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LOCAL DEPENDENCE FUNCTIONS FOR THE ELLIPTICALLY SYMMETRIC DISTRIBUTIONS

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SUMMARY. Bairamov *et al.* (2000) introduced a measure of local dependence which is a localized version of the Galton correlation coefficient. In this paper we: 1) provide a motivation for this new measure; 2) derive the exact form of the measure for the class of elliptically symmetric distributions; and, 3) provide an application of the new measure to the theory for ordering of bivariate dependence (this involves defining three new concepts for ordering of bivariate dependence and deriving certain asymptotic expansions). We illustrate the results for five examples of elliptically symmetric distributions.

1. Introduction

The study of the local dependence between two continuous random variables X and Y has attracted some interest in the past decade. In the early 1990s Bjerve and Doksum (1993), Doksum *et al.* (1994) and Blyth (1994a, b) constructed a "correlation curve". It is a generalization of the Galton correlation coefficient and is given by the formula

$$H(x) = \frac{Var(X)\left\{E(Y \mid X = x)\right\}^2}{Var(X)\left\{E(Y \mid X = x)\right\}^2 + Var(Y \mid X = x)}. \qquad (1)$$

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Evidently, H measures "the strength of the association between X and Y locally at $X = x$". Several useful properties of this correlation curve are given in the papers cited above. One property is that $H(x) \equiv \rho$ for all x when (X, Y) has the bivariate normal distribution with correlation coefficient ρ.

Jones (1996) observes the following downside of the correlation curve measure:

> correlation curve is shorthand for the clumsier ... Because $H(x)$ does not treat X and Y on an equal footing, but needs Y to be a response and X a predictor variable, the correlation curve is really a regression concept. A local dependence function, or surface, a function of both x and y, should measure the strength and direction of the association locally, treating variables symmetrically.

Holland and Wang (1987) and Wang (1993) introduced the only form yet known for a local dependence function, given by the formula:

$$H(x, y) = \frac{\partial^2 \log f(x, y)}{\partial x \partial y}, \tag{2}$$

where f denotes the joint probability density function (pdf) of (X, Y). The authors motivate this as the "limit of the local cross-ratio defined for adjacent cell probabilities, formed by a two-dimensional rectangular grid, when the length and width of the rectangles shrink to zero." This function shares many of the properties of the correlation curve, including that $H(x, y) \equiv \rho$ for all x and for all y when (X, Y) has the bivariate normal distribution. Two recent papers by Jones (1996, 1998) have studied this function further. Jones (1996) provides an alternative motivation of $H(x, y)$ based on localization of the Galton correlation coefficient while Jones (1998) identifies all distributions other than bivariate normal which have the property that $H(x, y) \equiv c$. The result of the latter paper is that the class of all distributions involving an exponential family conditional distribution with its canonical parameter being a linear function have constant local dependence. Jones (1998) concludes that this characterization is not very helpful.

A further weakness of (2) stems from its "margin-free" property: $H(x, y)$ is a function only of the conditional distribution of Y given X, or of X given Y. Hence, H appeals to all bivariate distributions with specified conditionals. But it is a known fact that conditionally specified joint distributions are extremely constrained. The book by Arnold *et al.* (1999) gives a thorough and up-to-date treatment of conditional specified models: on page 147 the authors quote:

> Two kinds of unexpected results have been encountered. On the one hand, the class of conditionally specified joint densities might be surprisingly constrained. For example, exponential conditionals models turn out to be always negatively associated. In some sense, then, specifying the form of the conditional distributions is more restrictive than we might have envisioned. On the other hand, the conditionally specified families often include unexpected models with anomalous properties ...

The authors go on to say:

> Instead of being given ... families of conditional densities we might be given ... regression function.

Hence, if we specify the local dependence function in terms of the conditional moments, $E(X \mid Y = y)$ and $E(Y \mid X = x)$, we can expect H to have the appeal of a much wider class of joint distributions. This is the main motive for the work in Bairamov *et al.* (2000); however, the paper does not specify this. The new expression they propose is:

$$H(x, y) = \frac{E\left[\{X - E(X \mid Y = y)\}\{Y - E(Y \mid X = x)\}\right]}{\sqrt{E\{X - E(X \mid Y = y)\}^2}\sqrt{E\{Y - E(Y \mid X = x)\}^2}}. \qquad (3)$$

This new form also provides a much more radical generalization of the Galton correlation coefficient in that, instead of $E(X)$ and $E(Y)$, $E(X \mid Y = y)$ and $E(Y \mid X = x)$ are utilized; thus, (3) characterizes the effect of X on Y (and vice versa) conditionally on (X, Y) being at the point (x, y). To the best of our knowledge this is the first modification of the correlation coefficient along these lines and also (3) is the first form for a local dependence function that applies to both continuous and discrete distributions (however, in this paper, we shall only consider the continuous case). Figure 1 provides a graphical comparison of the surface of (3) with those of (1) and (2) when (X, Y) has the bivariate normal distribution with correlation coefficient $\rho = 0.5$.

It is obvious that the new form (3) provides the only localized version. It is hard to see how (3) connects with (1) and (2) above, but two alternative representations of (3) are:

$$H(x, y) = \frac{Cov(X, Y) + \xi_X(y)\xi_Y(x)}{\sqrt{Var(X) + \xi_X^2(y)}\sqrt{Var(Y) + \xi_Y^2(x)}},$$

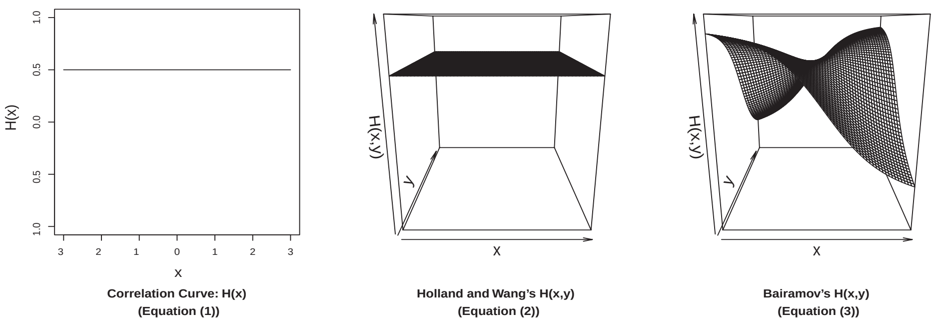

Figure 1. Various dependence functions for bivariate normal with $\rho = 0.5$.

where $\xi_X(y) = E(X \mid Y{=}y) - E(X)$ and $\xi_Y(x) = E(Y \mid X{=}x) - E(Y)$ are the deviations between the unconditional and conditional means, and

$$H(x,y) = \frac{\rho + \phi_X(y)\phi_Y(x)}{\sqrt{1+\phi_X^2(y)}\sqrt{1+\phi_Y^2(x)}},$$

where $\phi_X(y) = \xi_X(y)/\sigma_X$ and $\phi_Y(x) = \xi_Y(x)/\sigma_Y$ are the standardized deviations. One important application of (3) is in the theory for ordering of bivariate dependence. In Section 2 of this paper, we shall introduce three new concepts for ordering of bivariate dependence and derive asymptotic expansions (based on (3)) useful for checking the ordering. No doubt that (3) would also attract numerous applications in engineering, behavioral and medical sciences, where there might be interest in pinpointing variable values with strong or weak association.

2. Properties of the Local Dependence Function

The new local dependence function (3) retains many of the properties of (1) and (2). Here are some useful properties that it has – the proofs of which can be found in Bairamov *et al.* (2000):

- $|H(x,y)| \leq 1$ for all $(x,y) \in N_{X,Y}$, where $N_{X,Y}$ denotes the support of (X,Y);
- If $a \leq \phi_X(y) \leq b$ and $a \leq \phi_Y(x) \leq b$ (possibly including $a = -\infty$ and $b = \infty$) for all $(x,y) \in N_{X,Y}$ then $H(x,y)$ attains its maximums at $(\phi_Y^{-1}(a), \phi_X^{-1}(a))$ and $(\phi_Y^{-1}(b), \phi_X^{-1}(b))$ and attains its minimums at $(\phi_Y^{-1}(a), \phi_X^{-1}(b))$ and $(\phi_Y^{-1}(b), \phi_X^{-1}(a))$;
- The point (x^*, y^*) satisfying $\phi_X(y^*) = \phi_Y(x^*) = 0$ is a saddle point of $H(x,y)$ and at this point $H(x^*, y^*) = \rho$;

- If X and Y are totally independent then $H(x,y)=0$ for all $(x,y)\in N_{X,Y}$;
- If $H(x,y)=0$ for all $(x,y)\in N_{X,Y}$ then – assuming that $E(Y\mid X=x)$ and $E(X\mid Y=y)$ are differentiable functions of x and y, respectively – we must have either $E(Y\mid X=x)$ or $E(X\mid Y=y)$ or both constants;
- If $|H(x,y)|=1$ for some $(x,y)\in N_{X,Y}$ then $\rho\neq 0$;
- If $|\rho|=1$ then $|H(X,Y)|=1$ almost surely;
- If $Y=aX+b$ almost surely then $H(X,Y)=\operatorname{sign}(a)$ almost surely;
- If $\tilde{X}=aX+b$ and $\tilde{Y}=cY+d$ then $H_{X,Y}(x,y)=H_{\tilde{X},\tilde{Y}}(\tilde{x},\tilde{y})$, where $\tilde{x}=ax+b$ and $\tilde{y}=cy+d$.

Two further properties of (3) are (proofs are straightforward):

- If the distribution is axially symmetric, i.e. either (X,Y) and $(-X,Y)$ or (X,Y) and $(X,-Y)$ follow the same distribution, then $H(x,y)=0$ for all $(x,y)\in N_{X,Y}$;
- If X and Y are totally dependent then

$$H(x,y)=\frac{E\left(X^2\right)-(x+y)E(X)+xy}{\sqrt{E\left(X^2\right)-2xE(X)+x^2}\sqrt{E\left(X^2\right)-2yE(X)+y^2}}$$

and, up to the first order of approximation,

$$H(x,y)=1-\frac{(x-y)^2}{2E\left(X^2\right)}+\cdots.$$

However, $E(H(X,Y))=1$ holds exactly.

Bairamov *et al.* (2000) calculated expressions for $H(x,y)$ and $E(H(X,Y))$ for a number of distributions, including the bivariate normal distribution. Their calculations of $E(H(X,Y))$ are "rather cumbersome" and they provide only numerical approximations. In Section 3 we shall calculate the exact form of $H(x,y)$ for the entire class of elliptically symmetric distributions, which include the bivariate normal distribution as a particular case. In Section 4 we shall provide an application of the new local dependence and derive asymptotic expansions for $H(x,y)$, $E(H(X,Y))$ and related measures. In Section 5 we shall illustrate the results for five examples of elliptically symmetric distributions.

3. H for Elliptically Symmetric Distributions

The joint pdf of an elliptically symmetric bivariate distribution is of the form

$$f(x,y) = |\mathbf{\Sigma}|^{-\frac{1}{2}} g\left((x-\mu_x, y-\mu_y)\, \mathbf{\Sigma}^{-1} (x-\mu_x, y-\mu_y)^T \right), \tag{4}$$

where $g(\cdot)$ is a scale function referred to as the *density generator* and $\mathbf{\Sigma}$ is a 2×2 constant matrix of the structure $\mathbf{A}\mathbf{A}^T$. Without loss of generality, we shall assume that $\mu_x = \mu_y = 0$ and

$$\mathbf{\Sigma} = \mathbf{R} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \qquad -1 < \rho < 1.$$

Then, the joint pdf (4) becomes

$$f(x,y) = \frac{1}{\sqrt{1-\rho^2}} g\left(\frac{x^2+y^2-2\rho xy}{1-\rho^2} \right). \tag{5}$$

Setting

$$g(x) = \frac{\exp(-x/2)}{2\pi}, \tag{6}$$

we arrive at the familiar bivariate normal distribution. By Theorems 2.17 and 2.18 in Fang *et al.* (1990), the unconditional and conditional moments associated with (5) are

$$E(X) = E(Y) = 0,$$

$$Var(X) = Var(Y) = D_1/2, \qquad Cov(X,Y) = \rho D_1/2$$

and

$$E(X \mid Y=y) = \rho y, \qquad E(Y \mid X=x) = \rho x,$$

where

$$D_t = \pi \int_0^\infty x^t g(x) dx, \qquad t \geq 0. \tag{7}$$

Thus, the local dependence function associated with (5) becomes

$$H(x,y) = \frac{\rho + \frac{2\rho^2 xy}{D_1}}{\sqrt{1+\frac{2\rho^2 x^2}{D_1}}\sqrt{1+\frac{2\rho^2 y^2}{D_1}}}. \tag{8}$$

Note that $(0,0)$ is the saddle point and that $H(0,0) = \rho$. At the boundaries: $H(\infty,\infty) = H(-\infty,-\infty) = 1$ and $H(\infty,-\infty) = H(-\infty,\infty) = -1$. Along the diagonal $x=y$, $H(x,x)$ is an increasing function of $\mid x \mid$ while $H(x,-x)$

is a decreasing function of $\mid x \mid$. Thus, maximal local dependence is attained at (x, x) for x large.

4. Application to Ordering of Bivariate Dependence

Here we define three new concepts for ordering of bivariate dependence – each based on local dependence – and derive asymptotic expansions for the ordering to be checked. We stress that these new concepts and the basic idea for the use of asymptotic expansions can be applied to any bivariate distribution – either continuous or discrete (however, in this paper, discussion is limited to the class of elliptically symmetric distributions).

Consider two joint pdfs f_1 and f_2 of the form (4). Let H^1 and H^2 denote the respective local dependence functions. We say that f_1 is more positively dependent than f_2 with respect to:

- local dependence at (x, y) if

$$H^1(x, y) \geq H^2(x, y) \tag{9}$$

  for all $\mid \rho \mid < 1$;

- expected local dependence if

$$\begin{aligned} E\left(H^1(X,Y)\right) &= \int\int H^1(x,y) f_1(x,y) dy dx \\ &\geq \int\int H^2(x,y) f_2(x,y) dy dx = E\left(H^2(X,Y)\right) \end{aligned} \tag{10}$$

  for all $\mid \rho \mid < 1$;

- total local dependence if

$$\begin{aligned} PA\left(H^1(X,Y)\right) &= \int\int H^1(x,y) dy dx \\ &\geq \int\int H^2(x,y) dy dx = PA\left(H^2(X,Y)\right) \end{aligned} \tag{11}$$

  (PA stands for 'primitive average') for all $\mid \rho \mid < 1$; this is meaningful only when the support of (X, Y) is finite.

Each of these three conditions treats H^i, $i = 1, 2$ as functions of ρ; hence, it is useful to have a representation of H in terms of ρ.

Transform

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{R}^{1/2} \begin{pmatrix} U \\ V \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{1+\rho}+\sqrt{1-\rho} & \sqrt{1+\rho}-\sqrt{1-\rho} \\ \sqrt{1+\rho}-\sqrt{1-\rho} & \sqrt{1+\rho}+\sqrt{1-\rho} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}. \tag{12}$$

It follows by Theorem 2.16 in Fang *et al.* (1990) that (U, V) has the elliptically symmetric distribution, (4), with $\mu_x = \mu_y = 0$ and $\mathbf{\Sigma} = \mathbf{I}$, the identity matrix. Under the transformation,

$$H(X, Y) = A(U, V) / \sqrt{B(U, V)}, \tag{13}$$

where

$$A(U, V) = \rho + \frac{2UV}{D_1}\rho^2 + \frac{U^2 + V^2}{D_1}\rho^3$$

and

$$\begin{aligned} B(U, V) &= 1 + \frac{2\left(U^2 + V^2\right)}{D_1}\rho^2 + \frac{4UV}{D_1}\rho^3 + \frac{4U^2V^2}{D_1^2}\rho^4 \\ &\quad + \frac{4UV\left(U^2 + V^2\right)}{D_1^2}\rho^5 + \frac{\left(U^2 + V^2\right)^2}{D_1^2}\rho^6. \end{aligned}$$

Expanding (13) around the saddle point $(U, V) = (0, 0)$, we obtain the representation:

$$H(X, Y) = \sum_{j=1}^{\infty} H_j(U, V)\rho^j. \tag{14}$$

For the purposes of checking the ordering and since $|\rho| < 1$ we can approximate

$$\begin{aligned} H(X, Y) &\approx \sum_{j=1}^{N} H_j(U, V)\rho^j, \\ E\left(H(X, Y)\right) &\approx \sum_{j=1}^{N} E\left(H_j(U, V)\right)\rho^j, \\ PA\left(H(X, Y)\right) &\approx \sum_{j=1}^{N} PA\left(H_j(U, V)\right)\rho^j \end{aligned}$$

for sufficiently large N. We can then compute each of these approximations separately for the two joint pdfs f_1 and f_2 and plot them (as functions of ρ) on the same scale to check whether any of the conditions (9), (10) or (11) is satisfied. Hence, this provides an empirical means for checking the ordering of bivariate dependence.

Computation of the three approximations above requires explicit algebraic expressions for $H_j(U, V)$, $E(H_j(U, V))$ and $PA(H_j(U, V))$, $j = 1, \ldots, N$. Expressions for the coefficients $H_j(U, V)$ can be readily obtained by use of computer algebraic manipulation packages (such as Maple). We find that the first ten coefficients are:

$$H_1(U,V) = 1, \tag{15}$$

$$H_2(U,V) = \frac{2UV}{D_1}, \tag{16}$$

$$H_3(U,V) = 0, \tag{17}$$

$$H_4(U,V) = -\frac{2U^3V}{D_1^2} - \frac{2UV^3}{D_1^2} - \frac{2UV}{D_1}, \tag{18}$$

$$H_5(U,V) = -\frac{5U^2V^2}{D_1^2} + \frac{U^4}{2D_1^2} + \frac{V^4}{2D_1^2}, \tag{19}$$

$$H_6(U,V) = \frac{2U^3V}{D_1^2} + \frac{2UV^3}{D_1^2} + \frac{2U^3V^3}{D_1^3} + \frac{3U^5V}{D_1^3} + \frac{3UV^5}{D_1^3}, \tag{20}$$

$$H_7(U,V) = -\frac{U^4}{2D_1^2} + \frac{5U^2V^2}{D_1^2} - \frac{V^4}{2D_1^2} + \frac{9U^4V^2}{D_1^3} + \frac{9U^2V^4}{D_1^3} - \frac{U^6}{D_1^3} - \frac{V^6}{D_1^3}, \tag{21}$$

$$\begin{aligned} H_8(U,V) = & -\frac{6U^5V}{D_1^3} + \frac{12U^3V^3}{D_1^3} - \frac{6UV^5}{D_1^3} - \frac{3U^5V^3}{D_1^4} \\ & -\frac{3U^3V^5}{D_1^4} - \frac{5U^7V}{D_1^4} - \frac{5UV^7}{D_1^4}, \end{aligned} \tag{22}$$

$$\begin{aligned} H_9(U,V) = & -\frac{39U^6V^2}{2D_1^4} - \frac{51U^4V^4}{4D_1^4} - \frac{9U^2V^4}{D_1^3} + \frac{15U^8}{8D_1^4} \\ & -\frac{39U^2V^6}{2D_1^4} + \frac{U^6}{D_1^3} - \frac{9U^4V^2}{D_1^3} + \frac{15U^8}{8D_1^4} + \frac{V^6}{D_1^3}, \end{aligned} \tag{23}$$

$$\begin{aligned} H_{10}(U,V) = & \frac{35UV^9}{4D_1^5} + \frac{35U^9V}{4D_1^5} + \frac{5U^3V^7}{D_1^5} + \frac{5U^7V^3}{D_1^5} \\ & +\frac{9U^5V^5}{2D_1^5} - \frac{30U^3V^5}{D_1^4} - \frac{30U^5V^3}{D_1^4} \\ & +\frac{14UV^7}{D_1^4} - \frac{14U^3V^3}{D_1^3} + \frac{14U^7V}{D_1^4} + \frac{3UV^5}{D_1^3} + \frac{3U^5V}{D_1^3}. \end{aligned} \tag{24}$$

Using the facts that

$$E\left(U^{2m}U^{2n}\right) = \frac{1}{\pi} D_{m+n} B\left(\frac{1}{2}+m, \frac{1}{2}+n\right), \qquad m \geq 1, \quad n \geq 1$$

(Theorem 2.8, Fang *et al.*, 1990), where

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \alpha > 0, \quad \beta > 0,$$

and

$$E\left(U^{2m-1}U^{2n-1}\right) = 0, \qquad m \geq 1, \quad n \geq 1,$$

and equations (15)-(24), we can derive algebraic expressions also for the expectations $E(H_j(U,V))$. We find that the first ten expectations are:

$$\begin{aligned}
E\left(H_1(U,V)\right) &= 1, \\
E\left(H_j(U,V)\right) &= 0, \qquad j = 2,3,4,6,8,10, \\
E\left(H_5(U,V)\right) &= \frac{3D_4}{8D_1^2}, \\
E\left(H_7(U,V)\right) &= -\frac{3D_4}{8D_1^2} - \frac{7D_6}{8D_1^3}, \\
E\left(H_9(U,V)\right) &= \frac{7D_6}{8D_1^3} + \frac{987D_8}{512D_1^4}.
\end{aligned}$$

Thus, using equation (14) we have

$$E\left(H(X,Y)\right) = \rho + \alpha_1\rho^5 + \alpha_2\rho^7 + \alpha_3\rho^9 + o\left(\rho^{10}\right), \tag{25}$$

where

$$\alpha_1 = \frac{3D_4}{8D_1^2}, \qquad \alpha_2 = -\frac{3D_4}{8D_1^2} - \frac{7D_6}{8D_1^3}, \qquad \alpha_3 = \frac{7D_6}{8D_1^3} + \frac{987D_8}{512D_1^4}.$$

If (U,V) has a finite support, say $\{(U,V) : U^2 + V^2 < d\}$ with $d < \infty$, then we can derive algebraic expressions also for $PA(H_j(U,V))$. Integrating equations (15)-(24), we get

$$\begin{aligned}
PA\left(H_1(U,V)\right) &= 4d, \\
PA\left(H_j(U,V)\right) &= 0, \qquad j = 2,3,4,6,8,10, \\
PA\left(H_5(U,V)\right) &= -\frac{64d^3}{45D_1^2}, \\
PA\left(H_7(U,V)\right) &= \frac{64d^3}{45D_1^2} + \frac{128d^4}{35D_1^3}, \\
PA\left(H_9(U,V)\right) &= -\frac{128d^4}{35D_1^3} - \frac{4096d^5}{525D_1^4}.
\end{aligned}$$

Thus, using equation (14) we have

$$PA\left(H(X,Y)\right) = 4d\rho + \beta_1\rho^5 + \beta_2\rho^7 + \beta_3\rho^9 + o\left(\rho^{10}\right), \tag{26}$$

where

$$\beta_1 = -\frac{64d^3}{45D_1^2}, \qquad \beta_2 = \frac{64d^3}{45D_1^2} + \frac{128d^4}{35D_1^3}, \qquad \beta_3 = -\frac{128d^4}{35D_1^3} - \frac{4096d^5}{525D_1^4}.$$

5. Illustration of Results

Here we illustrate the results in Sections 3 and 4 for the following five examples of elliptically symmetric distributions: the symmetric Kotz type distributions (Section 5.1), the symmetric Pearson type VII distributions (Section 5.2), the symmetric Pearson type II distributions (Section 5.3), the symmetric Bessel distributions (Section 5.4), and the symmetric logistic distributions (Section 5.5).

5.1 *Symmetric Kotz type distribution.* This distribution is a generalization of the bivariate normal distribution. Take the generator g as

$$g(x) = \frac{h(x)}{\pi \int_0^\infty h(y)dy}, \tag{27}$$

where

$$h(y) = y^{N-1} \exp\left(-ry^s\right), \qquad y > 0, \quad r > 0, \quad s > 0, \quad N > 0$$

(compare with (6)). Since

$$\int_0^\infty h(y)dy = \int_0^\infty y^{N-1} \exp\left(-ry^s\right) dy = \frac{\Gamma\left(N/s\right)}{sr^{N/s}},$$

the joint pdf, (5), becomes

$$f(x,y) = \frac{sr^{N/s}\left(x^2+y^2-2\rho xy\right)^{N-1}}{\pi\Gamma\left(N/s\right)\left(1-\rho^2\right)^{N-1/2}} \exp\left\{-r\left(\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)^s\right\}. \tag{28}$$

When $s = 1$, this is the original Kotz distribution introduced in Kotz (1975). When $N = 1$, $s = 1$ and $r = 1/2$, (28) reduces to a bivariate normal density.

In (7),

$$D_t = \frac{\int_0^\infty y^t h(y)dy}{\int_0^\infty h(y)dy} = \frac{\int_0^\infty y^{t+N-1} \exp\left(-ry^s\right) dy}{\int_0^\infty y^{N-1} \exp\left(-ry^s\right) dy} = \frac{\Gamma\left(\frac{N+t}{s}\right)}{r^{t/s}\Gamma\left(\frac{N}{s}\right)}.$$

Thus, by (8), the local dependence function associated with (28) becomes

$$H(x,y) = \frac{\rho + \frac{2r^{1/s}\Gamma(N/s)\rho^2 xy}{\Gamma((N+1)/s)}}{\sqrt{1+\frac{2r^{1/s}\Gamma(N/s)\rho^2 x^2}{\Gamma((N+1)/s)}}\sqrt{1+\frac{2r^{1/s}\Gamma(N/s)\rho^2 y^2}{\Gamma((N+1)/s)}}},$$

and its expectation admits the expansion, (25), with

$$\begin{aligned}
\alpha_1 &= \frac{3\Gamma(N/s)\Gamma((N+4)/s)}{8r^{2/s}\Gamma^2((N+1)/s)}, \\
\alpha_2 &= -\frac{3\Gamma(N/s)\Gamma((N+4)/s)}{8r^{2/s}\Gamma^2((N+1)/s)} - \frac{7\Gamma^2(N/s)\Gamma((N+6)/s)}{8r^{3/s}\Gamma^3((N+1)/s)} \\
\text{and } \alpha_3 &= \frac{7\Gamma^2(N/s)\Gamma((N+6)/s)}{8r^{3/s}\Gamma^3((N+1)/s)} + \frac{987\Gamma^3(N/s)\Gamma((N+8)/s)}{512r^{4/s}\Gamma^4((N+1)/s)}.
\end{aligned}$$

5.2 *Symmetric Pearson type VII distribution.* For this distribution the generator g takes the form (27) with

$$h(y) = \left(1 + \frac{y}{m}\right)^{-N}, \qquad y > 0, \quad N > 1, \quad m > 0.$$

Since

$$\int_0^\infty h(y)dy = \frac{m}{N-1},$$

the joint pdf, (5), becomes

$$f(x,y) = \frac{N-1}{\pi m\sqrt{1-\rho^2}} \left(1 + \frac{x^2 + y^2 - 2\rho xy}{m(1-\rho^2)}\right)^{-N}. \tag{29}$$

The bivariate t-distribution and the bivariate Cauchy distribution are special cases of this for $N = (m+2)/2$ and $m = 1$, $N = 3/2$, respectively.

In (7),

$$D_t = \frac{\int_0^\infty y^t (1 + y/m)^{-N} dy}{\int_0^\infty (1 + y/m)^{-N} dy}$$

exists only if $t < N - 1$. By equation (3.194.3) in Gradshteyn and Ryzhik (2000), the numerator is:

$$\int_0^\infty y^t (1 + y/m)^{-N} dy = m^{t+1} B(t+1, N-t-1).$$

Thus,

$$D_t = \frac{m^{t+1}B(t+1, N-t-1)}{mB(1, N-1)} = \frac{m^t t!}{(N-t-1)\cdots(N-2)}, \quad N > t+1.$$

Hence, the local dependence function associated with (29) becomes

$$H(x,y) = \frac{\rho + \frac{2(N-2)\rho^2 xy}{m}}{\sqrt{1 + \frac{2(N-2)\rho^2 x^2}{m}}\sqrt{1 + \frac{2(N-2)\rho^2 y^2}{m}}},$$

provided that $N > 2$. Furthermore, $E(H(X,Y))$ admits the expansion, (25), with

$$\begin{aligned}\alpha_1 &= \frac{9m^2(N-2)}{(N-5)\cdots(N-3)},\\ \alpha_2 &= -\frac{9m^2(N-2)}{(N-5)\cdots(N-3)} - \frac{630m^3(N-2)^2}{(N-7)\cdots(N-3)}\\ \text{and } \alpha_3 &= \frac{630m^3(N-2)^2}{(N-7)\cdots(N-3)} + \frac{310905m^4(N-2)^3}{4(N-9)\cdots(N-3)},\end{aligned}$$

provided that $N \geq 10$.

5.3 *Symmetric Pearson type II distribution.* For this distribution the generator g takes the form (27) with

$$h(y) = (1-y)^N, \qquad 0 < y < 1, \quad N > -1.$$

Since

$$\int_0^1 h(y)dy = \frac{1}{N+1},$$

the joint pdf, (5), becomes

$$f(x,y) = \frac{N+1}{\pi\sqrt{1-\rho^2}}\left(1 - \frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)^N. \tag{30}$$

In (7),

$$D_t = (N+1)\int_0^1 y^t(1-y)^N dy = (N+1)B(N+1,t+1),$$

where the final step follows by equation (3.191.3) in Gradshteyn and Ryzhik (2000). Thus, the local dependence function associated with (30) becomes

$$H(x,y) = \frac{\rho + 2(N+2)\rho^2 xy}{\sqrt{1+2(N+2)\rho^2x^2}\sqrt{1+2(N+2)\rho^2y^2}},$$

and its expectation admits the expansion, (25), with

$$\begin{aligned}\alpha_1 &= \frac{9(N+2)}{(N+3)\cdots(N+5)},\\ \alpha_2 &= -\frac{9(N+2)\left(N^2+83N+182\right)}{(N+3)\cdots(N+7)}\\ \text{and } \alpha_3 &= \frac{315(N+2)^2\left(8N^2+1123N+2550\right)}{4(N+3)\cdots(N+9)}.\end{aligned}$$

Since (30) is defined on a finite support, we can also see that $PA(H(X,Y))$ admits the expansion, (26), with

$$\beta_1=-\frac{64(N+2)^2}{45},\beta_2=\frac{64(N+2)^2(18N+43)}{315},\beta_3=-\frac{128(N+2)^3(32N+79)}{525}.$$

5.4 *Symmetric Bessel distribution.* For this distribution the generator g takes the form (27) with

$$h(y) = \left(\sqrt{y}/b\right)^a K_a\left(\sqrt{y}/b\right), \qquad y>0, \quad a>-1, \quad b>0,$$

where $K_a(\cdot)$ denotes the modified Bessel function of the third kind, i.e.

$$K_a(z) = \frac{\pi}{2}\frac{I_{-a}(z)-I_a(z)}{\sin(a\pi)}, \qquad \mid \arg(z) \mid < \pi, \quad a=0,\pm 1,\pm 2,\ldots,$$

where

$$I_a(z) = \sum_{j=0}^{\infty}\frac{1}{j!\Gamma(j+a+1)}\left(\frac{z}{2}\right)^{a+2j}, \qquad \mid z \mid < \infty \quad \mid \arg(z) \mid < \pi.$$

By equation (6.561.16) in Gradshteyn and Ryzhik (2000),

$$\int_0^\infty h(y)dy = 2b^{-a}\int_0^\infty y^{a+1}K_a\left(y/b\right)dy = 2^{a+1}b^2\Gamma(a+1);$$

thus, the joint pdf, (5), becomes

$$f(x,y) = \frac{\left(x^2+y^2-2\rho xy\right)^{a/2}}{\pi 2^{a+1}\Gamma(a+1)b^{a+2}\left(1-\rho^2\right)^{(a+1)/2}}K_a\left(\frac{\sqrt{x^2+y^2-2\rho xy}}{b\sqrt{1-\rho^2}}\right). \quad (31)$$

The special case of (31) for $a=0$ and $b=\sigma/\sqrt{2}$ is the symmetric Laplace distribution.

In (7),

$$D_t = \frac{\int_0^\infty y^{a+2t+1}K_a\left(y/b\right)dy}{\int_0^\infty y^{a+1}K_a\left(y/b\right)dy} = \frac{2^{a+2t}b^{a+2t+2}\Gamma(a+t+1)\Gamma(t+1)}{2^a b^{a+2}\Gamma(a+1)} = \frac{4^t b^{2t}\Gamma(a+t+1)\Gamma(t+1)}{\Gamma(a+1)},$$

where the penultimate step follows by equation (6.561.16) in Gradshteyn and Ryzhik (2000). Thus, the local dependence function associated with (31) becomes

$$H(x,y) = \frac{\rho+\frac{\rho^2 xy}{2(a+1)b^2}}{\sqrt{1+\frac{\rho^2x^2}{2(a+1)b^2}}\sqrt{1+\frac{\rho^2y^2}{2(a+1)b^2}}},$$

and its expectation admits the expansion, (25), with

$$
\begin{aligned}
\alpha_1 &= \frac{144(a+2)\cdots(a+4)b^4}{(a+1)}, \\
\alpha_2 &= -\frac{144(a+2)\cdots(a+4)b^4}{(a+1)} - \frac{40320(a+2)\cdots(a+6)b^6}{(a+1)^2} \\
\text{and } \alpha_3 &= \frac{40320(a+2)\cdots(a+6)b^6}{(a+1)^2} + \frac{19897920(a+2)\cdots(a+8)b^8}{(a+1)^3}.
\end{aligned}
$$

5.5 *Symmetric logistic distribution.* For this distribution the generator g takes the form (27) with

$$h(y) = \frac{\exp(-y)}{\{1+\exp(-y)\}^2}, \qquad y > 0.$$

By substituting $z = \exp(-y)$,

$$\int_0^\infty h(y)dy = \int_0^1 \frac{dz}{(1+z)^2} = \frac{1}{2};$$

thus, the joint pdf, (5), becomes

$$
\begin{aligned}
f(x,y) &= \frac{2}{\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right) \\
&\quad \left\{1+\exp\left(-\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)\right\}^{-2}.
\end{aligned}
\tag{32}
$$

By equation (3.423.3) in Gradshteyn and Ryzhik (2000),

$$\int_0^\infty y^t h(y)dy = \Gamma(t+1)\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^t},$$

which is a convergent series for every $t \geq 1$. Thus,

$$D_t = 2\Gamma(t+1)\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^t}.$$

By equation (5.1.2.3) in Fikhtengol'ts (1949),

$$D_1 = 2\log 2, \qquad D_4 = \frac{7\pi^4}{15}, \qquad D_6 = \frac{31\pi^6}{21}, \qquad D_8 = \frac{127\pi^8}{15}.$$

Hence, the local dependence function associated with (32) becomes

$$H(x,y) = \frac{\rho + \frac{\rho^2 xy}{\log 2}}{\sqrt{1+\frac{\rho^2 x^2}{\log 2}}\sqrt{1+\frac{\rho^2 y^2}{\log 2}}},$$

and its expectation admits the expansion, (25), with

$$\begin{aligned} \alpha_1 &= \frac{7\pi^4}{160\,(\log 2)^2}, \quad \alpha_2 = -\frac{7\pi^4}{160\,(\log 2)^2} - \frac{31\pi^6}{192\,(\log 2)^3}, \\ \alpha_3 &= \frac{31\pi^6}{192\,(\log 2)^3} + \frac{41783\pi^8}{40960\,(\log 2)^4}. \end{aligned}$$

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