

ESTIMATION OF PROBABILITIES OF LINEAR INEQUALITIES FOR INDEPENDENT ELLIPTIC RANDOM VECTORS

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SUMMARY. The goal of the present paper is to derive the maximum likelihood estimators of probabilities $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$ where independent vectors \mathbf{X} and \mathbf{Y} have elliptical distributions. Also, a technique of Bayesian estimation is proposed which enables one to avoid high-dimensional integration and, also, to utilize results of Bayesian analysis conducted earlier. Estimators for familiar classes of elliptical distributions are constructed.

1. Introduction

The stress-strength model has numerous applications in reliability, economics, clinical trials, genetics, etc. Mathematically, the problem reduces to estimation of the probability of the inequality $P(X < Y)$ where X and Y have known or unknown distributions. The problem was pioneered by Birnbaum and McCarty (1958) and was the topic of about 150 publications in the last four decades. Although estimators have been derived for various distributions of X and Y , majority of papers deal with one of the following three situations: distributions of X and Y are unspecified, X and Y are exponential random variables, and X and Y have normal distributions. Some generalizations of the original stress-strength model have also been studied (see, for example, Gupta and Gupta (1988), Gupta and Gupta (1990), Reiser and Faraggi (1994), Rinco (1983), Yang and Mo (1985)).

The present paper investigates one more generalization of the stress-strength model where instead of a two-component vector (X, Y) we have two independent k_1 and k_2 -component random vectors $\mathbf{X} = (X^{(1)}, \dots, X^{(k_1)})$ and $\mathbf{Y} = (Y^{(1)}, \dots, Y^{(k_2)})$ and we are interested in estimation of the probability $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$ where \mathbf{A} and \mathbf{B} are a known k_1 and k_2 -dimensional vectors and C is a known scalar.

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Estimation of $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$ is very important in many practical situations. Consider a technical system which is functioning under a variety of random stresses $X^{(i)}$, $i = 1, \dots, k_1$, such that the total stress on the system is given by a known linear combination of the stresses $\mathbf{A}'\mathbf{X}$. This situation occurs when, for example, $X^{(i)}$ is the density of the vehicles of type i on the bridge (cars, buses, trucks and so on) and $A^{(i)}$ is the damage (stress) caused by a vehicle of type i . If the strength of the system is provided by several components (for example, special steel or concrete, extra strong supports for a bridge), then the strength of the system can be viewed as a linear combination of some random components $Y^{(i)}$, $i = 1, \dots, k_2$, that is $\mathbf{B}'\mathbf{Y}$. In this model, stress \mathbf{X} and strength \mathbf{Y} are independent. Reliability of the system is the probability that strength exceeds stress $P(\mathbf{A}'\mathbf{X} < \mathbf{B}'\mathbf{Y})$. If we are interested in estimating the probability that strength exceeds stress by a fixed value C , the problem reduces to estimating $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$.

The above problem and its minor variations were considered in the case when \mathbf{X} and \mathbf{Y} are normally distributed random vectors by Pensky (1982), Gupta and Gupta (1990) and Ivshin and Lumelskii (1994). It is easy to see that with $k_1 = k_2 = 1$, $C = 0$, $\mathbf{A} = (-1)$ and $\mathbf{B} = 1$, the problem reduces to the estimation of probability $P(X < Y)$ for independent random variables X and Y , while $k_1 = 2$, $\mathbf{A} = (-1, 1)$ and $\mathbf{B} = \mathbf{0}$ allows one to estimate $P(X < Y)$ for dependent variables X and Y . Pensky (1982) studied estimation of $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$ when $\mathbf{B} = \mathbf{0}$. Gupta and Gupta (1990) investigated a particular case of estimating $P(\mathbf{A}'\mathbf{X} + C > 0)$ with $C = 0$: although their model is written as $P(\mathbf{A}'\mathbf{X} > \mathbf{B}'\mathbf{Y})$, the normally distributed vectors \mathbf{X} and \mathbf{Y} are assumed to be dependent, so that vectors \mathbf{X} and \mathbf{Y} can be combined into one vector \mathbf{X} .

However, all results mentioned above cover only the situation when \mathbf{X} and \mathbf{Y} are normally distributed random vectors which may not cover the variety of situations in practice. The purpose of the present paper is to derive estimators for probabilities $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$ in the situation when vectors \mathbf{X} and \mathbf{Y} have elliptical distributions with the pdfs of the forms

$$p_j(\mathbf{z}|\boldsymbol{\theta}_j, \boldsymbol{\Sigma}_j) = |\boldsymbol{\Sigma}_j|^{-1/2} f_j\left((\mathbf{z} - \boldsymbol{\theta}_j)' \boldsymbol{\Sigma}_j^{-1} (\mathbf{z} - \boldsymbol{\theta}_j)\right), \quad j = 1, 2, \quad (1.1)$$

where $\boldsymbol{\Sigma}_j$ is a positive definite symmetric $k_j \times k_j$ matrix and $f_j(z) \geq 0$ is such that

$$\frac{\pi^{k_j/2}}{\Gamma(k_j/2)} \int_0^\infty z^{k_j/2-1} f_j(z) dz = 1, \quad j = 1, 2. \quad (1.2)$$

Here and in what follows, we denote vectors and matrices by bold characters, \mathbf{W}' is the transpose of matrix \mathbf{W} , $|\mathbf{W}|$ is the determinant of \mathbf{W} , $\text{tr}(\mathbf{W})$ is the trace of \mathbf{W} .

Elliptical distributions (1.1) is a class of symmetric distributions which has been thoroughly studied by Anderson, Fang and Hsu (1986), Fang, Kotz and Ng (1990), Sutradhar (1986), Sutradhar and Ali (1989) among others. It is easy to see that multivariate normal distribution is a particular case of (1.1) with $f_j(z) = (2\pi)^{-k_j/2} \exp\{-z/2\}$, $z > 0$. In this sense, the present paper continues the line of investigation devoted to the case when X and Y are normally distributed (see, for example, Downton (1973), Govindarajulu (1967), Gupta and Gupta (1990), Gupta *et al.* (1999), Ivshin and Lumelskii (1994), Pensky (1982), Nandi and Aich (1994, 1996), Reiser and Guttman (1986, 1987), Reiser and Fraggi (1994), Rinco (1983), Rukhin (1986), Singh (1991), Weerandi and Johnson (1992), Woodward and Kelley (1977), Yang and Mo (1985)). The last example in Section 4 deals with the case when \mathbf{X} and \mathbf{Y} are normally distributed random vectors. Although the maximum likelihood estimator (MLE) and the unbiased estimator for $P(\mathbf{A}, \mathbf{B}, C)$ in the case of the normal distribution have been derived previously by other authors (Anderson, Fang and Hsu (1986), Ivshin and Lumelskii (1994)) we felt that it may be interesting from didactic point of view to show how easily they follow from the general theory developed in the paper. The Bayes estimator of $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$ which is based on the inverse Wishart prior, to the best of our knowledge, is original. The only Bayes estimator known to the author is the one of Enis and Geisser (1971). The latter one, based on Jeffreys's noninformative prior, is a particular case of the Bayes estimator obtained in the present paper.

As far as we know, no results on estimation of $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$ have been obtained for the elliptical distribution of a general form. Also, no estimators are available for particular types of elliptical distributions rather than normal. The case when $\mathbf{B} = 0$ and \mathbf{X} has a multivariate T-distribution has been studied by Abusev and Kolegova (1998). However, the paper contains serious errors which make both the estimation procedure and the resulting estimators totally wrong.

In what follows, we assume that i.i.d. samples

$$\underline{\mathbf{X}} = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_m), \quad \underline{\mathbf{Y}} = (\mathbf{Y}_1, \dots, \mathbf{Y}_n) \quad (1.3)$$

having elliptic pdfs (1.1) are available. We construct the MLE and the Bayes estimators of $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$ based on these samples. To achieve this goal, we propose several algorithms which allow one to reduce the problem of calculation of $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$ to one or two-dimensional integration. These techniques are used to construct the MLEs by replacing θ_j and Σ_j , $j = 1, 2$, by their estimators. Moreover, in the case when unbiased estimators of the pdfs (1.1) exist and belong to the class of elliptic

distributions, this method automatically provides an unbiased estimator of $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$. We shall also consider estimation of $P(\mathbf{A}'\mathbf{X} + C > 0)$ since quite often it is possible to provide a closed form for the MLE of $P(\mathbf{A}'\mathbf{X} + C > 0)$ even if it is impossible to do so for $P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0)$.

2. The Maximum Likelihood Estimation

Denote

$$P(\mathbf{A}, \mathbf{B}, C) = P(\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} + C > 0), \quad P(\mathbf{A}, C) = P(\mathbf{A}'\mathbf{X} + C > 0) \quad (2.1)$$

and observe that

$$P(\mathbf{A}, \mathbf{B}, C) = \int_{R_K} I(\mathbf{A}'\mathbf{x} + \mathbf{B}'\mathbf{y} + C > 0) p_1(\mathbf{x}|\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1) p_2(\mathbf{y}|\boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2) d\mathbf{x}d\mathbf{y} \quad (2.2)$$

with $K = k_1 + k_2$, I being an indicator function and $p_j(\mathbf{z}|\boldsymbol{\theta}_j, \boldsymbol{\Sigma}_j)$, $j = 1, 2$, defined in (1.1). Hence, the MLE $\hat{P}(\mathbf{A}, \mathbf{B}, C)$ of $P(\mathbf{A}, \mathbf{B}, C)$ has the form (2.2) with $\boldsymbol{\theta}_j$ and $\boldsymbol{\Sigma}_j$ being replaced by their MLEs $\hat{\boldsymbol{\theta}}_j$ and $\hat{\boldsymbol{\Sigma}}_j$, $j = 1, 2$, respectively. The MLEs of the parameters $\boldsymbol{\theta}_j$ and $\boldsymbol{\Sigma}_j$ of the elliptical distributions have been derived by Anderson, Fang and Hsu (1986)

$$\begin{aligned} \hat{\boldsymbol{\theta}}_1 &= \bar{\mathbf{X}}, & \hat{\boldsymbol{\theta}}_2 &= \bar{\mathbf{Y}}, \\ \hat{\boldsymbol{\Sigma}}_1 &= k_1(\lambda_1 m)^{-1} \mathbf{S}_1, & \hat{\boldsymbol{\Sigma}}_2 &= k_2(\lambda_2 n)^{-1} \mathbf{S}_2, \end{aligned}$$

where

$$\begin{aligned} \bar{\mathbf{X}} &= m^{-1} \sum_{i=1}^m \mathbf{X}_i, & \bar{\mathbf{Y}} &= n^{-1} \sum_{i=1}^n \mathbf{Y}_i, \\ \mathbf{S}_1 &= \sum_{i=1}^m (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})', & \mathbf{S}_2 &= \sum_{i=1}^n (\mathbf{Y}_i - \bar{\mathbf{Y}})(\mathbf{Y}_i - \bar{\mathbf{Y}})' \end{aligned} \quad (2.3)$$

$$\lambda_j = \operatorname{argmax}\{z^{\frac{k_j}{2}} f_j(z)\}, \quad j = 1, 2. \quad (2.4)$$

Let us introduce new parameters

$$a = \sqrt{\mathbf{A}'\hat{\boldsymbol{\Sigma}}_1\mathbf{A}}, \quad b = \sqrt{\mathbf{B}'\hat{\boldsymbol{\Sigma}}_2\mathbf{B}}, \quad c = \mathbf{A}'\hat{\boldsymbol{\theta}}_1 + \mathbf{B}'\hat{\boldsymbol{\theta}}_2 + C, \quad (2.5)$$

with the MLEs

$$\hat{a} = \sqrt{k_1 \mathbf{A}'\mathbf{S}_1\mathbf{A}/(m\lambda_1)}, \quad \hat{b} = \sqrt{k_2 \mathbf{B}'\mathbf{S}_2\mathbf{B}/(n\lambda_2)}, \quad \hat{c} = \mathbf{A}'\bar{\mathbf{X}} + \mathbf{B}'\bar{\mathbf{Y}} + C, \quad (2.6)$$

where λ_1 and λ_2 are defined in (2.4).

Note that since Σ_j , $j = 1, 2$, are the positive definite symmetric matrices, there exist matrices \mathbf{V}_j such that $\mathbf{V}_j' \mathbf{V}_j = \Sigma_j$, $j = 1, 2$. Then, (see Fang, Kotz and Ng (1990)) vectors

$$\boldsymbol{\xi} = \mathbf{V}_1^{-1}(\mathbf{X} - \boldsymbol{\theta}_1), \quad \boldsymbol{\eta} = \mathbf{V}_2^{-1}(\mathbf{Y} - \boldsymbol{\theta}_2) \quad (2.7)$$

have spherical distributions with the pdfs $f_1(\mathbf{z}'\mathbf{z})$ and $f_2(\mathbf{z}'\mathbf{z})$, respectively, where $f_j(t)$, $j = 1, 2$, are defined in (1.1) and (1.2).

The following theorem and corollary allow one to obtain MLEs of $P(\mathbf{A}, \mathbf{B}, C)$ and $P(\mathbf{A}, C)$ (see (2.1)).

THEOREM 1. *The probability $P(\mathbf{A}, \mathbf{B}, C)$ and its MLE have, respectively, the forms*

$$P(\mathbf{A}, \mathbf{B}, C) = J(a, b, c), \quad \hat{P}(\mathbf{A}, \mathbf{B}, C) = J(\hat{a}, \hat{b}, \hat{c}). \quad (2.8)$$

Here, function $J(a, b, c)$ can be calculated as

$$J(a, b, c) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(ax + by + c > 0) f_{11}(x) f_{21}(y) dx dy, \quad (2.9)$$

where

$$f_{j1}(z) = \frac{\pi^{(k_j-1)/2}}{\Gamma((k_j-1)/2)} \int_0^{\infty} t^{(k_j-3)/2} f_j(z^2 + t) dt, \quad j = 1, 2, \quad (2.10)$$

are the pdfs of the first components $\xi^{(1)}$ and $\eta^{(1)}$ of vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ defined in (2.7).

If characteristic functions $\varphi_1(\omega)$ and $\varphi_2(\omega)$ of $\xi^{(1)}$ and $\eta^{(1)}$ are available

$$\varphi_j(\omega) = \int_{-\infty}^{\infty} e^{i\omega z} f_{j1}(z) dz, \quad j = 1, 2, \quad (2.11)$$

then $J(a, b, c)$ is given by

$$J(a, b, c) = \frac{1}{2} + \frac{1}{\pi} \text{Im} \int_0^{\infty} \frac{e^{ic\omega}}{\omega} \varphi_1(a\omega) \varphi_2(b\omega) d\omega. \quad (2.12)$$

Here, $\text{Im}(z)$ is the imaginary part of z .

PROOF. Denote by $\|\mathbf{z}\| = \sqrt{\mathbf{z}'\mathbf{z}}$ the norm of a vector \mathbf{z} . Observe that in terms of the new vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, defined in (2.7), inequality $\mathbf{A}'\mathbf{X} + \mathbf{B}'\mathbf{Y} +$

$C > 0$ takes the form $\mathbf{A}'\mathbf{V}'_1\xi + \mathbf{B}'\mathbf{V}'_2\eta + c > 0$, where c is given by (2.6). It follows from Theorem 2.4 of Fang, Kotz and Ng (1990) that

$$\mathbf{A}'\mathbf{V}'_1\xi = \|\mathbf{V}_1\mathbf{A}\|\xi^{(1)}, \quad \mathbf{B}'\mathbf{V}'_2\eta = \|\mathbf{V}_2\mathbf{B}\|\eta^{(1)},$$

where $\xi^{(1)}$ and $\eta^{(1)}$ are the first components of vectors ξ and η , respectively. These components have pdfs

$$f_{j1}(z) = \int f_j(t^2 + \sum_{i=1}^{k_j-1} z_i^2) \prod_{i=1}^{k_j-1} dz_i, \quad j = 1, 2.$$

With the help of formula 4.642 of Gradshteyn and Ryzhik (1980), the last integrals can be rewritten as (2.10). Now, to complete the first half of the proof one just need to notice that $\|\mathbf{V}_1\mathbf{A}\| = a$ and $\|\mathbf{V}_2\mathbf{B}\| = b$.

To prove the relation (2.12) use formula 3.721 of Gradshteyn and Ryzhik (1980) which implies that

$$I(ax + by + c > 0) = \frac{1}{2} + \frac{1}{\pi} \operatorname{Im} \int_0^\infty \frac{e^{i(ax+by+c)\omega}}{\omega} d\omega.$$

Therefore, it follows from (2.9) that

$$J(a, b, c) = \int_{-\infty}^\infty \int_{-\infty}^\infty f_{11}(x)f_{21}(y) \left[\frac{1}{2} + \frac{1}{\pi} \operatorname{Im} \int_0^\infty \frac{e^{i(ax+by+c)\omega}}{\omega} d\omega \right] dx dy. \quad (2.13)$$

Changing the order of integration in (2.13) results in (2.12). \square

COROLLARY 1. *The probability $P(\mathbf{A}, C)$ and its MLE can be written as*

$$P(\mathbf{A}, C) = J(a, c), \quad \hat{P}(\mathbf{A}, C) = J(\hat{a}, \hat{c})$$

where the function $J(a, c)$ has the form

$$J(a, c) = \int_{-\infty}^{c/a} f_{11}(x) dx = \frac{1}{2} + \int_0^{c/a} f_{11}(x) dx = \frac{1}{2} + \frac{1}{\pi} \operatorname{Im} \int_0^\infty \frac{e^{ic\omega}}{\omega} \varphi_1(a\omega) d\omega. \quad (2.14)$$

Here $f_{11}(x)$ and $\varphi_1(\omega)$ are defined in (2.10) and (2.11), respectively.

PROOF. To show that the first part of the equality (2.14) is true perform integration over y in (2.9) keeping in mind that $b = 0$ and $f_{11}(x)$ is an even function of x . Validity of the second part of (2.14) follows directly from (2.12). \square

3. Bayes estimation

For the sake of construction of a Bayes estimator of $P(\mathbf{A}, \mathbf{B}, C)$ one needs to choose a prior pdf $g(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2)$. Then, the Bayes estimator of $P(\mathbf{A}, \mathbf{B}, C)$ has the form

$$\begin{aligned} & \tilde{P}(\mathbf{A}, \mathbf{B}, C) \\ &= \frac{\int \cdots \int J(a, b, c) L_1(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1 | \underline{\mathbf{X}}) L_2(\boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2 | \underline{\mathbf{Y}}) g(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2) \prod_{j=1}^2 d\boldsymbol{\theta}_j d\boldsymbol{\Sigma}_j}{\int \cdots \int L_1(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1 | \underline{\mathbf{X}}) L_2(\boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2 | \underline{\mathbf{Y}}) g(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2) \prod_{j=1}^2 d\boldsymbol{\theta}_j d\boldsymbol{\Sigma}_j}. \end{aligned} \quad (3.1)$$

Here, a , b and c are given by (2.5), $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$ are samples on \mathbf{X} and \mathbf{Y} (see (1.3)), $L_1(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1 | \underline{\mathbf{X}})$ and $L_2(\boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2 | \underline{\mathbf{Y}})$ are the likelihood functions

$$L_1(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1 | \underline{\mathbf{X}}) = \prod_{i=1}^m p_1(\mathbf{X}_i | \boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1), \quad L_2(\boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2 | \underline{\mathbf{Y}}) = \prod_{i=1}^n p_2(\mathbf{Y}_i | \boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2), \quad (3.2)$$

where $p_j(\mathbf{X} | \boldsymbol{\theta}_j, \boldsymbol{\Sigma}_j)$, $j = 1, 2$, are defined in (1.1). The integrals in (3.1) are calculated over the space $R_{k_1} \times \mathcal{A}_{k_1} \times R_{k_2} \times \mathcal{A}_{k_2}$ where \mathcal{A}_k is the space of symmetric positive definite ($k \times k$) matrices in R_k .

It is easy to see that (3.1) leads to $[k_1 + k_2 + 0.5 k_1(k_1 + 1) + 0.5 k_2(k_2 + 1)]$ -dimensional integration which is almost computationally intractable since even in the case of $k_1 = k_2 = 2$ one needs to perform 10-dimensional integration. For this reason, in the case when $(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1)$ and $(\boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2)$ are a-priori independent, i.e.

$$g(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2) = g_1(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1) g_2(\boldsymbol{\theta}_2, \boldsymbol{\Sigma}_2), \quad (3.3)$$

we suggest an algorithm that allows one to avoid painful integration over $\boldsymbol{\theta}_j$ and $\boldsymbol{\Sigma}_j$, $j = 1, 2$.

THEOREM 2. *Assume that (3.3) is valid and denote the marginal pdfs of $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$ (see (1.3)) by $q_1(\underline{\mathbf{X}})$ and $q_2(\underline{\mathbf{Y}})$, respectively:*

$$q_1(\underline{\mathbf{X}}) \equiv q_1(\mathbf{X}_1, \dots, \mathbf{X}_m) = \int_{R_{k_1}} \int_{\mathcal{A}_{k_1}} L_1(\boldsymbol{\theta}, \boldsymbol{\Sigma} | \underline{\mathbf{X}}) g_1(\boldsymbol{\theta}, \boldsymbol{\Sigma}) d\boldsymbol{\theta} d\boldsymbol{\Sigma}, \quad (3.4)$$

$$q_2(\underline{\mathbf{Y}}) \equiv q_2(\mathbf{Y}_1, \dots, \mathbf{Y}_n) = \int_{R_{k_2}} \int_{\mathcal{A}_{k_2}} L_2(\boldsymbol{\theta}, \boldsymbol{\Sigma} | \underline{\mathbf{Y}}) g_2(\boldsymbol{\theta}, \boldsymbol{\Sigma}) d\boldsymbol{\theta} d\boldsymbol{\Sigma}.$$

Then the Bayes estimator (3.1) can be written as

$$\tilde{P}(\mathbf{A}, \mathbf{B}, C) = \int_{\Omega(\mathbf{A}, \mathbf{B}, C)} \frac{q_1(\mathbf{X}_1, \dots, \mathbf{X}_m, \mathbf{x})}{q_1(\mathbf{X}_1, \dots, \mathbf{X}_m)} \frac{q_2(\mathbf{Y}_1, \dots, \mathbf{Y}_n, \mathbf{y})}{q_2(\mathbf{Y}_1, \dots, \mathbf{Y}_n)} d\mathbf{x}d\mathbf{y}. \quad (3.5)$$

Here, the set $\Omega(\mathbf{A}, \mathbf{B}, C)$ is as follows:

$$\Omega(\mathbf{A}, \mathbf{B}, C) = \{ \mathbf{x} \in R_{k_1}, \mathbf{y} \in R_{k_2} : \mathbf{A}'\mathbf{x} + \mathbf{B}'\mathbf{y} + C \geq 0 \}. \quad (3.6)$$

PROOF. Note that $J(a, b, c) = P(\mathbf{A}, \mathbf{B}, C)$ is given by (2.2). Substituting (2.2) for $J(a, b, c)$ in (3.1) and changing the order of integration we arrive at (3.5). \square

Note that (3.5) requires $(k_1 + k_2)$ -dimensional integration only. For example, if $k_1 = k_2 = 2$, (3.5) results in 4-dimensional integration (compare with 10-dimensional integration in (3.1)). Moreover, since the value of the marginal density is essential for Bayes analysis, the expressions for $q_1(\underline{\mathbf{X}})$ and $q_2(\underline{\mathbf{Y}})$ may be available from Bayesian analysis conducted previously by another researcher. In the case of $\mathbf{B} = \mathbf{0}$, Theorem 2 reduces to

COROLLARY 2. If $\mathbf{B} = \mathbf{0}$, then the Bayes estimator (3.1) can be evaluated as

$$\tilde{P}(\mathbf{A}, C) = \int_{R_{k_1}} I(\mathbf{A}'\mathbf{x} + C \geq 0) \frac{q_1(\mathbf{X}_1, \dots, \mathbf{X}_m, \mathbf{x})}{q_1(\mathbf{X}_1, \dots, \mathbf{X}_m)} d\mathbf{x}. \quad (3.7)$$

Here, the marginal density $q_1(\underline{\mathbf{X}})$ is defined in (3.4).

PROOF. Formula (3.7) follows directly from (3.5). \square

4. Examples

EXAMPLE 1. The Pearson – type II distributions. Let \mathbf{X} and \mathbf{Y} have pdfs (1.1) with

$$f_j(z) = [\Gamma(\alpha_j)]^{-1} \pi^{-k_j/2} \Gamma(k_j/2 + \alpha_j) (1 - z)^{\alpha_j - 1} I(0 < z < 1), \quad j = 1, 2. \quad (4.1)$$

Using formulas (2.9) and (2.10), we derive the MLE of $P(\mathbf{A}, \mathbf{B}, C)$ of the form (2.8) with \hat{a} , \hat{b} and \hat{c} defined in (2.6) and $\lambda_j = k_j / (k_j + 2\alpha_j - 2)$:

$$J(a, b, c) = \frac{\int_{-1}^1 \int_{-1}^1 I(ax + by + c > 0) (1 - x^2)^{\alpha_1 + \frac{k_1 - 3}{2}} (1 - y^2)^{\alpha_2 + \frac{k_2 - 3}{2}} dx dy}{B(0.5, \alpha_1 + 0.5k_1 - 0.5) B(0.5, \alpha_2 + 0.5k_2 - 0.5)}. \quad (4.2)$$

Here $B(\alpha, \beta)$ is the beta function. It is easy to see that the last formula requires two-dimensional integration.

If it is desirable to reduce estimation of $P(\mathbf{A}, \mathbf{B}, C)$ to one-dimensional integration, one can use combination of formulae (2.10) – (2.12) and formula 3.771.8 of Gradshtein and Ryzhik (1980):

$$J(a, b, c) = \frac{1}{2} + \frac{1}{\pi} \Gamma\left(\frac{2\alpha_1 + k_1}{2}\right) \Gamma\left(\frac{2\alpha_2 + k_2}{2}\right) \left(\frac{2}{a}\right)^{\alpha_1 - 1 + \frac{k_1}{2}} \left(\frac{2}{b}\right)^{\alpha_2 - 1 + \frac{k_2}{2}} \times \int_0^\infty \sin(c\omega) \omega^{-(\alpha_1 + \alpha_2 - 1 + \frac{k_1 + k_2}{2})} J_{\alpha_1 - 1 + \frac{k_1}{2}}(a\omega) J_{\alpha_2 - 1 + \frac{k_2}{2}}(b\omega) d\omega, \quad (4.3)$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind which can be presented via infinite series (see 8.402 of Gradshtein and Ryzhik (1980))

$$J_\nu(z) = \frac{z^\nu}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)}.$$

Now, let us consider the case of $\mathbf{B} = \mathbf{0}$. In this situation, by (2.14),

$$J(a, c) = \begin{cases} \frac{1}{2} + \frac{c}{a} \left[B\left(\frac{1}{2}, \frac{2\alpha_1 + k_1 - 1}{2}\right) \right]^{-1} {}_2F_1\left(\frac{1}{2}, -\left(\alpha_1 + \frac{k_1 - 3}{2}\right); \frac{3}{2}; \frac{c^2}{a^2}\right), & \text{if } |c| \leq a, \\ \frac{1}{2} + \frac{1}{2} \text{sign}(c), & \text{if } |c| > a. \end{cases} \quad (4.4)$$

Here, ${}_2F_1(\alpha, \beta; \gamma; z)$ denotes the hypergeometric series

$${}_2F_1(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1) \cdot 2!} z^2 + \dots$$

Note that the hypergeometric series in (4.4) terminates if $2\alpha_1 + k_1 \geq 3$ is an odd integer.

EXAMPLE 2. The multivariate \mathbf{T} -distribution. Let \mathbf{X} and \mathbf{Y} have multivariate T -distributions with the pdfs (1.1) with

$$f_j(z) = \pi^{-\frac{k_j}{2}} \sigma_j^{-\frac{k_j}{2}} [\Gamma(\alpha_j/2)]^{-1} \Gamma((\alpha_j + k_j)/2) (1 + z/\sigma_j)^{-\frac{\alpha_j + k_j}{2}}, \quad \alpha_j, \sigma_j > 0. \quad (4.5)$$

Using formulas (2.9) and (2.10) we derive that

$$J(a, b, c) = 0.5 + [\sqrt{\sigma_1 \sigma_2} B(0.5\alpha_1, 0.5) B(0.5\alpha_2, 0.5)]^{-1} \times \int_0^\infty (1 + y^2/\sigma_2)^{-\frac{\alpha_2 + 1}{2}} \left[\int_0^{\frac{b}{a} y + \frac{c}{a}} (1 + x^2/\sigma_1)^{-\frac{\alpha_1 + 1}{2}} dx \right] dy. \quad (4.6)$$

Then $\hat{P}(\mathbf{A}, \mathbf{B}, C)$ has the form (2.8) with \hat{a} , \hat{b} , \hat{c} given by (2.6) and $\lambda_j = k_j \sigma_j / \alpha_j$, $j = 1, 2$.

If α_1 is even integer the inside integral in (4.6) can be reduced to a finite sum using formula 2.271.6 of Gradshteyn and Ryzhik (1980). Moreover, if α_1 and α_2 are both odd integers, we can derive a finite sum presentation for $J(a, b, c)$ using formulas (2.11), (2.12) and relation 3.737.1 of Gradshteyn and Ryzhik (1980)

$$J(a, b, c) = \frac{1}{2} + \frac{2^{2-\alpha_1-\alpha_2}}{\Gamma\left(\frac{\alpha_1}{2}\right)\Gamma\left(\frac{\alpha_2}{2}\right)} \times \sum_{i_1=0}^{\frac{\alpha_1-1}{2}} \sum_{i_2=0}^{\frac{\alpha_2-1}{2}} \frac{(\alpha_1-1-i_1)!(\alpha_2-1-i_2)! 2^{i_1+i_2} (\sqrt{\sigma_1}a)^{i_1} (\sqrt{\sigma_2}b)^{i_2}}{i_1! i_2! \left(\frac{\alpha_1-1}{2}-i_1\right)! \left(\frac{\alpha_2-1}{2}-i_2\right)!} \psi_{i_1, i_2}, \quad (4.7)$$

with

$$\psi_{i_1, i_2} = \begin{cases} \frac{\Gamma(i_1+i_2)}{[(\sqrt{\sigma_1}a+\sqrt{\sigma_2}b)^2+c^2]^{\frac{i_1+i_2}{2}}} \sin \left[(i_1+i_2) \arctan \left(\frac{c}{\sqrt{\sigma_1}a+\sqrt{\sigma_2}b} \right) \right], & \text{if } i_1+i_2 > 0, \\ \arctan \left(\frac{c}{\sqrt{\sigma_1}a+\sqrt{\sigma_2}b} \right), & \text{if } i_1+i_2 = 0. \end{cases} \quad (4.8)$$

It is easy to see that in the case of $\mathbf{B} = \mathbf{0}$, (4.6) reduces to

$$J(a, c) = \frac{1}{2} + \frac{1}{B(0.5\alpha_1, 0.5)} \frac{c}{\sqrt{\sigma_1 a^2 + c^2}} {}_2F_1 \left(\frac{1}{2}, \frac{2-\alpha_1}{2}; \frac{3}{2}; \frac{c^2}{\sigma_1 a^2 + c^2} \right). \quad (4.9)$$

Note that the hypergeometric series in (4.9) terminates if $\alpha_1 \geq 2$ is an even integer.

If α_1 is an odd integer, formulae (4.7) and (4.8) take the forms

$$J(a, c) = \frac{1}{2} + \frac{2^{1-\alpha_1}}{\Gamma(0.5\alpha_1)} \sum_{i=0}^{\frac{\alpha_1-1}{2}} \frac{(\alpha_1-1-i)! 2^i (\sqrt{\sigma_1}a)^i}{i! \left(\frac{\alpha_1-1}{2}-i\right)!} \psi_i, \quad (4.10)$$

where

$$\psi_i = \begin{cases} \frac{\Gamma(i)}{(\sigma_1 a^2 + c^2)^{i/2}} \sin \left(i \arctan \left(\frac{c}{a\sqrt{\sigma_1}} \right) \right), & \text{if } i > 0, \\ \arctan \left(\frac{c}{a\sqrt{\sigma_1}} \right), & \text{if } i = 0. \end{cases} \quad (4.11)$$

Observe that formulae (4.9) – (4.11) give the finite sum presentation for $\hat{P}(\mathbf{A}, C)$ if α_1 is an integer, even or odd.

We want to draw reader's attention to the fact that $J(a, c)$ can also be evaluated by using probability tables for the Student's t -distribution. In fact, by the first equality in (2.14),

$$J(a, c) = \frac{1}{\alpha_1 B(0.5\alpha_1, 0.5)} \int_{-\infty}^{\frac{c\sqrt{\alpha_1}}{a\sqrt{\sigma_1}}} \left(1 + \frac{z^2}{\alpha_1}\right)^{-\frac{\alpha_1+1}{2}} dz = P\left(t_{\alpha_1} < \frac{c\sqrt{\alpha_1}}{a\sqrt{\sigma_1}}\right), \quad (4.12)$$

where t_α has the Student's t -distribution with α degrees of freedom.

EXAMPLE 3. The multivariate Cauchy distribution. The multivariate Cauchy distribution is a particular form of a multivariate T -distribution with $\alpha_j = 1$, $j = 1, 2$. Then it follows from (4.7) and (4.8) that $P(\mathbf{A}, \mathbf{B}, C)$ can be written as

$$J(a, b, c) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{c}{a\sqrt{\sigma_1} + b\sqrt{\sigma_2}}\right). \quad (4.13)$$

To obtain an expression for $J(a, c)$ set $b = 0$ in (4.13).

EXAMPLE 4. The multivariate normal distribution. Let us derive the MLE, the unbiased and the Bayes estimators of the probability $P(\mathbf{A}, \mathbf{B}, C)$.

The maximum likelihood estimator has the form (2.8) where $J(a, b, c)$ is calculated by (2.12). Taking into account that $f_j(\mathbf{z}'\mathbf{z})$ in this case are the k_j -variate standard normal densities, we obtain that $\xi^{(1)}$ and $\eta^{(1)}$ have univariate standard normal distributions, so that $\varphi_j(\omega) = e^{-\omega^2/2}$, $j = 1, 2$. Then, it follows from (2.12), formula 3.952.6 of Gradshteyn and Ryzhik (1980) and formulas 7.1.1 and 7.1.5 of Abramowitz and Stegun (1992) that

$$\begin{aligned} J(a, b, c) &= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)i!} \left(\frac{c}{\sqrt{2}\sqrt{a^2+b^2}}\right)^{2i+1} \\ &= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{c}{\sqrt{2}\sqrt{a^2+b^2}}\right), \end{aligned} \quad (4.14)$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i z^{2i+1}}{(2i+1)i!}.$$

Using the probability integral $\Phi(z) = (\sqrt{2\pi})^{-1} \int_{-\infty}^z e^{-\frac{t^2}{2}} dt$ we can provide another presentation of (4.14):

$$J(a, b, c) = \Phi\left(\frac{c}{\sqrt{a^2+b^2}}\right). \quad (4.15)$$

To derive $J(a, c)$ just let $b = 0$ in (4.14) or (4.15).

To find an unbiased estimator of $P(\mathbf{A}, \mathbf{B}, C)$ observe that the unbiased estimator of k_j -variate normal density with unknown parameters $\boldsymbol{\theta}_j$ and $\boldsymbol{\Sigma}_j$ based on N_j observations has the form (see Voinov and Nikulin (1996))

$$\check{p}(\mathbf{x}|\hat{\boldsymbol{\theta}}_j, \mathbf{S}_j) = \frac{\Gamma\left(\frac{N_j-1}{2}\right) N_j^{k_j/2} |\mathbf{S}_j|^{-1/2}}{\Gamma\left(\frac{N_j-k_j-1}{2}\right) (N_j-1)^{k_j/2} \pi^{k_j/2}} F_j \left[(\mathbf{x} - \hat{\boldsymbol{\theta}}_j)' \mathbf{S}_j^{-1} (\mathbf{x} - \hat{\boldsymbol{\theta}}_j) \right], j = 1, 2, \quad (4.16)$$

with $F_j(z) = [1 - (N_j - 1)^{-1} N_j z]^{-\frac{N_j - k_j - 3}{2}} I(0 \leq z \leq N_j^{-1} (N_j - 1))$, $j = 1, 2$. It is easy to see that (4.16) are the pdfs of k_j -variate Pearson-type II distributions (see (4.1)) with the parameters $\boldsymbol{\theta}_j = \hat{\boldsymbol{\theta}}_j$, $\boldsymbol{\Sigma}_j = N_j^{-1} (N_j - 1) \mathbf{S}_j$, $\alpha_j = 0.5(N_j - k_j - 1)$, $j = 1, 2$. Since in our case, $N_1 = m$, $N_2 = n$, the unbiased estimator of $P(\mathbf{A}, \mathbf{B}, C)$ can be obtained as $\tilde{P}(\mathbf{A}, \mathbf{B}, C) = J(\check{a}, \check{b}, \check{c})$ with $\alpha_1 = 0.5(m - k_1 - 1)$, $\alpha_2 = 0.5(n - k_2 - 1)$,

$$\check{a} = \sqrt{\frac{m-1}{m}} \mathbf{A}' \mathbf{S}_1 \mathbf{A}, \quad \check{b} = \sqrt{\frac{n-1}{n}} \mathbf{B}' \mathbf{S}_2 \mathbf{B}, \quad \check{c} = \mathbf{A}' \bar{\mathbf{X}} + \mathbf{B}' \bar{\mathbf{Y}} + C \quad (4.17)$$

and $J(a, b, c)$ is given by (4.2) or (4.3). Note that the combination of (4.2) and (4.17) results in an unbiased estimator which coincides with the one provided by Ivshin and Lumelskii (1994).

If $\mathbf{B} = 0$, then the unbiased estimator of $P(\mathbf{A}, C)$ is $\tilde{P}(\mathbf{A}, C) = J(\check{a}, \check{c})$ where, \check{a} and \check{c} are defined in (4.17) and, by (4.4),

$$J(a, c) = \begin{cases} \frac{1}{2} + \frac{c}{a} \left[B\left(\frac{1}{2}, \frac{m}{2} - 1\right) \right]^{-1} {}_2F_1\left(\frac{1}{2}, -\frac{(m-4)}{2}; \frac{3}{2}; \frac{c^2}{a^2}\right), & \text{if } |c| \leq a, \\ \frac{1}{2} + \frac{1}{2} \text{sign}(c), & \text{if } |c| > a. \end{cases} \quad (4.18)$$

The estimator (4.18) has been derived in Pensky (1982). Estimator of Gupta and Gupta (1990) coincides with (4.18) with the only difference that it is expressed via incomplete beta function instead of hypergeometric series. Note that the hypergeometric series in (4.18) terminates provided $m \geq 4$ is even.

Now, let us derive the Bayes estimator. Let us assume that all parameters $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are a priori independent with the flat priors for $\boldsymbol{\theta}_j$ and the inverse Wishart priors or Jeffreys's priors for $\boldsymbol{\Sigma}_j$, $j = 1, 2$.

In the first case, $g_j(\boldsymbol{\theta}_j, \boldsymbol{\Sigma}_j)$, $j = 1, 2$, is the inverse Wishart pdf with the parameters r_j and \mathbf{W}_j , that is

$$g_j(\boldsymbol{\theta}_j, \boldsymbol{\Sigma}_j) \propto |\mathbf{W}_j|^{\frac{r_j+k_j-1}{2}} |\boldsymbol{\Sigma}_j|^{-\frac{r_j+2k_j}{2}} \exp\left\{-0.5 \text{tr}(\mathbf{W}_j \boldsymbol{\Sigma}_j^{-1})\right\}, \quad j = 1, 2, \quad (4.19)$$

where \propto means "proportional to". Let us derive $q_1(\underline{\mathbf{X}})$ since the expression for $q_2(\underline{\mathbf{Y}})$ is very similar (see (3.4)).

Integrating the likelihood function $L_1(\boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1 | \underline{\mathbf{X}})$ in (3.2) with respect to $\boldsymbol{\theta}_1$ and taking into account that $\sum_{i=1}^m (\mathbf{X}_i - \bar{\mathbf{X}})' \boldsymbol{\Sigma}_1^{-1} (\mathbf{X}_i - \bar{\mathbf{X}}) = \text{tr}(\boldsymbol{\Sigma}_1^{-1} \mathbf{S}_1)$ we obtain

$$p(\underline{\mathbf{X}} | \boldsymbol{\Sigma}_1) \propto \boldsymbol{\Sigma}_1^{-\frac{m-1}{2}} \exp \left\{ -0.5 \text{tr}(\boldsymbol{\Sigma}_1^{-1} \mathbf{S}_1) \right\}.$$

It is easy to notice that the product $p(\underline{\mathbf{X}} | \boldsymbol{\Sigma}_1) g_1(\mathbf{W}_1 | \boldsymbol{\theta}_1, \boldsymbol{\Sigma}_1)$ is proportional to the pdf of the inverse Wishart distribution with parameters $(r_j + m - 1)$ and $(\mathbf{S}_1 + \mathbf{W}_1)$. Hence, integration over $\boldsymbol{\Sigma}_1$ yields $q_1(\underline{\mathbf{X}}) \propto |\mathbf{S}_1 + \mathbf{W}_1|^{-\frac{(r_1+k_1+m-2)}{2}} \cdot |\mathbf{W}_1|^{\frac{r_1+k_1-1}{2}}$. Considering vector $(\underline{\mathbf{X}}, \mathbf{x})$ instead of $\underline{\mathbf{X}}$ and rearranging sufficient statistics, we obtain

$$\frac{q(\mathbf{X}_1, \dots, \mathbf{X}_m, \mathbf{x})}{q(\mathbf{X}_1, \dots, \mathbf{X}_m)} \propto \frac{|\mathbf{S}_1 + \mathbf{W}_1|^{\frac{(r_1+k_1+m-2)}{2}}}{\left| \mathbf{S}_1 + \mathbf{W}_1 + (m+1)^{-1} m(\mathbf{x} - \bar{\mathbf{X}})(\mathbf{x} - \bar{\mathbf{X}})' \right|^{\frac{(r_1+k_1+m-1)}{2}}}$$

and $q(\mathbf{Y}_1, \dots, \mathbf{Y}_n, \mathbf{y})/q(\mathbf{Y}_1, \dots, \mathbf{Y}_n)$ has a similar expression.

For any square positive definite matrix \mathbf{A} denote by $\sqrt{\mathbf{A}}$ a square positive definite matrix such that $\sqrt{\mathbf{A}}' \sqrt{\mathbf{A}} = \mathbf{A}$. Introducing new variables

$$\boldsymbol{\mu} = \frac{\sqrt{m}}{\sqrt{m+1}} [(\sqrt{\mathbf{S}_1 + \mathbf{W}_1})^{-1}]' (\mathbf{x} - \bar{\mathbf{X}}), \quad \boldsymbol{\nu} = \frac{\sqrt{n}}{\sqrt{n+1}} [(\sqrt{\mathbf{S}_2 + \mathbf{W}_2})^{-1}]' (\mathbf{y} - \bar{\mathbf{Y}}),$$

denoting

$$\tilde{\mathbf{A}} = \frac{\sqrt{m+1}}{\sqrt{m}} \sqrt{\mathbf{S}_1 + \mathbf{W}_1} \mathbf{A}, \quad \tilde{\mathbf{B}} = \frac{\sqrt{n+1}}{\sqrt{n}} \sqrt{\mathbf{S}_2 + \mathbf{W}_2} \mathbf{B}, \quad \tilde{C} = \mathbf{A}' \bar{\mathbf{X}} + \mathbf{B}' \bar{\mathbf{Y}} + C,$$

and using (3.5), we arrive at

$$\tilde{P}(\mathbf{A}, \mathbf{B}, C) = C_{12} \int_{\Omega(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{C})} [\boldsymbol{\mu}' \boldsymbol{\mu} + 1]^{-\frac{r_1+k_1+m-1}{2}} [\boldsymbol{\nu}' \boldsymbol{\nu} + 1]^{-\frac{r_2+k_2+n-1}{2}} d\boldsymbol{\mu} d\boldsymbol{\nu}. \quad (4.20)$$

Here the set $\Omega(\mathbf{A}, \mathbf{B}, C)$ is defined in (3.6) and

$$C_{12} = \pi^{-\frac{(k_1+k_2)}{2}} \frac{\Gamma(0.5(r_1 + k_1 + m - 1))}{\Gamma(0.5(r_1 + m - 1))} \frac{\Gamma(0.5(r_2 + k_2 + n - 1))}{\Gamma(0.5(r_2 + n - 1))}.$$

Therefore, we obtain that $\tilde{P}(\mathbf{A}, \mathbf{B}, C) = J(\tilde{a}, \tilde{b}, \tilde{c})$ where

$$\tilde{a} = \sqrt{\frac{m+1}{m}} \mathbf{A}' (\mathbf{S}_1 + \mathbf{W}_1) \mathbf{A}, \quad \tilde{b} = \sqrt{\frac{n+1}{n}} \mathbf{B}' (\mathbf{S}_2 + \mathbf{W}_2) \mathbf{B}, \quad \tilde{c} = \mathbf{A}' \bar{\mathbf{X}} + \mathbf{B}' \bar{\mathbf{Y}} + C, \quad (4.21)$$

and $J(a, b, c)$ is given by formula (4.6) with

$$\sigma_1 = \sigma_2 = 1, \quad \alpha_1 = m + r_1 - 1, \quad \alpha_2 = n + r_2 - 1. \quad (4.22)$$

If we choose r_1 and r_2 so that $m + r_1 \geq 2$ and $n + r_2 \geq 2$ are even, then there exists a finite sum presentation of $J(a, b, c)$ given in (4.7) and (4.8).

If $\mathbf{B} = \mathbf{0}$, then $\tilde{P}(\mathbf{A}, C) = J(\tilde{a}, \tilde{c})$ where $J(a, c)$ is given by (4.9), or by (4.12), or by combination of (4.10) and (4.11), \tilde{a} , \tilde{c} , σ_1 , σ_2 , α_1 and α_2 are defined in (4.21) and (4.22), respectively.

Similarly, we can derive the Bayes estimator when Σ_j is distributed according to Jeffreys's prior: $g_j(\mathbf{W}_j | \boldsymbol{\theta}_j, \Sigma_j) \propto |\Sigma_j|^{-\frac{k_j+1}{2}}$, $j = 1, 2$ (see Yang and Berger (1994)). Note that Jeffreys's prior is a particular case of (4.19) with $\mathbf{W}_j = 0$ and $r_j = 1 - k_j$. Let $m \geq k_1$ and $n \geq k_2$. Then $\tilde{P}(\mathbf{A}, \mathbf{B}, C) = J(\tilde{a}^*, \tilde{b}^*, \tilde{c})$ where $\tilde{a}^* = \sqrt{(m+1)\mathbf{A}'\mathbf{S}_1\mathbf{A}/m}$, $\tilde{b}^* = \sqrt{(n+1)\mathbf{B}'\mathbf{S}_2\mathbf{B}/n}$, \tilde{c} is defined in (4.21) and $J(a, b, c)$ is given by formula (4.6) with $\sigma_1 = \sigma_2 = 1$, $\alpha_1 = m - k_1$, $\alpha_2 = n - k_2$. If $\mathbf{B} = \mathbf{0}$, the Bayes estimator based on Jeffreys's prior reduces to (see (4.9) and (4.12))

$$\tilde{P}(\mathbf{A}, C) = \frac{1}{2} + \frac{\tilde{c}}{\sqrt{(\tilde{a}^*)^2 + \tilde{c}^2}} \frac{1}{B(0.5(m - k_1), 0.5)} {}_2F_1\left(\frac{1}{2}, \frac{2+k_1-m}{2}; \frac{3}{2}; \frac{\tilde{c}^2}{(\tilde{a}^*)^2 + \tilde{c}^2}\right), \quad (4.23)$$

or

$$\tilde{P}(\mathbf{A}, C) = P\left(t_{m-k_1} < \frac{(\mathbf{A}'\bar{\mathbf{X}} + C)\sqrt{m(m-k_1)}}{\sqrt{(m+1)\mathbf{A}'\mathbf{S}_1\mathbf{A}}}\right). \quad (4.24)$$

Note that the Bayes estimator of Enis and Geisser (1971) is the particular case of (4.24) when $C = 0$. The estimators coincide in this case. The seeming discrepancy between the two estimators is due to the difference in notations: \mathbf{S}_1 in (4.23) is given by (2.3) while Enis and Geisser (1971) defined \mathbf{S}_1 as $\mathbf{S}_1 = (m-1)^{-1} \sum_{i=1}^m (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})'$.

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