitrary, this proves (3.1).

Conversely, assume that (3.1) holds for all bounded, Borel measurable $g : X \mapsto [0,1]$. Let f be a Γ-excessive function on X into $[0,1]$. Choose a Borel measurable function $g : X \mapsto [0,1]$ such that $g = f$ a.s. (μ) and $g \leq f$. Let Q be a Borel measurable selector for Γ^c. Then, for each x,

$$\int f(y)\, Q(x,dy) \leq f(x)$$

because each Γ-excessive function is also Γ^c-excessive (MSa, Lemma 4.2). Consequently,

$$\int\int f(y)\, Q(x,dy)\, \lambda(dx) \leq \lambda(f).$$

Taking the sup over all Borel measurable selectors Q for Γ^c, we get

$$\sup\{\lambda Q(f) \,:\, Q \text{ is a Borel measurable selector for } \Gamma^c\} \leq \lambda(f). \tag{3.3}$$

Hence

$$\begin{aligned}\mu(f) = \mu(g) &\leq \sup\{\lambda Q(g) : Q \text{ is a Borel measurable selector for } \Gamma^c\} \\ &\leq \sup\{\lambda Q(f) : Q \text{ is a Borel measurable selector for } \Gamma^c\} \leq \lambda(f),\end{aligned}$$

where the first inequality is by (3.1), the second because $g \leq f$, and the last is by (3.3). It follows that $\mu \vdash \lambda$. □

For $\lambda \in \mathbb{P}(X)$ and $\Gamma' \subseteq X \times \mathbb{P}(X)$, denote by $\Gamma'(\lambda)$ the set of all $\mu \in \mathbb{P}(X)$ such that $\mu = \lambda Q$ for some Borel measurable selector Q for Γ'.

Lemma 3.2. *If $\lambda \in \mathbb{P}(X)$ and Γ' is an analytic gambling house such that $\delta(x) \in \Gamma'(x)$ for every $x \in X$, then $\Gamma'(\lambda)$ is an analytic subset of $\mathbb{P}(X)$.*

Proof. Denote by π_i the projection on $X \times X$ to the i-th coordinate, $i = 1, 2$. Next observe that $\gamma \in \Gamma'(\lambda)$ if and only if

$$\exists \nu \in \mathbb{P}(X \times X) : [\nu\pi_1^{-1} = \lambda \;\&\; \lambda(\{x \in X \,:\, \nu(x) \in \Gamma'(x)\}) = 1 \;\&\; \nu\pi_2^{-1} = \gamma].$$

Plainly, the condition in ν, γ within $[\cdot]$ is analytic. So $\Gamma'(\lambda)$ is analytic. □

Theorem 3.3. *For μ, $\lambda \in \mathbb{P}(X)$, $\mu \vdash \lambda$ if and only if $\mu \in \overline{\operatorname{sco}}\, \Gamma^c(\lambda)$. Consequently, the relation $\vdash$ is analytic.*

Proof. Using Lemma 3.2 and arguing as in MSa (Lemma 2.3), one proves easily that $\operatorname{sco} \Gamma^c(\lambda)$ is analytic. Obviously, it is strongly convex. Rewrite condition (3.1) as

$$\mu(g) \leq \sup\{\gamma(g) \,:\, \gamma \in \Gamma^c(\lambda)\}$$

for all Borel measurable $g : X \mapsto [0, 1]$. The last condition clearly implies the condition

$$\mu(g) \leq \sup\{\gamma(g) \,:\, \gamma \in \operatorname{sco} \Gamma^c(\lambda)\} \tag{3.4}$$

for all Borel measurable $g : X \mapsto [0, 1]$. The first assertion now follows from a result of Mokobodzki (MSa, Corollary 2.2), (3.4), (3.1) and Theorem 3.1. The second assertion can be proved by arguing one more time as in MSa, Lemma 2.3. □

The next result gives an alternative description of $\operatorname{sco} \Gamma^c(\lambda)$ in which the order of randomization is reversed.

Theorem 3.4. *Let $\lambda \in \mathbb{P}(X)$. Then*

$$\operatorname{sco} \Gamma^c(\lambda) = (\operatorname{sco} \Gamma^c)(\lambda).$$

PROOF. Let $\gamma \in \operatorname{sco}\Gamma^c(\lambda)$. Choose $m \in \mathbb{P}(\mathbb{P}(X))$ such that $m(\Gamma^c(\lambda)) = 1$ and $\gamma(\cdot) = \int \mu(\cdot)\, m(d\mu)$. Let

$$M = \{\nu \in \mathbb{P}(X \times X) : \nu\pi_1^{-1} = \lambda \;\&\; \lambda(\{x \in X : \nu(x) \in \Gamma^c(x)\}) = 1\}.$$

Note that M is analytic and that $\Gamma^c(\lambda)$ is the image of M under the Borel mapping ψ defined by $\psi(\nu) = \nu\pi_2^{-1}$. So, by Dellacherie and Meyer (1975, III 45), we can lift m to the Borel subsets of $\mathbb{P}(X \times X)$; i.e., there is $\bar{m} \in \mathbb{P}(\mathbb{P}(X \times X))$ such that $\bar{m}(M) = 1$ and $\bar{m}\psi^{-1} = m$. Define

$$\begin{aligned} \eta(\cdot) &= \int \nu(\cdot)\, \bar{m}(d\nu) \\ \text{and } \; Q(x,\cdot) &= \int \nu(x)(\cdot)\, \bar{m}(d\nu), \qquad x \in X. \end{aligned}$$

Then $\nu \in \mathbb{P}(X \times X)$ and Q is a Borel measurable transition function. Consider now the set

$$L = \{(\nu, x) \in M \times X : \nu(x) \in \Gamma^c(x)\}.$$

Then L is analytic. For each $\nu \in M$, the λ-measure of the ν-section of L is one. So, by an application of Fubini's theorem to the product measure $\bar{m} \times \lambda$, the x-section of L has $\bar{m}$-measure one a.s.(λ). Consequently, $Q(x,\cdot) \in \operatorname{sco}\Gamma^c(x)$ a.s.(λ). We can assume, by redefining $Q(x,\cdot) = \delta(x)$ on a λ-null Borel set, if necessary, that Q is a Borel measurable selector for $\operatorname{sco}\Gamma^c$. Now, for Borel subsets A, B of X, we have

$$\begin{aligned} \eta(A \times B) = \int_M \nu(A \times B)\, \bar{m}(d\nu) &= \int_M \int_A \nu(x)(B)\, \lambda(dx)\, \bar{m}(d\nu) \\ = \int_A \int_M \nu(x)(B)\, \bar{m}(d\nu)\, \lambda(dx) &= \int_A Q(x, B)\, \lambda(dx). \end{aligned}$$

Hence, if B is a Borel subset of X, then

$$\gamma(B) = \int \mu(B)\, m(d\mu) = \int \nu(X \times B)\, \bar{m}(d\nu) = \eta(X \times B) = \int Q(x, B)\, \lambda(dx),$$

where the second equality is by the change of variable theorem. So $\gamma = \lambda Q$, and hence $\gamma \in (\operatorname{sco}\Gamma^c)(\lambda)$.

Conversely, suppose that $\gamma \in (\operatorname{sco}\Gamma^c)(\lambda)$. So $\gamma = \lambda Q$ for some Borel measurable selector Q for $\operatorname{sco}\Gamma^c$. By von Neumann's selection theorem (Kechris, 1995, 29.9), there is a universally measurable function $\tau : X \mapsto \mathbb{P}(\mathbb{P}(X))$ such that for all x, (a) $\tau(x)(\Gamma^c(x)) = 1$ and (b) $Q(x,\cdot) = \int \mu(\cdot)\, \tau(x)(d\mu)$. By

Maitra and Sudderth (2000b, Lemma 3.1), there is a universally measurable function $\phi : X \times [0,1] \mapsto \mathbb{P}(X)$ such that $\tau(x) = l\phi(x,\cdot)^{-1}$, l being Lebesgue measure on $[0,1]$. Consider now the set

$$K = \{(x,s) \in X \times [0,1] \,:\, \phi(x,s) \in \Gamma^c(x)\}.$$

Then K is universally measurable. For each x, the x-section of K has Lebesgue measure one because $l\phi(x,\cdot)^{-1} = \tau(x)$ and $\tau(x)(\Gamma^c(x)) = 1$. Hence, $(\lambda \times l)(K) = 1$. Find a Borel set $K' \subseteq K$ such that $(\lambda \times l)(K') = 1$ and the restriction of ϕ to K' is Borel measurable. Outside K', redefine $\phi(x,s)$ and set it equal to $\delta(x)$. Then ϕ is Borel measurable, $l\phi(x,\cdot)^{-1} = \tau(x)$ a.s.(λ), and $\phi(x,s) \in \Gamma^c(x)$ for all x, s. Now define

$$\eta(s)(\cdot) = \int \phi(x,s)(\cdot)\,\lambda(dx), \qquad s \in [0,1],$$

so that η is a Borel measurable function from $[0,1] \mapsto \mathbb{P}(X)$ such that $\eta(s) \in \Gamma^c(\lambda)$ for all $s \in [0,1]$. Hence, for a Borel subset B of X,

$$\begin{aligned}\gamma(B) &= \int Q(x,B)\,\lambda(dx) \;= \int\!\!\int \mu(B)\,\tau(x)(d\mu)\,\lambda(dx)\\ &= \int\!\!\int \mu(B)\,(l\phi(x,\cdot)^{-1})(d\mu)\,\lambda(dx) \;= \int\!\!\int \phi(x,s)(B)\,l(ds)\,\lambda(dx)\\ &= \int\!\!\int \phi(x,s)(B)\,\lambda(dx)\,l(ds) \;= \int \eta(s)(B)\,l(ds) \;= \int \mu(B)\,l\eta^{-1}(d\mu),\end{aligned}$$

where the fourth and seventh equalities are by virtue of the change of variable theorem. Since $l\eta^{-1}(\Gamma^c(\lambda)) = 1$, it follows that $\gamma \in \operatorname{sco}(\Gamma^c(\lambda))$. This completes the proof. □

Since $\Gamma^s = \overline{\operatorname{sco}}\,\Gamma^c$ according to Theorem 2.2, we will write, for $\lambda \in \mathbb{P}(X)$, $\Gamma^s(\lambda)$ for $(\overline{\operatorname{sco}}\,\Gamma^c)(\lambda)$. We will now investigate whether $\Gamma^s(\lambda)$ can be expressed as $\overline{\operatorname{sco}}\;\Gamma^c(\lambda)$.

THEOREM 3.5. *Let $\mu, \lambda \in \mathbb{P}(X)$. Suppose that $\mu = \lambda Q$ for some Borel measurable selector Q for Γ^s. Then there exist Borel measurable selectors Q_k for $\operatorname{sco}\Gamma^c$ such that ν_k converges in norm to ν, where ν_k (respectively, ν) is the measure on $X \times X$ such that $\nu_k\pi_1^{-1} = \lambda$ (respectively, $\nu\pi_1^{-1} = \lambda$) and $\nu_k(x) = Q_k(x,\cdot)$ a.s.(λ) (respectively, $\nu(x) = Q(x,\cdot)$ a.s.(λ)).*

PROOF. Let

$$\Gamma_k = \{(x,\gamma) \in \operatorname{sco}\Gamma^c \,:\, \|\gamma - Q(x,\cdot)\| < 1/k\}, \qquad k \geq 1.$$

Note that each $\Gamma_k(x)$ is nonempty because $Q(x,\cdot) \in \Gamma^s(x) = \overline{\text{sco}}\,\Gamma^c(x)$. Also Γ_k is analytic. Let Q_k be a Borel measurable selector for $\text{sco}\,\Gamma^c$ such that $\|Q_k(x,\cdot)-Q(x,\cdot)\| < 1/k$ a.s.(λ). Let ν_k and ν be as defined in the statement of the theorem. Let (E_k, E_k^c) be a Jordan-Hahn decomposition of $X \times X$ for $\nu_k - \nu$. So

$$\begin{aligned}(\nu_k - \nu)^+(X \times X) = \nu_k(E_k) - \nu(E_k) \;&= \int [Q_k(x, E_{k,x}) - Q(x, E_{k,x})]\,\lambda(dx)\\ &\leq \int |Q_k(x, E_{k,x}) - Q(x, E_{k,x})|\,\lambda(dx) \;\leq 1/k.\end{aligned}$$

Similarly, $(\nu_k - \nu)^-(X \times X) \leq 1/k$. Consequently, $\|\nu_k - \nu\| \to 0$ as $k \to \infty$. □

The next result is an immediate consequence of Theorems 3.4 and 3.5.

COROLLARY 3.6. *For* $\lambda \in \mathbb{P}(X)$, $\Gamma^s(\lambda) \subseteq \overline{\text{sco}}\,\Gamma^c(\lambda)$.

From Theorems 3.3 and 3.4, we get the following.

COROLLARY 3.7. *(Compare Dellacherie and Meyer, 1983, XI 40.) For* $\mu, \lambda \in \mathbb{P}(X)$, $\mu \vdash \lambda$ *if and only if there exist* $\gamma_k \in (\text{sco}\,\Gamma^c)(\lambda)$ *such that* $\gamma_k \to \mu$ *in norm.*

The reverse inclusion in Corollary 3.6 fails as the following example of Dellacherie and Meyer (1983, XI 40) shows.

EXAMPLE 3.8. Let $X = [0, \infty)$, and let

$$\Gamma(x) = \{\gamma \in \mathbb{P}(x) \,:\, \gamma(\{x\} \cup (x+1, \infty)) = 1\}, \qquad x \in X.$$

Then Γ is a Borel gambling house. It is easily checked that Γ is closed under composition, so $\Gamma^c \subseteq \Gamma$, and, since, for each x, $\Gamma(x)$ is strongly convex and norm-closed, it follows that $\overline{\text{sco}}\,\Gamma^c(x) \subseteq \Gamma(x)$. So $\Gamma = \Gamma^s$.

Let λ and μ be Lebesgue measures restricted to [0,1] and [1,2], respectively. Suppose that f is Γ-excessive. Clearly, $f(x+y) \leq f(x)$ for all $y > 1$. So, for $\epsilon > 0$,

$$\int f\,d\lambda = \int_0^1 f(x)\,dx \geq \int_0^1 f(x+1+\epsilon)\,dx = \int_{1+\epsilon}^{2+\epsilon} f(y)\,dy.$$

Hence,

$$\int f\,d\lambda \geq \lim_{\epsilon \downarrow 0} \int_{1+\epsilon}^{2+\epsilon} f(y)\,dy = \int_1^2 f(y)\,dy = \int f\,d\mu.$$

It follows that $\mu \vdash \lambda$. So, from Theorem 3.3, we have: $\mu \in \overline{\text{sco}}\,\Gamma^c(\lambda)$.

We now show that $\mu \notin \Gamma^s(\lambda)$. Towards a contradiction, assume $\mu \in \Gamma^s(\lambda)$, i.e. $\mu = \lambda Q$ for some Borel measurable selector Q for $\Gamma^s = \Gamma$. Since $\mu([1,2]) = 1$, it follows that $Q(x,[1,2]) = 1$ a.s.(λ), so $Q(x,(x+1,2]) = 1$ a.s.(λ). It follows that

$$\int_1^2 y\, Q(x,dy) > x+1 \text{ a.s. } (\lambda).$$

Consequently,

$$\int y\, d\mu = \int \int_1^2 y\, Q(x,dy)\lambda(dx) > \int (x+1)\,\lambda(dx) = \int_0^1 (x+1)\,dx = 3/2,$$

which contradicts $\int y\,d\mu = \int_1^2 y\,dy = 3/2$.

Given μ, $\lambda \in \mathbb{P}(X)$, Theorem 3.5 gave necessary conditions for μ to belong to $\Gamma^s(\lambda)$. The next result is a sort of converse to Theorem 3.5 - it gives sufficient conditions for μ to belong to $\Gamma^s(\lambda)$.

THEOREM 3.9. *Let μ, $\lambda \in \mathbb{P}(X)$. Suppose Q_k is a Borel measurable selector for Γ^s and let $\nu_k \in \mathbb{P}(X \times X)$ be such that $\nu_k\pi_1^{-1} = \lambda$ and $\nu_k(x) = Q_k(x,\cdot)$ a.s. (λ). If ν_k converges in norm to $\nu \in \mathbb{P}(X\times X)$ and $\mu = \nu\pi_2^{-1}$, then $\mu = \lambda Q$ for some Borel measurable selector Q for Γ^s.*

PROOF. Since $\nu_k\pi_1^{-1}$ converges in norm to $\nu\pi_1^{-1}$, it follows that $\nu\pi_1^{-1} = \lambda$. Set $Q(x,\cdot) = \nu(x)$, $x \in X$. Then Q is a Borel measurable transition function. We will show that $Q(x,\cdot) \in \Gamma^s(x)$ a.s. (λ). Towards a contradiction, assume that there is a Borel set $B \subseteq X$ such that $\lambda(B) > 0$ and $Q(x,\cdot) \notin \Gamma^s(x)$ for $x \in B$. Since $\Gamma^s(x)$ is analytic, strongly convex, and norm-closed, it follows by combining Corollary 2.2 of MSa with the bold-face version of Theorem 2.7 of MSa, that there is a Borel measurable function $g : B \times X \mapsto [0,1]$ such that

$$\sup_{\gamma\in\Gamma^s(x)} \int g(x,y)\,\gamma(dy) < \int g(x,y)\,Q(x,dy)$$

for all $x \in B$. Hence,

$$\int_{B\times X} g\,d\nu = \int_B \int g(x,y)\,Q(x,dy)\lambda(dx) > \int_B (G_{\Gamma^s}(g_x)(x)\,\lambda(dx), \tag{3.5}$$

where g_x is the x-section of g. On the other hand,

$$\int_{B\times X} g\,d\nu_k = \int_B \int g(x,y)\,Q_k(x,dy)\lambda(dx) \le \int_B (G_{\Gamma^s}(g_x)(x)\,\lambda(dx), \tag{3.6}$$

where the last inequality follows from the fact that $Q_k(x,\cdot) \in \Gamma^s(x)$. Now let $k \to \infty$ in (3.6) to obtain

$$\int_{B\times X} g\, d\nu \le \int_B (G_{\Gamma^s}(g_x)(x)\, \lambda(dx), \tag{3.7}$$

which contradicts (3.5).

Thus, $Q(x,\cdot) \in \Gamma^s(x)$ a.s. (λ). By setting $Q(x,\cdot) = \delta(x)$ on a λ-null Borel set if necessary, we can ensure Q is a Borel measurable selector for Γ^s and $Q(x,\cdot) = \nu(x)$ a.s. (λ). Since $\mu = \nu\pi_2^{-1}$, it follows that $\mu = \lambda Q$. □

REMARK 3.10. The proof of Theorem 3.9 shows that the set

$$M^s = \{\nu \in \mathbb{P}(X\times X) \,:\, \nu\pi_1^{-1} = \lambda \;\&\; \lambda(\{x \,:\, \nu(x) \in \Gamma^s(x)\}) = 1\} \tag{3.8}$$

is norm-closed. Indeed, it is closed in the strong, or Feller, topology on $\mathbb{P}(X\times X)$, that is, the smallest topology on $\mathbb{P}(X\times X)$ for which the functions $\nu \mapsto \nu(f)$ are continuous for all bounded, Borel measurable functions f on $X\times X$.

For our next result, we endow $\mathbb{P}(X\times X)$ with the smallest topology for which the functions $\nu \mapsto \nu(f)$ are continuous for all bounded, Caratheodory functions f on $X\times X$. Following Schäl (1975), we call this the *ws-topology* on $\mathbb{P}(X\times X)$. Recall that a function $f : X\times X \mapsto \mathbb{R}$ is a *Caratheodory* function if (i) $f(x,\cdot)$ is continuous for each x, and (ii) $f(\cdot,y)$ is Borel measurable for each y.

THEOREM 3.11. *Suppose X is compact metric and $\Gamma^s(x)$ is closed in the weak topology for every x. Then the set M^s, defined by (3.8), is ws-closed.*

PROOF. The proof is similar to that of Theorem 3.9, only simpler. Recall that the proof of Theorem 3.9 turned on the fact that we could "separate" the gamble $Q(x,\cdot)$ from $\Gamma^s(x)$ by a bounded, Borel measurable function uniformly in x for x in a Borel set of positive λ-measure. To prove the present theorem we have to "separate" $Q(x,\cdot)$ from $\Gamma^s(x)$ by a continuous function uniformly in x.

Suppose, then, ν_α is a net in M^s converging in the ws-topology to ν. Plainly then, $\nu\pi_1^{-1} = \lambda$, because for any bounded, Borel measurable function f on X,

$$\int f\, d(\nu\pi_1^{-1}) = \int f\circ\pi_1\, d\nu = \lim_\alpha \int f\circ\pi_1\, d\nu_\alpha = \lim_\alpha \int f\, d(\nu_\alpha\pi_1^{-1}) = \int f\, d\lambda.$$

Towards a contradiction, assume, as in the proof of Theorem 3.9, that there is a Borel set $B \subseteq X$ such that $\lambda(B) > 0$ and $Q(x,\cdot) \notin \Gamma^s(x)$ for all $x \in B$, where $Q(x,\cdot) = \nu(x)$, $x \in X$. Let

$$N = \{(x,f) \in B \times C(X) \,:\, \int f(y)\, Q(x,dy) > (G_{\Gamma^s} f)(x)\}.$$

Then it is easy to check that N is universally measurable. Also each x-section N_x is nonempty for $x \in B$ by virtue of the Hahn-Banach theorem. Let $\{f_m \,:\, m \geq 1\}$ be a dense set in $C(X)$. It is easy to see that, for every $x \in B$, there is $m \geq 1$ such that $(x, f_m) \in N$. It follows that there is a universally measurable function $\Phi : B \mapsto C(X)$ such that $(x, \Phi(x)) \in N$ for all $x \in B$. Modify Φ on a λ-null Borel set so that Φ becomes Borel measurable. Set

$$g(x,y) = \Phi(x)(y), \qquad x \in B,\, y \in X,$$

so g is Borel measurable, $g(x,\cdot)$ is continuous for $x \in B$ and (3.5) holds. Also, setting $Q_\alpha(x,\cdot) = \nu_\alpha(x)$, $x \in X$, we see that (3.6) holds with k replaced by α. Using the hypothesis that $\nu_\alpha \to \nu$ in the ws-topology, we get (3.7), which contradicts (3.5). So $Q(x,\cdot) \in \Gamma^s(x)$ a.s. (λ). Hence, $\nu \in M^s$, completing the proof. □

As a consequence we derive the following.

COROLLARY 3.12. *Under the same hypotheses as in Theorem 3.11, if* $\lambda \in \mathbb{P}(X)$, *then*

$$\Gamma^s(\lambda) = \overline{\mathrm{sco}}\, \Gamma^c(\lambda).$$

PROOF. The inclusion $\subseteq$ is by virtue of Corollary 3.6. Conversely, suppose $\mu \in \overline{\mathrm{sco}}\, \Gamma^c(\lambda)$. So there exist $\mu_k \in \mathrm{sco}\, \Gamma^c(\lambda)$ such that $\mu_k \to \mu$ in norm. Since $\mu_k \in \mathrm{sco}\, \Gamma^c(\lambda)$, by Theorem 3.4, there is a Borel measurable selector Q_k for $\mathrm{sco}\, \Gamma^c$ such that $\mu_k = \lambda Q_k$. Let $\nu_k \in \mathbb{P}(X \times X)$ be defined such that $\nu_k \pi_1^{-1} = \lambda$ and $\nu_k(x) = Q_k(x,\cdot)$ a.s. (λ). Then $\nu_k \in M^s$, $k \geq 1$, where M^s is defined by (3.8). It follows from Theorem 3.11 and Schäl (1975, Theorem 3.10 and Remark 3.11) that M^s is sequentially compact in the ws-topology. So there is a subsequence ν_{k_i} converging in the ws-topology to $\nu \in M^s$. Write $\nu \pi_2^{-1} = \lambda Q$ for some Borel measurable selector Q for Γ^s. Since $\nu_{k_i} \to \nu$ in the ws-topology and $\mu_{k_i} = \nu_{k_i} \pi_2^{-1}$, it follows that $\mu_{k_i} \to \lambda Q$ in the weak topology. But also, since $\mu_{k_i} \to \mu$ in norm, $\mu_{k_i} \to \mu$ in the weak topology. So $\mu = \lambda Q$, which completes the proof. □

Combining Corollary 3.12 with Theorem 3.3, we get the following.

COROLLARY 3.13. *Under the same hypotheses as in Theorem 3.11, if* $\lambda \in \mathbb{P}(X)$, *then*

$$\Gamma^s(\lambda) = \{\mu \in \mathbb{P}(X) \,:\, \mu \vdash \lambda\}.$$

Corollary 3.13 was proved by Grecea (2001, Theorem 1.2). Grecea's proof is different from ours, being based on Strassen's theorem on the measurable disintegration of continuous linear functionals on separable Banach spaces (Strassen, 1965, Theorem 1).

Though our methods do not appear to yield it, the following extension of Grecea's result can be obtained by using another result of Strassen.

THEOREM 3.14. *Let* X *be a Polish space and suppose that* $\Gamma^s(x)$ *is closed in the weak topology for every* x. *If* $\lambda \in \mathbb{P}(X)$, *then*

$$\Gamma^s(\lambda) = \{\mu \in \mathbb{P}(X) \,:\, \mu \vdash \lambda\}.$$

PROOF. The inclusion $\subseteq$ is immediate from the definition of $\Gamma^s(\lambda)$. For the reverse inclusion, suppose $\mu \vdash \lambda$. By Theorems 2.1 and 2.2, R_Γ and G_{Γ^s} agree on bounded, upper analytic functions on X. Thus, if f is a bounded, upper analytic function on X, then

$$R_\Gamma f(x) = \sup\{\gamma(f) \,:\, \gamma \in \Gamma^s(x)\}, \qquad x \in X. \tag{3.9}$$

Moreover, since $f \le R_\Gamma f$ and $R_\Gamma f$ is Γ-excessive (MSa, Theorem 1.1), we have

$$\int f \, d\mu \le \int R_\Gamma f \, d\mu \le \int R_\Gamma f \, d\lambda. \tag{3.10}$$

Consequently, the hypotheses of Theorem 3 in Strassen (1965) being satisfied (indeed, we only need (3.9) and (3.10) to hold for bounded, continuous functions on X), it yields a Borel measurable selector Q for Γ^s such that $\mu = \lambda Q$, i.e. $\mu \in \Gamma^s(\lambda)$. This completes the proof. □

4. Examples

We will close this article with some examples which serve to illustrate the theory developed in MSa and the present article.

EXAMPLE 4.1. Let $X = [0, \infty)$ and let $\Gamma(x) = \{\gamma \in \mathbb{P}(X) \,:\, \int_0^\infty y \, d\gamma \le x\}$, $x \in X$, that is, $\Gamma(x)$ is the set of "subfair" gambles at x. It is easily checked that Γ is a Borel gambling house. Since $\int_0^\infty y \wedge n \, d\gamma(y) \uparrow \int_0^\infty y \, d\gamma(y)$, it follows that the function $\gamma \mapsto \int_0^\infty y \, d\gamma(y)$ is lower semicontinuous in the weak topology. So, for fixed x, $\Gamma(x)$ is closed in the weak topology, hence closed in the norm topology. Also, it is easily verified that $\Gamma(x)$ is strongly convex and closed under composition for every $x \in X$, the latter fact implying that $\Gamma^c \subseteq \Gamma$, as was mentionned in section 2. So, by Theorem 2.2, $\Gamma = \Gamma^s$, i.e., Γ is saturated.

It is proved in Maitra and Sudderth (1996, pp.32-33) that the Γ-excessive functions are the bounded, nondecreasing, concave functions on $[0, \infty)$. Such functions can only be discontinuous at $x = 0$ (see Hewitt and Stromberg, 1965, 13.34). Hence the balayage order defined by Γ is as follows:

$$\mu \vdash \lambda \qquad \text{if and only if} \qquad \mu(f) \leq \lambda(f) \tag{4.1}$$

for every bounded, nondecreasing, concave function f on $[0, \infty)$. So, by Theorem 3.14, we have

THEOREM 4.2. *If $\mu, \lambda \in \mathbb{P}([0, \infty))$ and $\vdash$ is defined by (4.1), then $\mu \vdash \lambda$ if and only if $\mu = \lambda Q$ for some Borel measurable transition function Q such that $Q(x, \cdot)$ is a "subfair" gamble at x for each x.*

The next example is a much studied one. See for example Dellacherie and Meyer (1983, pp.28-40) and Grecea (2001). We sketch it here for completeness.

EXAMPLE 4.3. Let X be a compact metric space and let F be a convex cone of continuous functions on X containing constants and closed under $\wedge$, the pointwise minimum operation. Define, for $\mu, \lambda \in \mathbb{P}(X)$,

$$\mu \vdash_F \lambda \qquad \text{if and only if} \qquad \mu(f) \leq \lambda(f)$$

for every $f \in F$.

To describe $\vdash_F$ as a balayage order defined by a gambling house, let

$$\Gamma(x) = \{\gamma \in \mathbb{P}(X) \,:\, \gamma \vdash_F \delta(x)\},\ x \in X. \tag{4.2}$$

Then Γ is a compact subset of $X \times \mathbb{P}(X)$, when $\mathbb{P}(X)$ is given the weak topology. It is easily verified, as in Example 4.1, that $\Gamma = \Gamma^s$. Finally, it is not difficult to show, as is done in Dellacherie and Meyer (1983, X 39), that $\mu \vdash_F \lambda$ implies $\mu \vdash_\Gamma \lambda$. The converse is of course trivial. Corollary 3.13 now yields the following result:

THEOREM 4.4. *If $\mu, \lambda \in \mathbb{P}(X)$ then $\mu \vdash_F \lambda$ if and only if $\mu = \lambda Q$ for some Borel measurable selector for Γ, defined by (4.2).*

It is to be noted that in proving Theorem 4.4 we have not used the disintegration theorem of Strassen.

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