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DAMAGE MODELS — A MARTIN BOUNDARY CONNECTION

By C.R. RAO
Pennsylvania State University, USA
M. BHASKARA RAO
North Dakota State University, USA and
D.N. SHANBHAG
University of Sheffield, UK

SUMMARY. This article is predominantly a review paper of the literature bringing Martin Boundary theory into the ambit of Damage Models. More specifically, it concerns the Martin Boundary in the environment of non-negative matrices with the inherent extreme point methods that is linked to Damage Models. Included in this paper are some new observations on certain results on damage models, which were obtained earlier following random walk and branching processes methods, amongst other things. De Finetti's theorem for exchangeable random variables has already been known to have links with certain results on the Integrated Cauchy Functional Equation (ICFE) (Shanbhag, 1977 and Lau and Rao, 1982). A special version of ICFE, or of de Finetti's theorem for discrete random variables plays a crucial role in the damage model studies. We bring the Martin boundary theory into the fold of damage model studies.

1. Introduction

The concept of a damage model was first introduced by Rao (1965), which can be described mathematically as follows. Let X be an integer-valued nonnegative random variable such that $P(X = 0) < 1$. Suppose that a damage mechanism reduces X to Y according to a binomial damage process, i.e., for some $0 < \pi < 1$,

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$$P(Y = r|X = n) = \binom{n}{r}\pi^r(1-\pi)^{n-r},\ r = 0, 1, \ldots, n \text{ and } n \geq 0.$$

Rao (196) observed that if X has a Poisson distribution, then

$$P(Y = r) = P(Y = r|X = Y), r = 0, 1, \ldots \tag{1.1}$$

and raised the question whether Property (1.1) implies the distribution of X to be Poisson. He showed that it is so if (1.1) holds for all π, the parameter of the binomial distribution and the marginal distribution of X is free of π. Later in collaboration with Rubin (Rao and Rubin, 1964), Rao has shown that the result remains valid if (1.1) holds for just one fixed value of π. It was pointed out by Shanbhag (1977) that the Rao-Rubin result can be deduced from the solution to a general recurrence relation of the form

$$v_m = \sum_{n=0}^{\infty} \omega_n v_{m+n},\ m = 0, 1, 2, \ldots, \tag{1.2}$$

where $\{\omega_m : m \geq 0\}$ is a given sequence of nonnegative real numbers with $\omega_1 > 0$ and $\{v_n : n \geq 0\}$ is a sequence of nonnegative real numbers to be determined. Shanbhag obtained a complete solution to (1.2), which provided a unified approach to a variety of characterizations of discrete distributions including, in particular, those related to damage models, strong memoryless property, order statistics, record values, etc. Inspired by this work, Lau and Rao (1982) studied a more general version of (1.2), which can be described as follows.

Let μ be a σ-finite Borel measure on $\mathbb{R}^+$ with $\mu(\{0\}) < 1$ and $H : \mathbb{R}^+ \to \mathbb{R}^+$ be a locally integrable (with respect to Lebesgue measure) Borel measurable function satisfying the integral equation,

$$H(x) = \int_{\mathbb{R}^+} H(x + y)\mu(dy) \text{ for almost all [Lebesgue] } x \in \mathbb{R}^+. \tag{1.3}$$

Since (1.3) involves implicitly the well known Cauchy equation, Rao named it as the Integrated Cauchy Functional Equation (ICFE). A complete solution to (1.3) was obtained by Lau and Rao (1982). Equation (1.3) provides one with solutions to a variety of characterization problems involving continuous as well as discrete distributions. Since the findings of Lau-Rao and Shanbhag, various papers have appeared on applications of ICFE, providing variations or extensions of the aforementioned characterizations in the discrete case. Incidentally, commenting on a paper of Laczkovich (1986), Rao and Shanbhag (1998) have shown that the Lau-Rao theorem holds in the case when μ is concentrated on a countable set, even when it is not assumed a priori that H in (1.3) be locally integrable.

Choquet and Deny (1960) and Deny (1961) had earlier identified the solutions to a certain convolution equation on some groups. These results have applications to renewal theory and allied topics. The specialized version of Deny's theorem for groups such as $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ could be viewed as variations of the Lau-Rao-Shanbhag theorems.

The literature prior to 1994 on the integral equations of the type considered here has been reviewed and unified by Rao and Shanbhag (1994). The above is simply a gist of this comprehensive review. See also Rao and Shanbhag (1986) and Ramachandran and Lau (1991) for some relevant material on this topic. More recently, Asadi et al. (2001) have unified several results in characterization theory via approaches based on these integral equations and also arrived at new characterizations of the Generalized Pareto distributions, among other results.

Apart from Shanbhag's (1977) extension of the original Rao-Rubin problem, there are other modifications and variations of the problem dealt with in the literature. See, for example, Talwalker (1980), Rao et al. (1980), Alzaid et al. (1986, 1987a), and Alzaid et al. (1988). In these papers, the authors deal mostly with situations when certain changes are made to Equation (1.1) and link the damage models to certain functional equations in random walk and branching processes. The literature on damage models prior to 1980 has been reviewed by Panaretos (1977) and Rao and Shanbhag (1982).

De Finetti's theorem for exchangeable random variables also plays an important role in linking the damage models or the relevant functional equations to stochastic processes. In particular, Ressel (1985), Alzaid et al. (1987b), Shanbhag (1991), and Rao and Shanbhag (1991) have shown that arguments based on de Finetti's theorem (or its specialized versions) can be used to solve (1.3) or certain of its extended versions or variations. Rao and Shanbhag (1994, 1998) have discussed and unified most of these developments.

The purpose of the present paper is to make further revelations on functional equations of relevance especially to damage models and link these to certain extreme point results stemming from the Martin boundary and Choquet theories. A connection has been alluded in Rao and Shanbhag (1994) but has not been established beyond doubt. The connection facilitates a unified approach to most of the important results on Rao's damage model and its extensions. Among various observations we have made in this paper, there is one in which we provide a new proof of a specialized version of de

Finetti's theorem using some results on non-negative matrices. It is worth pointing out here that Alzaid et al. (1984, 1986) have also linked damage models to nonnegative matrices in a somewhat different context, involving, in particular, the Perron-Frobenius theorem; most of these latter findings have been reproduced in Chapter 7 of Rao and Shanbhag (1994) and, in conjunction with the material covered in the present paper, should tell the audience as to how important the role of nonnegative matrices is in damage model studies.

2. Martin Boundary and Choquet Theoretic Results

The starting point in our study is the following theorem of Rao and Shanbhag (1994, Theorem 4.4.1), which, in turn, is a slight generalization of a theorem of Williams (1979, Theorem 48.7). The proof of the theorem we have given here essentially follows the thread of Williams's proof, and has also appeared, but for some notational changes, in Rao and Shanbhag (1994, p.98); we reproduce it here for ready reference.

THEOREM 2.1. *Let I be a countable set and Π be a nonnegative (in the sense of Seneta, 1981) $I \times I$ matrix. Assume that there exists a reference point b in I such that $\sup_n \Pi^n(b,j) > 0$ for each j in I, where $\Pi^n(b,j)$ denotes the (b,j)-th element of the matrix Π^n. If we denote by Φ the class of functions $f : I \to \mathbb{R}^+$ such that $f(b) = 1$ and $\Pi f \leq f$, then Φ is a compact convex subset of the locally convex linear topological space $\mathbb{R}^I$. If Φ_ϵ is the set of all extreme points of Φ, then Φ_ϵ is a G_δ-subset of Φ. Moreover, given any f in Φ, there exists a probability measure ν on the Borel σ-field of Φ_ϵ such that for each i in I,*

$$f(i) = \int_{\Phi_\epsilon} \xi(i)\nu(d\xi). \tag{2.1}$$

(Note that the map $\xi \to \xi(i)$ is continuous on Φ for each i in I.)

PROOF. Denote $\sup_n \Pi^n(b,j)$ by $\theta(j)$. By the assumption in the theorem, we have $\theta(j) > 0$ for each j. Let $f \in \Phi$. Then $f \geq \Pi f \geq \Pi^2 f \geq \ldots$. Hence

$$f(b) \geq \theta(j)f(j) \text{ for all } j. \tag{2.2}$$

(Note that $\theta(j)$ could be equal to ∞. In that case $f(j)$ equals zero.) In view of Fatou's lemma, from (2.2), essentially as observed by Williams (1979), we have that Φ satisfies the hypothesis of Choquet's theorem in the metrizable case. For the details of Choquet's theorem, see Phelps (1966). Hence, we have the present theorem. □

REMARK 1. Any nonnegative function f on I with $\Pi f \leq f$, i.e. $f(i) \geq \sum_j \Pi(i,j)f(j)$ for each i, where $\Pi(i,j)$ is the (i,j)-th element of the matrix Π, is referred to as an excessive function (super regular function in the sense of Seneta, 1981). An excessive function is referred to as regular if $\Pi f = f$.

REMARK 2. If we take f so that we have

$$f(j) \geq \sum_i \Pi(i,j)f(i) \text{ for each } j$$

in place of

$$f(i) \geq \sum_j \Pi(i,j)f(j) \text{ for each } i,$$

Theorem 2.1 still holds for f with the obvious modifications in the stated assumption. Subinvariant measures considered in Seneta (1981) and stationary measures studied in Athreya and Ney (1972) are special cases of such functions.

REMARK 3. If we are given a function $f_0 : I \to (0,\infty)$ such that $\Pi f_0 \leq f_0$ (with Π as in Theorem 2.1), then there is no loss of generality in assuming that Π is a substochastic matrix. Define Π^* on $I \times I$ by

$$\Pi^*(i,j) = \frac{\Pi(i,j)f_0(j)}{f_0(i)}, \; i,j \in I.$$

It is clear that Π^* is substochastic. Now, let $f : I \to \mathbb{R}^+$ be a function such that

$$f(i) \geq \sum_j \Pi(i,j)f(j) \text{ for each } i.$$

The integral representation problem for f in the framework of Π can be translated into an integral representation problem for a related function in the framework of a substochastic matrix. Let $f^*(i) = \dfrac{f(i)}{f_0(i)}, i \in I$. Then

$$f^*(i) \geq \sum_j \Pi(i,j)\frac{f(j)}{f_0(i)} = \sum_j \frac{\Pi(i,j)f_0(j)}{f_0(i)}\frac{f(j)}{f_0(j)} = \sum_j \Pi^*(i,j)f^*(j) \text{ for each } i.$$

Thus we are now in the environment of a substochastic matrix.

REMARK 4. In the integral representation of a regular function f in Theorem 2.1, we can take the measure ν to be a probability measure concentrated on the set of extreme points of Φ that are regular.

Next, we focus on proving a version of de Finetti's theorem. We need some notation. Let $\mathbb{J}$ be any countable set. Let $K = \cup_{n\geq 1}\mathbb{J}^n$, where $\mathbb{J}^n$ is the n-fold Cartesian product of $\mathbb{J}$. Let $f : K \to \mathbb{R}^+$ be any function with the following properties:

(i) $1 \geq \sum_{j\in\mathbb{J}} f(j)$

(ii) $f(j_1, j_2, \ldots, j_n) \geq \sum_{j_{n+1}\in\mathbb{J}} f(j_1, j_2, \ldots, j_{n+1})$
for all $j_1, j_2, \ldots, j_n \in \mathbb{J}, n \geq 1$.

(iii) $f(j_1, j_2, \ldots, j_n) = f(j_{i_1}, j_{i_2}, \ldots, j_{i_n}), j_1, j_2, \ldots, j_n \in \mathbb{J}$,

for all permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$, and $n \geq 1$. (For notational convenience, we denote here, for each $n \geq 1$, the values of the restriction to $\mathbb{J}^n$ of f by $f(j_1, j_2, \ldots, j_n)$ respectively.) We call such f a symmetric substochastic function; any symmetric substochastic function f is referred to as a symmetric stochastic function if it satisfies (i) and (ii) with "=" in place of "≥". Let $\mathcal{G}$ be the collection of all probability distributions on $\mathbb{J}$ (i.e. of all of the functions $g : \mathbb{J} \to \mathbb{R}^+$ such that $\sum_{j\in\mathbb{J}} g(j) = 1$) equipped with the usual topology. We now have the following result as a corollary of Theorem 2.1. This result is a version of the celebrated de Finetti's theorem for exchangeable random variables. For a good exposition on de Finetti's theorem, see Aldous (1985).

COROLLARY 2.1. *Let f be a symmetric stochastic function as defined above. Then there exists a probability measure P on the Borel σ-field of $\mathcal{G}$ such that*

$$f(j_1, j_2, \ldots, j_n) = \int_{\mathcal{G}} \left(\prod_{i=1}^{n} g(j_i)\right) P(dg) \tag{2.3}$$

for all $j_1, j_2, \ldots, j_n \in \mathbb{J}$ and $n \geq 1$.

PROOF. Consider Φ to be the class of all symmetric substochastic functions on K. Adjoin a reference point b to K and extend each member of Φ to $K \cup \{b\}$ such that its value at b equals 1. The class Φ contains f in the statement of the corollary and the extension of f meets the requirements of the function in the statement of Theorem 2.1, with Π appropriately chosen. In view of Remark 4, the measure ν in the integral representation of the extension of f is concentrated on the set of extreme points that are regular. We now need to identify extreme symmetric stochastic functions. Let e be any extreme symmetric stochastic function on K. It is clear that for any $j_1, j_2, \ldots, j_n, j_{n+1} \in \mathbb{J}$ and $n \geq 1$,

$$e(j_1, j_2, \ldots, j_n, j_{n+1}) = e(j_1, j_2, \ldots, j_n)e(j_{n+1}) = \prod_{i=1}^{n+1} e(j_i). \tag{2.4}$$

(For e, (ii) holds with "$\geq$" replaced by "$=$" in conjunction with (i) modified appropriately and (iii) and hence implies easily that the first identity in (2.4) holds whenever $e(j_{n+1}) \neq 0$; it also implies when taken in conjunction with (iii) that the left-hand side of the identity in (2.4) equals zero if $e(j_{n+1}) = 0$, and hence that the identity referred to holds if $e(j_{n+1}) = 0$.) The measure P is taken to be the restriction to the Borel σ-field of K, of ν. The proof is complete. □

REMARK 5. If f is a symmetric stochastic function on K, then, in view of the Kolmogorov extension theorem (appearing in some references as the Daniell-Kolmogorov theorem), there exists a probability space $(\mathbb{J}^{\infty}, B(\mathbb{J}^{\infty}), P')$ such that the projection maps are exchangeable and the joint distribution of the first n projection maps is given by the restriction of f to $\mathbb{J}^n$.

COROLLARY 2.2. *Let S be a countable Abelian semigroup with zero element and $\{(v_x, w_x) : x \in S\}$ be a collection of 2-tuples with nonnegative real components with at least one v_x nonzero, $w_0 < 1$ and satisfying*

$$v_x = \sum_{y \in S} w_y v_{x+y}, x \in S.$$

Let S^* be the subsemigroup of S, with zero element, generated by $\{x \in S : w_x > 0\}$. Then (i) the class $\mathcal{E}$ of all exponential functions e on S^* (i.e. where e is nonnegative and $e(x+y) = e(x)e(y), x, y$ in S^*) satisfying $\sum_{y \in S^*} e(y) w_y = 1$ is non-empty and (ii) for each $x \in S$ there exists a probability measure P_x on $\mathcal{B}(\mathcal{E})$ such that

$$v_{x+y} = v_x \int_{\mathcal{E}} e(y) P_x(de), x \in S \quad and\ y \in S^*. \tag{2.5}$$

PROOF. Since there exists at least one v_x which is nonzero, there is no loss of generality if we assume that every v_x is nonzero. For each $x \in S$, define $f_x : S^* \to \mathbb{R}^+$ by

$$f_x(i) = v_{x+i}/v_x,\ i \in S^*$$

and the nonnegative $S^* \times S^*$ matrix Π by

$$\Pi(i, j) = \sum_{\{y \in S^* : i+y=j\}} w_y,\ i, j \in S^*.$$

Clearly, f_x and Π meet the requirements of Theorem 2.1 with $I = S^*$. The regular extreme points e_x here are such that

$$e_x(i) = \sum_{j \in S^*} e_x(i+j) w_j,\ i \in S^*,$$

implying that $\mathcal{E}$ is non-empty and e_x belongs to $\mathcal{E}$. Consequently, the corollary follows from Theorem 2.1. □

REMARK 6. The proof of Corollary 2.2 is essentially due to Rao and Shanbhag (1994). The result also follows from Corollary 2.1, which can be seen as follows. Take $\mathbb{J} = S^*$ and f_x as in Corollary 2.2. Define $h_x : K \to \mathbb{R}^+$ by

$$h_x(j_1, j_2, \ldots, j_n) = f_x(j_1 + j_2 + \ldots + j_n) \prod_{i=1}^{n} w_{j_i}, j_1, j_2, \ldots, j_n \in \mathbb{J}.$$

We see that h_x meets the requirements of a symmetric stochastic function on K. In this case, the measure P (i.e., P_x) of (2.3) is concentrated on the set of g's that are concentrated on the set $\{y \in S^* : w_y > 0\}$. The relevant extreme points e on S^* have the form implied by

$$e(y_1 + y_2 + \ldots + y_n) = \prod_{i=1}^{n} \frac{g(y_i)}{w_{y_i}}, y_i \in \{y \in S^* : w_y > 0\},\ i = 1, 2, \ldots, n,$$

for each $n \geq 1$ and are members of $\mathcal{E}$.

REMARK 7. Various other observations on the integral equation of the type stated in Corollary 2.2 appear in Rao and Shanbhag (1994, 1998). In particular, if $\mathcal{E}$ is a singleton (as in the case of $S = \{0, 1, 2, \ldots\}$) or $S^* = S$, Corollary 2.2 simplifies. The simplified versions have applications in damage models, see, for example, Shanbhag (1977) and Rao and Shanbhag (1994).

COROLLARY 2.3. *Let $\{(v_x, w_x) : x = 0, \pm 1, \ldots\}$ be a sequence of 2-tuples with nonnegative real components such that $w_0 < 1$ and at least one v_x and one w_x for $x \geq 0$ are nonzero and*

$$v_x = \sum_{y=-\infty}^{\infty} w_y v_{x+y},\ x = 0, 1, \ldots. \tag{2.6}$$

Let the largest common divisor of $\{y : w_y > 0\}$ be equal to 1. Also, let f_{x0} for $x \geq 1$ and f_{xy} for $x \geq 0$ and $y < 0$ be the first passage and the absorption measures (in obvious notation) relative to the $\mathbb{Z} \times \mathbb{Z}$ nonnegative matrix Π *such that*

$$\Pi(x, y) = \begin{cases} w_{y-x} & \text{if } x \geq 0 \text{ and } y \in \mathbb{Z}, \\ \delta_{xy} & \text{if } x < 0 \text{ and } y \in \mathbb{Z}, \end{cases}$$

where δ_{xy} is the Kronecker delta. Then, for some nonnegative constant α and positive constant β, we have

$$v_x = \sum_{y=-\infty}^{-1} f_{xy} v_y + \alpha \sum_{y=0}^{x} f_{y0} \beta^y,\ x = 0, 1, \ldots, \tag{2.7}$$

where $f_{00} = 1$.

PROOF. In view of what is revealed in Remark 7, the result follows trivially when $w_x = 0$ for all $x < 0$. We may hence assume that $w_x > 0$ for some $x < 0$; there is no loss of generality in assuming also that $w_x > 0$ for some $x > 0$. Assume then that $w_x > 0$ for some $x > 0$ and for some $x < 0$. Using essentially a standard argument (on Markov chains) involving the Lebesgue dominated convergence theorem, we can see that

$$u_x = \sum_{y=-x}^{\infty} w_y u_{x+y}, \ x = 0, 1, \ldots, \tag{2.8}$$

and

$$u'_x = \sum_{y=-x+1}^{\infty} w_y u'_{x+y}, \ x = 1, 2, \ldots, \tag{2.9}$$

where

$$u_x = v_x - \sum_{y=-\infty}^{-1} f_{xy} v_y \geq 0, \ x = 0, 1, \ldots,$$

and

$$u'_x = u_x - f_{x0} u_0 \geq 0, \ x = 1, 2, \ldots.$$

In view of the assumptions, it is now clear that $u_x \equiv 0$ unless $u_0 \neq 0$. If $u_0 \neq 0$, then taking zero as the reference point and appealing to Theorem 2.1, we have the integral representation for $u_x/u_0, x \geq 0$, in terms of the regular extreme points. From our assumptions it is clear that for a sufficiently large k, we have a $\gamma > 0$ such that $\Pi^k(x, x+1) > \gamma$ for all $x \geq 0$; from (2.8) it then follows that

$$u_x \geq \gamma u_{x+1}, \ x = 0, 1, \ldots,$$

and hence that

$$u_x = \gamma u'_{x+1} + u''_x, \ x = 0, 1, \ldots, \tag{2.10}$$

with $\{u''_x\}$ as a nonnegative sequence satisfying (2.8). In view of (2.8), (2.9) and (2.10), it follows that if e is a regular extreme point referred to above, we have for some $\beta > 0$,

$$\beta e(x) = e(x+1) - f_{x+1,0}, \ x = 0, 1, \ldots, \tag{2.11}$$

which implies that

$$e(x) = \sum_{y=0}^{x} f_{y0} \beta^{x-y}, \ x = 0, 1, \ldots.$$

Hence, the corollary follows. □

REMARK 8. Corollary 2.3 follows essentially from Theorem 2.3.4 of Rao and Shanbhag (1994); see also Alzaid et al. (1986). However, our proof here involves new ideas; this proof simplifies slightly if $w_1 > 0$. It may further be verified that in the case of $\alpha \neq 0$, β satisfies

$$\sum_{x=0}^{\infty} \beta^x \tau(x) = 1,$$

where τ is the weak ascending ladder height measure as in Alzaid et al. (1988).

REMARK 9. Defining for the Π of Theorem 2.1

$$\Gamma(i,j) = \sum_{n=0}^{\infty} \Pi^n(i,j),\ i,j \in I,$$

we can see that if $\Gamma(b,j) < \infty$ for all $j \in I$, essentially as in Williams (1979), we can define the Martin kernel $k : I \times I \to \mathbb{R}^+$ such that

$$k(i,j) = \Gamma(i,j)/\Gamma(b,j) \leq (\theta(i))^{-1} \text{ for all } i,j \in I,$$

where θ is as defined in the proof of Theorem 2.1. In this case, even without the assumption that Π be substochastic the discussion on Martin compactification and various results (including the result of Doob and Hunt given in Williams, 1979, Theorem 48.21) hold. (Incidentally, in the notation of Williams, the set $K \setminus I$, where K is the Martin compactification of I achieved via the Martin kernel k, is referred to as the Martin boundary of I.) If we have additionally the situation of Remark 3, some further results including those on identifying extreme points follow from the relevant results discussed in the cited reference.

3. Applications to Damage Models

Shanbhag (1977), Alzaid et al. (1986), Alzaid et al. (1988) and Rao and Shanbhag (1994, Chapter 7) have given or implied various applications of the results met in the previous section. In particular, from Rao and Shanbhag (1994, p.167), it is clear that Corollary 2.2 (with the situation as in Remark 7) gives us the following theorem, which subsumes Shanbhag's (1977) generalization of the Rao-Rubin theorem.

Again, we do not claim here that the results appearing in this section are new. Our aim is to review and discuss certain results on damage models scattered in various places in the monograph of Rao and Shanbhag (1994) and elsewhere, and highlight systematically a thread developing on a link between the results discussed in the last section on nonnegative matrices and certain relevant findings on damage models.

THEOREM 3.2. *Let (X, Y) be a random vector such that X and Y are k-component vector satisfying*

$$P(X = n, Y = r) = g_n S(r|n),\ r \in [0, n] \cap \mathbb{N}_0^k,\ n \in \mathbb{N}_0^k$$

with $\{g_n : n \in \mathbb{N}_0^k\}$ as a probability distribution and, for each n for which $g_n > 0$,

$$S(r|n) = \frac{a_r b_{n-r}}{c_n},\ r \in [0, n] \cap \mathbb{N}_0^k,$$

where $\{a_n : n \in \mathbb{N}_0^k\}$ and $\{b_n : n \in \mathbb{N}_0^k\}$ are respectively positive and non-negative real sequences with $b_0 > 0$ and $b_n > 0$ if n is of unit length, and $\{c_n : n \in \mathbb{N}_0^k\}$ is the convolution of these two sequences. Then

$$P(Y = r) = P(Y = r|X = Y),\ r \in \mathbb{N}_0^k \tag{3.1}$$

if and only if (in obvious notation)

$$g_n/c_n = \int_{\mathbb{R}_+^k} \left(\prod_{i=1}^k \lambda_i^{n_i}\right) \nu(d\lambda),\ n \in \mathbb{N}_0^k, \tag{3.2}$$

where $0^0 = 1$ and ν is a uniquely determined finite measure on $\mathbb{R}_+^k$ (i.e. by the left-hand side of (3.2)) such that it is concentrated for some $\beta > 0$, on $\{\lambda : \Sigma_n b_n \prod_{i=1}^k \lambda_i^{n_i} = \beta\}$.

The above theorem follows on noting amongst other things that (3.1) is equivalent to

$$g_x/c_x \propto \sum_{y \in \mathbb{N}_0^k} b_y (g_{x+y}/c_{x+y}),\ x \in \mathbb{N}_0^k.$$

Extensions of the Rao-Rubin (1964) or of the above theorem for $k = 1$ involving an extended Rao-Rubin condition

$$P(Y = r) = P(Y = r|X - Y = k),\ r = 0, 1, \ldots,$$

have been studied by Alzaid et al. (1986) and Alzaid et al. (1988); Rao and Shanbhag (1994, Chapter 7) have discussed and unified most of these results. Corollary 2.3 plays a crucial role in these studies, since these depend mainly on the problem of solving, under appropriate conditions, the system of equations

$$v_{m+k} = \sum_{n=0}^{\infty} w_n v_{m+n}, \ m = 0, 1, \ldots .$$

There are also other types of extensions of the Rao-Rubin problem with modified conditions. Suppose we define (X, Y) as in Theorem 3.2 (but with $g_0 < 1$) and (X', Y') such that the marginal distribution of X' is the same as that of X and for each n for which $g_n \neq 0$, we have

$$P(Y' = r | X' = n) = \frac{a'_r b'_{n-r}}{c'_n}, \ r \in [0, n] \cap \mathbb{N}_0^k,$$

with $\{(a'_r, b'_r, c'_r)\}$ of the form of $\{(a_r, b_r, c_r)\}$. Then,

$$P(Y' = r | X' = Y') = P(Y = r), \ r = 0, 1, \ldots, \tag{3.3}$$

gives us yet another generalization of the Rao-Rubin condition. Specialized versions of variations of (3.3) have been studied by Talwalker (1980), Rao et al. (1980) and Alzaid et al. (1987a). Given $\{(a_r, b_r)\}$, $\{a'_r, b'_r)\}$ and $P(X' = Y')$, the problem of identifying the solution to (3.3) is a specialized version of that of identifying the nonnegative solution $\{v_x\}$ that is not identically equal to zero, to

$$v_x = u_x \sum_{y \in \mathbb{N}_0^k} w_y v_{x+y}, \ x \in \mathbb{N}_0^k, \tag{3.4}$$

where $\{u_x\}$ and $\{w_x\}$ are given positive and nonnegative sequences respectively with $w_x > 0$ whenever x is of unit length. If it is assumed that $u_x w_0 < 1$ for all x, then (3.4) implies that whenever there is a solution $\{v_x\}$, we have the situation of Remark 9 met with zero as the reference point and we can identify the solution to (3.4) as implied in the remark. (Note also that if there is a solution $\{v_x\}$ to (3.4) vanishing outside a finite set, then it is determined by v_0.) We can hence identify the solution of (3.4) under appropriate constraints.

While we are in the process of reviewing applications of Martin boundary related results on nonnegative matrices to damage models, it would not be out of place to discuss briefly an extended version of Spitzer's theorem given by Alzaid et al. (1987a) while solving a generalized Rao-Rubin equation met earlier in Talwalker (1980) and Rao et al. (1980). Consider now, for a given $c \in (0, 1]$, a modified discrete branching process (with state space $\{0, 1, \ldots\}$) whose substochastic matrix $(P_{ij}), i, j = 1, 2, \ldots$ (in obvious notation) is c times that of a Bienaymé-Galton-Watson branching process with offspring distribution $\{p_j : j = 0, 1, \ldots\}$ satisfying $0 < p_0 < 1$, and define $m = \sum_{j=0}^{\infty} j p_j$, $m^* = \sum_{j=1}^{\infty} j(\log j) p_j$ and f as the generating function of $\{p_j\}$. Defining stationary measures (for the modified branching process) as in Athreya and Ney (1972; Chapter II, Section 2) and using Bernstein's theorem for absolutely monotonic functions, amongst other things, Alzaid et al. (1987a) have established the following theorem:

THEOREM 3.3. (An extended version of Spitzer's theorem). *If $m < 1$ and $m^* < \infty$, then for every probability measure ν on $[0,1)$*

$$U(s) = K \int_{[0,1)} U(s,t)\ d\nu(t) \tag{3.5}$$

is the generating function of a stationary measure, where

$$U(s,t) = \sum_{n=-\infty}^{\infty} [\exp\{(\mathcal{B}(s) - 1)m^{n-t}\} - \exp\{-m^{n-t}\}]c^{n-t},$$

with $\mathcal{B}(s)$ as the unique probability generating function among those vanishing at $s = 0$ and satisfying the equation

$$\mathcal{B}(f(s)) = m\mathcal{B}(s) + 1 - m, \tag{3.6}$$

and K is the appropriate normalizing constant. Conversely, every stationary measure has the representation (3.5) for some probability measure ν on $[0,1)$.

(For an interpretation of $\mathcal{B}$ in Bienaymé-Galton-Watson branching processes, see Athreya and Ney (1972, p.17).)

The following remarks provide us with further information of relevance to Theorem 3.2, which indicates in particular as to how the theorem is linked to the Martin Boundary related results.

REMARK 10. Bernstein's theorem referred to above is a corollary to Choquet's theorem and also to a general result on an integral equation, and it, in turn, gives as corollaries several interesting results in distribution theory including the Rao-Rubin (1964) result mentioned earlier, see, for more details, Rao and Shanbhag (1994; pages 72, 73 and 165).

REMARK 11. Remark 2 in Alzaid et al. (1987a) sketches, assuming implicitly that $p_1 > 0$, a proof for Theorem 3.2, based on the Martin Boundary theory. The proof implied here is obviously a minor variation of the proof appearing in Athreya and Ney (1972) essentially for the specialized version of Theorem 3.2 with $c = 1$.

REMARK 12. The argument implied in Remark 2 of Alzaid et al. (1987a) holds irrespectively of whether or not $p_1 > 0$, if, for example, we replace, in particular, the two denominators in line 5 of the remark respectively by $G(k_i, j^*)$ and the coefficient of s^{j^*} in $\tilde{U}(s,t)$, where j^* is the smallest positive support point of $\{p_j\}$. It is perhaps useful to observe here to make the argument more accessible to the reader, that, for each $s \in [0,1)$, as $k_i \to \infty$,

$$c^{(\log k_i)/(\log m)} \sum_{j=1}^{\infty} G(k_i, j) s^j \to \tilde{U}(s,t)$$

with the notation as specified.

REMARK 13. Various versions of the Martin Boundary theory involving substochastic or nonnegative matrices (and exit or entrance boundaries or otherwise) have appeared, each with some interesting individual features, in Kemeny et al. (1966) and several other places, including those that we have cited in the previous section namely Williams (1979) and Seneta (1981). The proof in Athreya and Ney (1972) referred to in Remark 11 appeals to Kemeny et al. (1966) for a result in the theory. Remarks 2, 3 and 9 of the previous section throw implicitly light on the underlying mechanism of this proof and relate it to relevant studies in Williams (1979) and Seneta (1981).

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C.R. Rao
Department of Statistics
The Pennsylvania State University
University Park, PA 16802, USA
E-mail: crr1@psu.edu

M. Bhaskara Rao
Department of Statistics
North Dakota State University
Fargo, ND 58105, USA
E-mail: mb.rao@ndsu.nodak.edu

D.N. Shanbhag
Department of Probability and Statistics
University of Sheffield
Sheffield S3 7RH, UK
E-mail: d.shanbhag@btopenworld.com