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TESTING EXPONENTIALITY AGAINST LIKELIHOOD RATIO BEHAVIOUR USING KERNEL METHODS

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SUMMARY. In this work, testing exponentiality against monotone likelihood ratio is taken up as well as testing exponentiality in a goodness-of-fit setting. The procedures are based on the celebrated "kernel" density estimation of probability density functions and some of its derivatives. The limiting null and nonnull distributions of the test statistics are normal and the null variances are calculated exactly. Small samples null critical values are obtained via simulation. The efficacies of the test statistic used for testing against monotone likelihood are calculated for some common alternatives and are compared to some other procedures. The powers of test statistic used for the goodness-of-fit testing are obtained for some well-known alternatives via simulations and are shown to compare favorably against other more involved tests.

1. Introduction and Tests Formulation

Testing exponentiality against a specific life aging behavior or in a general goodness-of-fit setting has been extensively studied over the last five decades. Of the former, we see testing exponentiality versus positively aging classes of life distributions such as "increasing failure rate" (IFR), "increasing failure rate average" (IFRA), "new is better than used" (NBU), "new is better than used in convex ordering" (NBUC), "new is better than used of specified age" (NBU-t_0), "new is better than used in expectation" (NBUE),

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"harmonic new is better than used in expectation" (HNBUE), and "decreasing mean residual lifetime" (DMRL), are among the better known ones. For a recent review, see Lai (1994). While in goodness-of-fit setting, testing exponentiality has a vast literature, see reviews by Ascher (1990), D'Agostino and Stephens (1986), Doksum and Yandell (1984), and Spurrier (1984). In both problems, most of the procedures are based on using the empirical distribution function in estimating functionals that measure the departure from $H^{(0)}$: the distribution is exponential in favor of the alternative at hand. For example, in the goodness of fit setting, an L_2 norm of some measure of departure from H_0 is often the way to proceed. This procedure is highly unrobust and may be too complicated to pursue and/or may result in null distributions that are difficult to track. When a simplified approach is found, it is often of low power, efficiency, or both. Note here that the Pitman's asymptotic efficiency concepts are usually used to compare procedures of testing exponentiality against some restricted positive aging family while the asymptotic power is usually used to compare different approaches of exponentiality goodness-of-fit testing since, in this case, the efficiency concept does not work.

Recently, another approach began to be applied successfully into some general goodness-of-fit problems as well as model specification problems. This approach is based on the very popular "kernel method" of density estimation. In the problems it is used in, this approach provides more robust and powerful techniques than the empirical distribution methods. Also, it is often possible to establish that the null distribution is asymptotically normal, cf. Ahmad and Li (1997) and (1998), Bowman (1992), Fan (1994), Fan and Li (1995), Hart (1997), Jayasuriya (1996), Li (1996), and Wooldridge (1992), among recent works.

Thus, we are prompted to use this approach to test $H^{(0)}$: the distribution is exponential against a positive aging property not considered previously, which is the alternative $H^{(1)}$: The distribution has monotone likelihood ratio. We are also prompted to develop a goodness-of-fit testing of exponentiality based on a measure of departure from $H^{(0)}$, which uses the likelihood ratio property. The class of life distributions with monotone likelihood ratio is a subclass of the class of increasing failure rate and has not been tested to the best of our knowledge.

Definition 1.1. A nonnegative continuous random variable X with probability density function f is said to have monotone increasing (decreasing) likelihood ratio if $\ln f(x)$ is a concave (convex) function on its support.

Distributions satisfying the above definition are also known as "Polya

frequency functions of order 2" (PF_2) cf. Karlin (1968), Ross (1983), and Shaked and Shantikumar (1994) for mathematical properties of this class and possible applications. Here we are interested in testing $H^{(0)}$: f is exponential (having constant likelihood ratio) against $H^{(1)}$: f has monotone increasing likelihood ratio (PF_2) and not exponential. Assume throughout that f is twice differentiable. Thus, f is PF_2 if and only if $\frac{d^2}{dx^2} \ln f(x) \leq 0$ for all $x \geq 0$. That is, if and only if $f(x)f''(x) - (f'(x))^2 \leq 0$, for all $x \geq 0$. Thus, we may take the measure of departure of f from $H^{(0)}$ in favor of $H^{(1)}$ as follows:

$$\Delta_f = \frac{1}{\mu^3} \int_0^\infty \Big[(f'(x))^2 - f(x)f''(x)\Big]\, dx, \tag{1.1}$$

where $\mu = \int_0^\infty\ xdF(x)$, provided that the integral exists and is finite. Note that dividing by μ^3 is to make the measure (1.1) scale invariant, i.e., the null limiting distribution is parameter free. Note also that under $H^{(0)}, \Delta_f = 0$ and is positive under $H^{(1)}$. To estimate, Δ_f, we let $X_1, \ldots, X_n$ be a random sample from f and let k be a known symmetric, bounded probability density such that $\lim_{|u|\to\infty} |u|k(u) = 0$ and having zero mean and finite variance σ_k^2. Further, let $\{a_n\}$ be a sequence of positive real numbers such that $a_n \to 0$ and $na_n \to \infty$ as The "kernel" estimate of $f(x)$ is given by

$$\hat{f}_n(x) = \frac{1}{na_n} \sum_{i=1}^n k\left(\frac{x - X_i}{a_n}\right). \tag{1.2}$$

Thus, if k and f have r-th derivatives $k^{(r)}$ and $f^{(r)}$, respectively, then the "kernel" estimate of $f^{(r)}(x)$ is

$$\hat{f}_n^{(r)}(x) = \frac{1}{na_n^{r+1}} \sum_{i=1}^n k^{(r)}\left(\frac{x - X_i}{a_n}\right), \quad r \geq 0. \tag{1.3}$$

Hence, an estimate of Δ_f may be given as

$$\hat{\Delta}_{\hat{f}_n} = \left\{\int_0^\infty (\hat{f}'_n(x))^2 dx - \int_0^\infty \hat{f}''_n(x) dF_n(x)\right\} / \bar{X}_n^3, \tag{1.4}$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $F_n(x) = \frac{1}{n}\sum_{i=1}^n I(x - X_i)$ is the usual empirical distribution.

Note that $\hat{\Delta}_{fn}$ can be written in an asymptotically equivalent form as follows:

$$\begin{aligned}\hat{\Delta}_{\hat{f}_n} \quad = \quad & \Bigg\{\frac{1}{n(n-1)a_n^4} \sum\sum_{i\neq j} \int_{-\infty}^\infty k'\left(\frac{x - X_i}{a_n}\right) k'\left(\frac{x - X_j}{a_n}\right) dx \\ & - \frac{1}{n(n-1)a_n^3} \sum\sum_{i\neq j} k''\left(\frac{X_i - X_j}{a_n}\right)\Bigg\} / \bar{X}_n^3.\end{aligned} \tag{1.5}$$

For computational purposes, $\hat{\Delta}_{\hat{f}_n}$ is written as:

$$\hat{\Delta}_{\hat{f}_n} = \left[\frac{1}{n(n-1)a_n^3} \sum_{i \neq j}\sum \left\{ k' * k' \left(\frac{X_i - X_j}{a_n} \right) - k'' \left(\frac{X_i - X_j}{a_n} \right) \right\} \right] / \bar{X}_n^3, \tag{1.6}$$

where $k' * k'$ denote the convolution of k' with itself.

We reject $H^{(0)}$ if $\hat{\Delta}_{\hat{f}_n}$ is large. In Section 2, we establish the null and nonnull asymptotic distribution of $\hat{\Delta}_{\hat{f}_n}$ and give Monte-Carlo critical points of this test. We also compare this test with others in the literature in terms of Pitman's asymptotic relative efficiency and show that our procedure is highly competitive.

Next, the general problem of testing $H^{(0)}$ against the alternative $H^{(2)} : f$ is not exponential may be addressed using the measure of departure from $H^{(0)}$ given by:

$$\delta_f = \frac{1}{\mu^7} \int_{-\infty}^{\infty} \left[(f'(x))^2 - f''(x) f(x) \right]^2 dx. \tag{1.7}$$

The definition of δ_f in (1.7) is in the spirit of the huge class of L_2 norm-based functionals such as the Cramer-von Mises Class. Again, note that dividing by μ^7 is to make the measure scale invariant. Note also that under $H^{(0)}, \delta_f = 0$, and is positive under $H^{(2)}$. Hence, as above, one can estimate δ_j by plunging in the estimates of $f(x), f'(x)$ and $f''(x)$, but this results in a degenerate estimate which is $o_p(n^{-\frac{1}{2}})$. To avoid this degeneracy, we perturb some of the estimates using weights $C_{i,n}(\gamma)$ such that $\frac{1}{n}\sum_{i=1}^n C_{i,n}(\gamma) \to 1$ and $\frac{1}{n}\sum_{i=1}^n C_{i,n}^2(\gamma) \to C^2(\gamma) > 1$. An obvious choice of these weights, cf. Ahmad (1993), is to choose $C_{i,n}(\gamma) = 1 + \gamma$ if i is odd and $C_{i,n}(\gamma) = 1 - \gamma$ if i is even for some $0 < \gamma \leq 1$. Thus, we estimate δ_f by:

$$\begin{aligned} \hat{\delta}_{\hat{f}_n}(\gamma) &= \tfrac{1}{\bar{X}^7} \left\{ \int (\hat{f}_n'(x))^4 dx - 2 \int (\hat{f}_{n,\gamma}'(x))^2 \hat{f}''n, \gamma(x) dF_{n,\gamma}(x) \right. \\ &\quad \left. + \int (\hat{f}_n''(x))^2 \hat{f}_n(x) dF_n(x) \right\}, \end{aligned} \tag{1.8}$$

where $\hat{f}_{n,\gamma}^{(r)}(x) = \frac{1}{n a_n^{r+1}} \sum_{i=1}^n C_{i,n}(\gamma) k^{(r)} \left(\frac{x - X_i}{a_n} \right)$ and $F_{n,\gamma}(x) = \frac{1}{n}\sum_{i=1}^n C_{i,n}(\gamma)$ $I(x - X_i)$ for some $0 \leq \gamma < 1$, and we use the estimate (1.3) for $\hat{f}_n^{(r)}(x)$ and $F_n(x)$ is the usual empirical estimate of $F(x)$. Thus, an asymptotically equivalent form may be given as follows:

$$\hat{\delta}_{\hat{f}_n}(\gamma) = \frac{1}{\bar{X}^7} \left\{ \frac{1}{n(n-1)(n-2)(n-3)a_n^8} \sum\sum_{i_1 \neq i_2 \neq i_3 \neq i_4}\sum\sum \int k' \left(\frac{x - X_{i_1}}{a_n} \right) \right.$$

$$\times k'\left(\frac{x-X_{i_2}}{a_n}\right)k'\left(\frac{x-X_{i_3}}{a_n}\right)k'\left(\frac{x-X_{i_4}}{a_n}\right)dx$$
$$-\frac{2}{n(n-1)(n-2)(n-3)a_n^7}\sum\sum\sum_{i_1\neq i_2\neq i_3\neq i_4}\sum\prod_{j=1}^{4}C_{i_{j,n}}(\gamma)$$
$$\times k'\left(\frac{X_{i_1}-X_{i_2}}{a_n}\right)k'\left(\frac{X_{i_1}-X_{i_3}}{a_n}\right)k''\left(\frac{X_{i_1}-X_{i_4}}{a_n}\right)$$
$$+\frac{1}{n(n-1)(n-2)(n-3)a_n^7}\sum\sum\sum_{i_1\neq i_2\neq i_3\neq i_4}\sum$$
$$\times k''\left(\frac{X_{i_1}-X_{i_2}}{a_n}\right)k''\left(\frac{X_{i_1}-X_{i_3}}{a_n}\right)k\left(\frac{X_{i_1}-X_{i_4}}{a_n}\right)\Bigg\}. \tag{1.9}$$

While (1.8) and (1.9) are asymptotically equivalent, (1.9) is better for computational purposes. In Section 3, we obtain the null and nonnull distribution of $\hat{\delta}_{\hat{f}_n}(\gamma)$ above, give a table of Monte Carlo critical points, provide the asymptotic power of the test under some common alternatives, and show that it compares favorably to other procedures. We conclude in Section 4 by showing how the test procedures of the second and third sections extend to the two-sample case. In this case, if X and Y are two independent random variables with densities f and g, respectively, we say that X is larger than Y in the likelihood ratio sense if $\frac{f(x)}{g(x)}$ is nondecreasing in x. Thus, to test $H^{(0)}: f = g$ against $H^{(3)}$: X is larger than Y in the likelihood, we take the measure

$$\Delta_{f,g} = \int_{-\infty}^{\infty}[f'(x)g(x) - g'(x)f(x)]dx \tag{1.10}$$

and based on two independent samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ from f and g, respectively, and two sequences of reals a_m and b_n we estimate $\Delta_{f,g}$ by:

$$\hat{\Delta}_{\hat{f}_m,\hat{g}_n} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\frac{1}{a_m^2}k'\left(\frac{Y_j-X_i}{a_m}\right) - \frac{1}{b_n^2}k'\left(\frac{X_i-Y_j}{b_n}\right)\right\}. \tag{1.11}$$

We establish the asymptotic normality of $[\min\ (m,n)]^{\frac{1}{2}}\left[\hat{\Delta}_{f_{m},g_{n}} - \Delta_{f,g}\right]$, both under the null and nonnull alternatives. Finally, testing $H^{(0)}: f = g$ against the general alternative $H^{(4)}: f \neq g$, may be based on the measure of departure from $H^{(0)}$ given by

$$\delta_{f,g} = \int_{-\infty}^{\infty}[f'(x)g(x) - g'(x)f(x)]^2dx \tag{1.12}$$

Thus, a test statistic may be based on the estimate $\hat{\delta}_{\hat{f}_m,\hat{g}_n}(\gamma)$ given by:

$$\begin{aligned}\hat{\delta}_{\hat{f}_m,\hat{g}_n}(\gamma) &= \int_{-\infty}^{\infty} \hat{f}'^2_m(x)\hat{g}_n(x)dG_n(x) \\ &- \int_{-\infty}^{\infty} \hat{f}'_{m,\gamma}(x)\hat{g}'_{n,\gamma}(x)\hat{f}'_{m,\gamma}(x)dG_{n,\gamma}(x) \\ &- \int_{-\infty}^{\infty} \hat{f}'_{m,\gamma}(x)\hat{g}'_{n,\gamma}(x)\hat{g}_{n,\gamma}(x)dF_{n,\gamma}(x) \\ &+ \int_{-\infty}^{\infty} \hat{f}_m(x)\hat{g}'2_n(x)dF_m(x). \end{aligned} \tag{1.13}$$

We obtain the limiting behavior of $[\min\ (m,n)]^{\frac{1}{2}}\left[\hat{\delta}_{\hat{f}_{m,\hat{g}_n}}(\gamma) - \delta_{f,g}\right]$, both under the null and under the nonnull hypotheses. We also offer consistent estimates of the null variances of the two cases above so that tests may be carried out. Proofs in this case are only sketched.

For the rest of this investigation, we shall write a (and b) for a_n (and b_n), and, when the integration limits are the entire real line, they will not be given. For smooth flowing of the material, all proofs are deferred into the appendix.

2. Testing Exponentiality Against Monotone Likelihood Ratio

In this section, we want to test $H^{(0)}$: F is exponential (μ) against $H_1^{(1)}$: F has monotone increasing likelihood ratio (or is PF_2) and not exponential. Based on Δ_f in (1.1) and its estimate $\hat{\Delta}_{\hat{f}_n}$ of (1.4) or (1.5) or (1.6) we can state and prove the following:

Theorem 2.1. *If $na \to \infty$ and $na^4 \to 0$ as $n \to \infty$, if f has up to the fourth derivative where the second is bounded, if $\int f'(x)f^{(3)}(x)dx < \infty$ and $\int f(x)f^{(4)}(x)dx < \infty$ then $\sqrt{n}(\hat{\Delta}_{\hat{f}_n} - \Delta_f)$ is asymptotically normal with 0 mean and variance given in (A.11). Under $H^{(0)}$, the variance is $\frac{4}{3}$.*

In order to conduct the test, calculate $\sqrt{3n}\hat{\Delta}_{\hat{f}_n}/2$ and reject if this is larger than z_a, the standard normal variate. Clearly, the test is consistent and unbiased. To compare this procedure to other in the literature such as new better than used (NBU), we calculate the efficacy of our test for the following alternative:

1) Linear failure rate: $f_\theta(x) = (1+\theta x)e^{-x-\frac{\theta}{2}x^2}, x \geq 0, \theta \geq 0.$

2) Makeham: $f_\theta(x) = [1+\theta(1+e^x)]e^{-x-\theta(x+e^{-x}-1)}, x \geq 0, \theta \geq 0.$

Note that the Weibull distribution does not have monotone likelihood ratio. The efficacy is equal to $\left\{\left(\frac{\partial}{\partial\theta}\Delta_{f_\theta}\right) \; at \; \theta = \theta_0\right\}/\sigma_0$. For the above two distributions, the efficacies are, respectively, 0.433 and 0.866. For testing against NBU alternative, the test developed by Hollander and Proschan (1972) has efficacies for the above alternatives given by 0.613 and 0.258 giving relative efficacies of our test to there's of 0.706 and 3.357, respectively.

In calculating small sample critical values of the test given above and its power at $\alpha = .95$ for the above two alternatives (the linear failure rate and the Makeham), we use simulation for sample sizes 5(1)25 and based on 10,000 replications. In these simulations, we used the standard normal kernel and the normal scale rule to choose the bandwidth a cf. Jones and Wand (1995), p.60. Hence, $a = cn^{-1/\theta}$ where θ is an integer. We choose $\theta = 2, 3, 4$ and find that there are no appreciative differences between all three cases. However, we report the case $\theta = 2$ here since this gives the closest estimated size to the nominal levels, confirming the general trend that under smoothing gives better values, cf. Marron and Wand (1992). The simulated critical points and power calculations are given in the following tables.

Table 1. Critical values of $\hat{\Delta}_{f_n}$.

n	%90	%95	%98	%99
5	4.1874	8.7421	21.5875	38.7887
6	2.7450	5.2701	11.2167	20.1689
7	2.1867	4.0384	08.0266	13.5907
8	1.7399	3.0812	06.1376	09.3373
9	1.5082	2.4719	04.7058	06.9611
10	1.3363	2.1313	03.6316	04.9083
11	1.1866	1.8535	03.0713	04.3164
12	1.1023	1.6654	02.7787	03.7266
13	1.0185	1.5355	02.5804	03.5806
14	0.8784	1.2556	01.9248	02.6861
15	0.8591	1.2328	01.8575	02.5462
16	0.8116	1.1941	01.7667	02.3632
17	0.7720	1.0733	01.5620	01.9951
18	0.7575	1.0393	01.5552	02.0777
19	0.7079	0.9778	01.4068	01.8542
20	0.6856	0.9184	01.3209	01.6721
21	0.6802	0.9171	01.2144	01.5272
22	0.6472	0.8711	01.2358	01.5043
23	0.6310	0.8413	01.1929	01.5724
24	0.6004	0.8025	01.1036	01.3715
25	0.5884	0.7647	01.0406	01.2727

TABLE 2. POWER OF $\hat{\Delta}_{f_n}$ AT 95% LEVEL.

	Alternative		
Sample Size	Theta	LFR	Makeham
	2	0.206	0.132
10	3	0.387	0.262
	4	0.577	0.366
	2	0.496	0.263
20	3	0.828	0.555
	4	0.957	0.773
	2	0.635	0.339
25	3	0.918	0.675
	4	0.990	0.898

3. A Goodness of Fit Test for Exponentiality

In this section, we test $H^{(0)}$: F is exponential against $H^{(2)}$: F is not exponential based on δ_f in (1.6) and its estimate $\hat{\delta}^{(r)}_{\hat{f}_n}$ in (1.7) or (1.8). We state the following result.

THEOREM 3.1. *If the conditions of Theorem 2.1 hold, if*

$$\int_0^\infty f(x)(f'(x))^2 f^{(4)}(x)dx < \infty, \int_0^\infty f(x)f'(x)f''(x)f'''(x)dx < \infty,$$

$$\int f(x)(f''(x))^2 dx < \infty \quad \textit{and} \quad \int f^2(x)f''(x)f^{(4)}dx < \infty$$

then $\sqrt{n}[\hat{\delta}_{\hat{f}_n}(\gamma) - \delta_f]$ *is asymptotically normal with mean zero and variance given in (A.24). Under* $H_0, \delta_f = 0$ *and null variance is* $\frac{36}{7}(C^2(\gamma) - 1)$.

In order to carry this test, we must choose the kernel k, the bandwidth a, the sequence $\{C_{i,n}(\gamma)\}$, and the value of $\gamma \in (0, 1]$. The choices of k and a are done as in the previous section. A simple choice of $C_{i,n}(\gamma)$ is to take it $1 + \gamma$ for odd indexes and $1 - \gamma$ for even indexes in the sample. We also choose γ via simulation where the range $\gamma \in [0.4, 0.8]$ gives values that result in empirical size of the test fairly close to the nominal size for most usable levels of α such as $\alpha = .05, .025, .02, .01, .005, .001$. Another way to proceed is to calculate $Z_\gamma = \sqrt{n}\hat{\delta}_{\hat{f}_n}(\gamma)/\sigma_0(\gamma)$ as a function of γ and search for the values of γ such that $Z_\gamma > z_{\alpha/2}$ (the standard normal variate). Values of γ guaranteeing rejection should be in the range [.4,.8] other values should be dismissed. A computer program to calculate $\hat{\delta}_{\hat{f}_n}(\gamma)$ with k the standard normal, a chosen as in Section 2 $C_{i,n}(\gamma) = 1 + \gamma$ if i is odd and $1 - \gamma$ if i is even is available from the authors.

For small samples 5(1)15 and 20(5)25 using Monte Carlo methods, critical values of the above test are calculated for various of $\gamma \in [.4,.8]$. These values are reported in Table 3. To facilitate comparing the power of the above procedure to others in the literature, we selected the exact alternatives as in Ebrahimi, Habibullah and Soofi (1992) and for the same choices of the parameters. These alternatives are:

1. Weibull: $f(x, \lambda, \beta) = \beta\lambda^{\beta}x^{\beta-1}\ \exp\ [-(\lambda x)^{\beta}], \beta > 0, \lambda > 0, x \geq 0$.
   Choose $\lambda = \Gamma\left(\frac{\beta+1}{\beta}\right)$

2. Lognormal: $f(x, v, \sigma^2) = (x\sigma\sqrt{2\pi})^{-1}\ \exp\ [-(\ln\ x - v)^2/2\sigma^2], -\infty < v < \infty, \sigma > 0, \lambda > 0$.
   Choose $v = \frac{-\sigma^2}{2}$.

The power of the test at 90% and at 95% is given in Table 4. Observe that the power of our test is better than the test of Ebrahimi, et. al. (1992) and most others and also is not much affected by changing the value of γ.

TABLE 3. CRITICAL VALUES OF $\hat{\delta}_{\hat{f}_n}(\gamma)$

		$1-\alpha$			
n	γ	90	95	98	99
	.4	15.8054	90.8460	1164.9879	6578.5688
	.5	17.9685	94.1396	825.3184	2908.8470
5	.6	20.9876	138.1132	1759.4464	6172.5980
	.7	25.4904	170.6604	1318.5339	4701.8287
	.8	34.3602	208.6007	1893.3246	7363.3694
	.4	7.2294	33.7676	220.8241	1012.6272
	.5	7.2010	35.3940	242.5321	1094.3152
6	.6	9.7296	40.2773	294.8820	1174.6378
	.7	10.2165	50.6670	424.4192	1597.9458
	.8	14.3974	70.1793	374.3397	1442.3712
	.4	4.1717	15.9328	62.8356	183.3114
	.5	4.5212	17.6040	104.5054	365.4225
7	.6	5.8354	27.9373	146.7407	413.1641
	.7	6.4822	23.3617	116.3952	354.1686
	.8	6.3412	29.3435	155.8216	549.8752
	.4	2.3550	9.0426	37.3966	98.8601
	.5	2.6660	8.7240	41.7683	110.4875
8	.6	3.0382	10.9127	61.1726	181.7385
	.7	3.9063	12.5973	55.3669	149.5062
	.8	4.8335	17.6928	65.3543	212.6568

TABLE 3. CRITICAL VALUES OF $\hat{\delta}_{\hat{f}_n}(\gamma)$ (CONTD.)

		$1-\alpha$			
n	γ	90	95	98	99
	.4	1.5688	5.0811	21.1470	43.1866
	.5	1.9139	6.0690	29.0109	65.4148
9	.6	2.1135	6.8466	24.8511	79.7730
	.7	2.5527	9.0103	37.6589	91.0570
	.8	2.7892	8.6742	41.0742	102.0252
	.4	1.2469	4.1794	16.8669	31.4339
	.5	1.6784	5.5816	21.0134	54.2175
10	.6	1.6432	4.6504	19.9288	53.7478
	.7	1.7587	5.0371	18.1796	62.3695
	.8	2.4847	7.4898	25.7349	70.5956
	.4	0.9351	2.8861	8.8372	20.4767
	.5	1.0068	3.0266	9.0414	23.6144
11	.6	1.1545	3.4947	12.4128	25.8148
	.7	1.3856	3.8467	11.3654	21.2258
	.8	1.5387	4.3914	13.2836	29.2666
	.4	0.6957	1.7710	5.6741	14.9870
	.5	0.8941	2.4266	7.7168	14.5975
12	.6	0.9427	2.6928	7.9841	15.6513
	.7	1.1521	3.0817	9.1677	24.0933
	.8	1.3571	3.6774	13.6407	29.7818
	.4	0.5942	1.4690	4.5271	11.7877
	.5	0.7773	2.0466	5.7683	12.3147
13	.6	0.7861	1.9357	5.2504	12.7672
	.7	1.0652	2.5529	6.9326	14.3241
	.8	1.0674	2.7069	7.3761	16.7349
	.4	0.5653	1.3538	4.2132	8.5010
	.5	0.6645	1.5671	4.5386	8.6081
14	.6	0.6755	1.7038	4.8225	9.0659
	.7	0.7641	1.8432	4.8888	10.2578
	.8	0.9637	2.3917	5.9166	11.4515
	.4	0.5050	1.1455	2.9813	5.4441
	.5	0.4836	1.1667	2.7199	5.4367
15	.6	0.6366	1.5403	4.9892	10.4348
	.7	0.6752	1.5045	4.3655	9.0488
	.8	0.8032	1.6654	5.2820	9.4750
	.4	0.2655	0.5555	1.2827	2.2972
	.5	0.3145	0.6573	1.4385	2.3521
20	.6	0.3497	0.7052	1.7494	2.8377
	.7	0.4174	0.8640	1.9509	3.7536
	.8	0.4869	1.0242	2.2586	3.6393
	.4	0.2010	0.4078	0.8798	1.3160
	.5	0.2384	0.4372	0.8769	1.3586
25	.6	0.2551	0.5007	1.0187	1.6227
	.7	0.2934	0.5973	1.3057	2.2356
	.8	0.3662	0.6743	1.3404	2.2902

TABLE 4(a). POWER OF $\hat{\delta}_{\hat{f}_n}(\gamma)$ AGAINST THE WEIBULL

		β					
		2		3		4	
n	γ	%90	%95	%90	% 95	%90	%95
	.4	.3128	.1540	.6596	.3856	.8824	.6568
	.5	.3110	.1558	.6620	.3970	.8722	.6512
5	.6	.3196	.1456	.6430	.3528	.8772	.6234
	.7	.3100	.1452	.6402	.3548	.8728	.6080
	.8	.2860	.1288	.6198	.3398	.8632	.5992
	.4	.6336	.3644	.9812	.8694	1.0000	.9940
	.5	.5926	.3306	.9722	.8456	1.0000	.9922
10	.6	.6314	.3982	.9826	.8994	1.0000	.9966
	.7	.6656	.4286	.9830	.9020	1.0000	.9962
	.8	.6160	.3676	.9790	.8716	1.0000	.9936
	.4	.8384	.6322	.9998	.9932	1.0000	1.0000
	.5	.8748	.6798	.9448	.9954	1.0000	1.0000
15	.6	.8486	.6338	.9496	.9944	1.0000	1.0000
	.7	.8760	.6926	1.0000	.9472	1.0000	1.0000
	.8	.8718	.6984	1.0000	.9971	1.0000	1.0000
	.4	.9626	.8482	1.0000	1.0000	1.0000	1.0000
	.5	.9514	.8280	1.0000	1.0000	1.0000	1.0000
20	.6	.9632	.8650	1.0000	1.0000	1.0000	1.0000
	.7	.9562	.8314	1.0000	1.0000	1.0000	1.0000
	.8	.9632	.8386	1.0000	1.0000	1.0000	1.0000

4. The Two Sample Case

The methodology used in Sections 2 and 3 can be used to establish that $\hat{\Delta}_{\hat{f}_n,\hat{g}_n}$ and $\hat{\delta}_{\hat{f}_n,\hat{g}_n}(\gamma)$ are asymptotically normal. We briefly outline this. For the former, observe that writing

$$\Psi_n(X_1,Y_1)=\frac{1}{a^2}k'\left(\frac{X_1-Y_1}{a}\right)-\frac{1}{b^2}k'\left(\frac{Y_1-X_1}{b}\right). \tag{4.1}$$

Then, $\Psi_{1n}(X_1)=E[\Psi_n(X_1,Y_1)|X_1]$ can be approximated by:

$$\Psi_{1n}(X_1)=2g'(X_1)+\frac{g''(X_1)}{2}(a^2+b^2)+o_p(a^2+b^2). \tag{4.2}$$

Similarly, if $\Psi_{2n}(Y_1)=E[\Psi_n(X_1,Y_1)|Y_1]^2$, then we have

$$\Psi_{2n}(Y_1)=2f'(Y_1)+\frac{f''(Y_1)}{2}(a^2+b^2)+o_p(a^2+b^2). \tag{4.3}$$

TABLE 4(b). POWER OF $\hat{\delta}_{\hat{f}_n}(\gamma)$ AGAINST THE LOGNORMAL

		v					
		− 0.3		− 0.2		− 0.1	
n	γ	%90	%95	%90	% 95	%90	%95
	.4	0.1954	0.1090	0.2464	0.1316	0.4212	0.2350
	.5	0.1878	0.0988	0.2336	0.1320	0.4150	0.2432
5	.6	0.1946	0.0984	0.2472	0.1234	0.4174	0.2244
	.7	0.1926	0.0944	0.2352	0.1182	0.4210	0.2228
	.8	0.1912	0.0980	0.2282	0.1200	0.4032	0.2208
	.4	0.2474	0.1455	0.3748	0.2090	0.7150	0.5072
	.5	0.2366	0.1288	0.3218	0.1826	0.6758	0.4584
10	.6	0.2508	0.1566	0.3788	0.2358	0.7058	0.5168
	.7	0.2580	0.1596	0.3766	0.2394	0.7304	0.5434
	.8	0.2472	0.1436	0.3618	0.2089	0.6852	0.4864
	.4	0.2846	0.1836	0.4440	0.3030	0.8552	0.7310
	.5	0.3046	0.1910	0.4460	0.3360	0.8772	0.7466
15	.6	0.2780	0.1680	0.4650	0.2988	0.8896	0.7184
	.7	0.2946	0.1955	0.4754	0.3326	0.8724	0.7522
	.8	0.2974	0.2044	0.4744	0.3450	0.8750	0.7542
	.4	0.3338	0.2160	0.5632	0.4026	0.9416	0.8638
	.5	0.3238	0.2068	0.5442	0.3900	0.9498	0.8610
20	.6	0.3350	0.2218	0.5552	0.4058	0.9476	0.8728
	.7	0.3268	0.2136	0.5400	0.3808	0.9416	0.8610
	.8	0.3336	0.2206	0.5506	0.3940	0.9348	0.8638

Hence, $\sqrt{\min(m,n)}(\hat{\delta}_{\hat{f}_n,\hat{g}_n} - \Delta_{f,g})$ is asymptotically normal with mean 0 and variance $4\{V(f'(Y_1))+V(g'(X_1))\}$. Under H_0, the variance is $8V(f'(X_1))$ $= 8\{\int f'^2(x)f(x)dx - (\int f'(x)f(x)dx)^2\}$. To use this test, one needs to estimate the null variance. But this is immediate by pooling the two samples (let $m+n=N$) and proposing the estimate

$$\hat{\sigma}_0^2 = 8\left\{\int \hat{f}_N'^2(x)dF_N(x) - \left(\int \hat{f}_N'(x)dF_N(x)\right)^2\right\}. \tag{4.4}$$

Next, using the methodology of Section 3, it is not difficult to establish that $[\min(m,n)]^{\frac{1}{2}}\left[\hat{\delta}_{\hat{f}_n,\hat{g}_n}(\gamma) - \delta_{f,g}\right]$ is asymptotically normal with mean 0 and variance given by

$$\begin{aligned}\sigma_1^2 \;=\; & 4(C^2(\gamma)-1)\{V[f'(X_1)g'(X_1)+g'^2(X_1)f(X_1)] \\ & +V[f'(Y_1)g'(Y_1)+f'^2(Y_1)g(Y_1)]\}.\end{aligned} \tag{4.5}$$

Under $H_0, f=g$ and the null variance is:

$$\sigma_{01}^2 = 16(C^2(\gamma)-1)V[f'^2(X_1)f(X_1)], \tag{4.6}$$

which is estimated consistently by:

$$\hat{\sigma}_{01}^2=16(C^2(\gamma)-1)\left\{\int f_N'^4(x)f_N^2(x)dF_N(x)-\left(\int f_N'^2(x)f_N(x)dF_N(x)\right)^2\right\}. \tag{4.7}$$

As usual, the above two sample tests are only asymptotically distribution free. Power comparisons of these tests to others in the literature are possible to obtain, but, for the sake of brevity, will not be given here.

Appendix

PROOF OF THEOREM 2.1. Write $\Delta_f = \theta_f/\mu^3$, $\hat{\Delta}_{f_n} = \hat{\theta}_{\hat{f}_n}/\bar{X}^3$ and using the notation "$\stackrel{AD}{=}$" to mean two sides having same limiting distributions, we have

$$\begin{aligned}\hat{\Delta}_{f_n}-\Delta_f &= \left(\hat{\theta}_{f_n}-\theta_f\right)/\bar{X}^3+\theta_f\left(\frac{1}{\bar{X}^3}-\frac{1}{\mu^3}\right) \stackrel{AD}{=} \frac{1}{\mu^3}\left(\hat{\theta}_{f_n}-\theta_f\right)\\ &\quad -\frac{\Delta_f}{\mu^3}(\bar{X}^3-\mu^3) \stackrel{AD}{=} \frac{1}{\mu^3}\left(\hat{\theta}_{f_n}-\theta_f\right)-\frac{3\Delta_f}{\mu}(\bar{X}-\mu). \end{aligned} \tag{A.1}$$

Now, $\theta_f = \int_0^\infty (f'(x))^2dx - \int_0^\infty f''(x)dF(x) = I_1 - I_2$, say. Similarly, $\hat{\theta}_{f_n} = \int_0^\infty (\hat{f}_n'(x))^2dx - \int_0^\infty \hat{f}_n''(x)dF_n(x) = \hat{I}_{1n} - \hat{I}_{2n}$ say. First, we see that

$$\sqrt{n}(E\hat{\theta}_{f_n}-\theta_f) = \sqrt{n}(\hat{\theta}_{f_n}-E\hat{\theta}_{f_n}) + \sqrt{n}(E\hat{\theta}_{f_n}-\theta_f). \tag{A.2}$$

Now, we show that, $\sqrt{n}(E\hat{\theta}_{f_n}-\theta_f) = o(1)$. Note that by integration by parts:

$$\begin{aligned}E\hat{I}_1 &= a^{-4}\int\int\int k'\left(\frac{x-u}{a}\right)k'\left(\frac{x-v}{a}\right)f(u)f(v)dudvdx\\ &= a^{-2}\int\int\int k'(u)k'(v)f(x-au^*)f(x-av^*)du^*dv^*dx\\ &= \int\int\int k'(u^*)k'(v^*)f'(x-au^*)f'(x-av^*)du^*dv^*dx\\ &= \int\int\int k'(u^*)k'(v^*)\{f'(x)-au^*f''(x)+\frac{a^2}{2}(u^*)^2f'''(x)\}\\ &\quad \times\{f'(x)-av^*f''(x)+\frac{a^2}{2}(v^*)^2f'''(x)\}du^*dv^*dx\\ &= I_1+a^2\sigma_k^2\int f'(x)f'''(x)dx+o(a^2).\end{aligned} \tag{A.3}$$

Similarly,

$$\begin{aligned}
E\hat{I}_2 &= a^{-3}\int\int k'\left(\frac{x-y}{a}\right) f(x)f(y)dxdy \\
&= a^{-2}\int\int k''(u)f(x)f(x-au)dxdu \\
&= a^{-2}\int f(x)\{\int k''(u)f(x-au)du\}dx \\
&= \int f(x)\int k(u)f''(x-au)dudx \\
&= \int f(x)\int k(u)\{f''(x)-auf'''(x)+\frac{a^2}{2}u^2f^4(x)\}dudx \\
&= I_2+\frac{a^2}{2}\sigma_k^2\int f'(x)f^{(4)}(x)dx+o(a^2). \qquad \text{(A.4)}
\end{aligned}$$

From (2.3) and (2.4), we get that

$$\sqrt{n}(E\hat{\Delta}_{f_n}-\Delta_f)=\sqrt{na^4}\left\{\frac{\sigma_k^2}{2}\right\}\left\{2\int_0^\infty f'(x)f'''(x)dx-\int_0^\infty f(x)f^{(4)}(x)dx\right\}$$

$= o(1)$, by assumptions. Next, we look at $\sqrt{n}(\hat{\Delta}_{f_n}-E\hat{\Delta}_{f_n})$. Set

$$\begin{aligned}
\phi_n(X_1,X_2) &= a^{-4}\int k'\left(\frac{x-X_1}{a}\right)k'\left(\frac{x-X_2}{a}\right)dx-a^{-3}k''\left(\frac{X_1-X_2}{a}\right) \\
&= \Psi_{1n}(X_1,X_2)-\Psi_{2n}(X_1,X_2), \text{ say} \qquad \text{(A.5)}
\end{aligned}$$

Look at $E[\Psi_{in}(X_1,X_2)|X_1]$ and $E[\Psi_{in}(X_1,X_2)|X_2], i=1,2$. Now,

$$\begin{aligned}
E[\Psi_{1n}(X_1,X_2)|X_1] &= a^{-4}\int\int k'\left(\frac{x-X_1}{a}\right)k'\left(\frac{x-y}{a}\right)f(y)dydx \\
&= a^{-3}\int\int k'\left(\frac{x-X_1}{a}\right)k'(u)f(x-au)dudx \\
&= a^{-2}\int\int k'\left(\frac{x-X_1}{a}\right)k(u)f'(x-au)dudx \\
&= a^{-2}\int\int k'\left(\frac{x-X_1}{a}\right)k(u)\{f'(x)-auf''(x)+\frac{a^2u^2}{2}f^{(3)}(x)\}dudx \\
&= a^{-2}\int\int k'\left(\frac{x-X_1}{a}\right)f'(x)dx+\frac{\sigma_k^2}{2}\int k'\left(\frac{x-X_1}{a}\right)f^{(3)}(x)dx \\
&= L_{1n}+L_{2n}, \text{ say.} \qquad \text{(A.6)}
\end{aligned}$$

Now,

$$L_{1n} = a^{-1}\int k'(w)f'(au+X_1)dw=-\int k(w)f''(aw+X_1)dw$$

$$
\begin{aligned}
&= f''(X_1) - \frac{a^2\sigma_k^2}{2} f^{(4)}(X_1), \\
L_{2n} &= -\frac{a^2\sigma_k^2}{2} \int k(w) f^{(4)}(aw + aX_i)dw = \frac{a^2\sigma_k^2}{2} f^{(4)}(X_1).
\end{aligned}
$$

Hence,

$$
E[\Psi_{1n}(X_1, X_2)|X_1] = -f''(X_1) - a^2\sigma_k^2 f^{(4)}(X_1). \tag{A.7}
$$

Next,

$$
E[\Psi_{2n}(X_1, X_2)|X_1] = a^3 \int k'' \left(\frac{X_1 - x}{a}\right) f(x)dx = -f''(X_1) - \frac{a^2\sigma_k^2}{2} f^{(4)}(X_1). \tag{A.8}
$$

Hence,

$$
E[\Psi_n(X_1, X_2)|X_1] = -2f''(X_1) - \frac{3a^2\sigma_k^2}{2} f^{(4)}(X_1) = 2f''(X_1) + o_p\left(\frac{1}{\sqrt{n}}\right). \tag{A.9}
$$

Similarly, we can show that $E[\Psi_n(X_1, X_2)|X_2] = -2f''(X_2) + o_p\left(\frac{1}{\sqrt{n}}\right)$. Using the theory of U-statistics and differential statistics we have that

$$
\sqrt{n}\left(\hat{\Delta}_{\hat{f}_n} - E\hat{\Delta}_{\hat{f}_n}\right) = \frac{-4}{\sqrt{n}} \sum_{i=1}^{n} [f''(X_i) - Ef''(X_i)] - \frac{3\Delta_f\sqrt{n}}{\mu}(\bar{X} - \mu) + o_p(1). \tag{A.10}
$$

Therefore, $\sqrt{n}(\hat{\Delta}_{\hat{f}_n} - E\hat{\Delta}_{\hat{f}_n})$ is asymptotically normal with mean 0 and variance

$$
\sigma^2 = 16V(f''(X_1)) + \frac{9\Delta_f^2}{\mu^2} V(X_1) - \frac{24\Delta_f}{\mu} cov(f''(X_1), X_1). \tag{A.11}
$$

Under $H_0, f(x) = e^{-x}$ and the null variance follows by simple calculation. The theorem is now proved.

Proof of Theorem 3.1. Write $\delta_f = \eta_f/\mu^7$ and $\hat{\delta}_{\hat{f}_n}(\gamma) = \hat{\eta}_{\hat{f}_n}(\gamma)/\bar{X}^7$ we can see as in Theorem 2.1,

$$
\hat{\delta}_{f_n} - \delta_f \overset{AD}{=} \frac{1}{\mu^7}(\hat{\eta}_{f_n}(\gamma) - \eta_f) - \frac{\delta_f}{\mu^7}(\bar{X} - \mu). \tag{A.12}
$$

Now, $\eta_f = \int_0^\infty [(f'(x))^2 - f''(x)f(x))]^2 dx = I_1 - 2I_2 + I_3$, say. Similarly, we can write $\hat{\eta}_{\hat{f}_n}(\gamma) = \hat{I}_{1n} - 2\hat{I}_{2n}(\gamma) + \hat{I}_{3n}$, say. Now, $\sqrt{n}(\hat{\eta}_{\hat{f}_n}(\gamma) - \eta_f) = \sqrt{n}(\hat{\eta}_{\hat{f}_n}(\gamma) - E(\hat{\eta}_{\hat{f}_n}(\gamma)) + \sqrt{n}(E\hat{\eta}_{\hat{f}_n}(\gamma) - \eta_f)$. Analogous to Theorem 2.1, we can show that

$$
E\hat{I}_{1n} = I_1 + 2a^2\sigma_k^2 \int f'(x)f''(x)dx + o(a^2). \tag{A.13}
$$

Next,

$$\begin{aligned}
E\hat{I}_{2n}(\gamma) &= \int\int\int\int k'\left(\frac{x-y_1}{a}\right) k'\left(\frac{x-y_2}{a}\right) k'\left(\frac{x-y_2}{a}\right) \\
&\qquad f(x)f(y_1)f(y_2)f(y_3)dxdy_1dy_2dy_3 \\
&= \int\int\int\int f(x)k'(u)k'(v)k'(w)f(x-au)f(x-av)f(x-aw)dxdudvdw \\
&= \iiiint f(x)k'(u)k'(v)k'(w)\{f'(x) - auf''(x) + \frac{a^2u^2}{2}f'''(x)\} \\
&\qquad .\{f'(x) - avf''(x) + \frac{a^2y^2}{2}f'''(x)\} \\
&\qquad .\{f''(x) - auf'''(x) + 2\frac{a^2u^2}{2}f^{(4)}(x)\}dxdudvdw \\
&= I_2 + \frac{a^2\sigma_k^2}{2}\int f(x)\{(f'(x))^2 f^{(4)}(x)+2f'(x)f''(x)f'''(x)\}dx + o(a^2).
\end{aligned} \tag{A.14}$$

Finally,

$$E\hat{I}_{3n} = I_3 + \frac{a^2\sigma_k^2}{2}\int f(x)\{(f''(x))^3 + 2f(x)f''(x)f^{(4)}\}dx + o(a^2). \tag{A.15}$$

Hence, under our conditions $\sqrt{n}(E\hat{\eta}_{\hat{f}_n}(\gamma)-\eta_f) = o(1)$. Next, we approximate $\sqrt{n}(\hat{\eta}_{\hat{f}_n}(\gamma) - E\hat{\eta}_{\hat{f}_n}(\gamma))$. Set

$$\begin{aligned}
&\phi(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}) \\
&= \frac{1}{a^8}\int k'\left(\frac{x-X_{i_1}}{a}\right) k'\left(\frac{x-X_{i_2}}{a}\right) k'\left(\frac{x-X_{i_3}}{a}\right) k'\left(\frac{x-X_{i_4}}{a}\right) dx \\
&\qquad -\frac{2}{a^7}\prod_{j=1}^{4} C_{i_j,n}(\gamma) k'\left(\frac{X_{i_1}-X_{i_2}}{a}\right) k'\left(\frac{X_{i_1}-X_{i_3}}{a}\right) k''\left(\frac{X_{i_3}-X_{i_4}}{a}\right) \\
&\qquad +\frac{1}{a^7}k''\left(\frac{X_{i_1}-X_{i_2}}{a}\right) k''\left(\frac{X_{i_1}-X_{i_3}}{a}\right) k\left(\frac{X_{i_3}-X_{i_4}}{a}\right).
\end{aligned} \tag{A.16}$$

Let us evaluate $E[\phi_n(X_{i_1}, \ldots, X_{i_4})|X_{i_1}] = \phi_{11n}(X_{i_1})+\phi_{13n}(X_{i_1})-2\phi_{12n}(X_{i_1})$, say, where

$$\begin{aligned}
\phi_{11n}(X_{i_1}) &= \frac{1}{a^8}\int\int\int\int k'\left(\frac{x-X_{i_1}}{a}\right) k'\left(\frac{x-u}{a}\right) k'\left(\frac{x-v}{a}\right) k'\left(\frac{x-w}{a}\right) \\
&\qquad f(u)f(v)f(w)dudvdwdx \\
&= \frac{1}{a^2}\int k'\left(\frac{x-X_{i_j}}{a}\right)(f'(x))^3 dx = -3(f'(X_{i_1}))^2 f(X_{i_1}),
\end{aligned} \tag{A.17}$$

$$\begin{aligned}
\phi_{12n}(X_{i_1}) &= \frac{1}{a^7}\prod_{j=1}^{4} C_{i_{j,n}}(\gamma)\int\int\int k'\left(\frac{X_{i_1}-x}{a}\right)k'\left(\frac{X_{i_1}-y}{a}\right) \\
&\qquad k''\left(\frac{X_{i_1}-z}{a}\right)f(x)f(y)f(z)dxdydz \\
&= \frac{1}{a^4}\prod_{j=1}^{4} C_{i_{j,n}}(\gamma)\int\int\int k'(u)k'(v)k'(w) \\
&\qquad f(X_{i_1}-au)f(X_{i_1}-av)f(X_{i_1}-aw)dudvdw \\
&= \prod_{j=1}^{4} C_{i_{j,n}}(\gamma)(f'(X_{i_1}))^2 f''(X_{i_1}), \qquad \text{(A.18)}
\end{aligned}$$

and similarly,

$$\phi_{13n}(X_{i_1}) = (f''(X_{i_1}))^2 f(X_{i_1}), \qquad \text{(A.19)}$$

Then

$$\begin{aligned}
E[\phi_n(X_{i_1},\ldots,X_{i_4})|X_{i_1}] &= -3(f'(X_{i_1}))^2 f''(X_{i_1}) - 2C_{i_{1,n}}(\gamma)(f''(X_{i_1}))^2 f''(X_{i_1}) \\
&\quad +(f''X_{i_1}))^2 f(X_{i_1}) + o_p\left(\frac{1}{\sqrt{n}}\right). \qquad \text{(A.20)}
\end{aligned}$$

In a similar fashion, we can show that:

$$\begin{aligned}
&E[\phi_n(X_{i_1},\ldots,X_{i_4})|X_{i_2}] \\
&= -(f'(X_{i_2}))^2 f''(X_{i_2}) + 2C_{i_{2,n}}(\gamma) \\
&\quad \times[f'^2(X_{i_2}))^2 f''(X_{i_2}) + f(X_{i_2})(f''(X_{i_2}))^2 + f(X_{i_2})f'(X_{i_2})f'''(X_{i_2})] \\
&\quad +2f(X_{i_2})(f''(X_{i_2}))^2 + 2f'(X_{i_2}))^2 f''(X_{i_2}) \\
&\quad +f^2(X_{i_2})f^{(4)}(X_{i_2}) + 4f(X_{i_2})f'(X_{i_2})f''(X_{i_2}) + o_p\left(\frac{1}{\sqrt{n}}\right). \qquad \text{(A.21)}
\end{aligned}$$

Next, $E[\phi_n(X_{i_1},\ldots,X_{i_4})|X_{i_3}]$ has the same form as (3.10). Finally,

$$\begin{aligned}
&E[\phi_n(X_{i_1},\ldots,X_{i_4})|X_{i_4}] \\
&= -3(f'(X_{i_4}))^2 f''(X_{i_4}) - 2C_{i_{4,n}}(\gamma) \\
&\quad \times[5(f'(X_i))^2 f''(X_{i_4}) + 2f(X_{i_4})(f''(X_{i_4}))^2 + 2f(X_{i_4})f'(X_{i_4})f''(X_{i_4})] \\
&\quad +f(X_{i_4})(f''(X_{i_4}))^2 + o_p\left(\frac{1}{\sqrt{n}}\right). \qquad \text{(A.22)}
\end{aligned}$$

Hence,

$$\begin{aligned}\sqrt{n}(\hat{\delta}_{fn}(\gamma) - E\hat{\delta}_{f_n}(\gamma)) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{-12(f'(X_i))^2 f''(X_i) \\ &\quad -8C_{i,n}(\gamma)(f'(X_i))^2 f''(X_i) + 6f(X_i)(f''(X_i))^2 \\ &\quad +4(f'(X_i))^2 f''(X_i) + 2f^2(X_i)f^{(4)}(X_i) \\ &\quad +8f(X_i)f'(X_i)f'''(X_0)\} - \frac{7\delta_f}{\mu}\sqrt{n}(\bar{X} - \mu) \\ &\quad +o_p\left(\frac{1}{\sqrt{n}}\right), \end{aligned} \tag{A.23}$$

which is asymptotically normal with mean 0 and variance

$$\begin{aligned} &\text{Var}\ \Bigg\{\frac{1}{n}\sum_{i=1}^{n}[-12f'^2(X_i) - 8C_{i,n}(\gamma)f'^2(X_i)f''(X_i) \\ &+6f'^2(X_i)f''^2(X_i) + 4f'^2(X_i)f''(X_i) \\ &+2f^2(X_i)f^{(4)}(X_i) + 8f(X_i)f'(X_i)f''(X_i) - \frac{7\delta_f}{\mu}(X_i - \mu)]\Bigg\} \end{aligned} \tag{A.24}$$

Under $H_0, f(x) = e^{-x}$ and the variance reduces to

$$\lim_n Var\left[\frac{1}{n}\sum_{i=1}^{n}\{(1 - C_{i,n}(\gamma))8e^{-3x_i}\}\right] = (C^2(\gamma) - 1).\frac{36}{7}.$$

The theorem is thus proved.

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