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One-Dimensional Stochastic Differential Equations with Singular and Degenerate Coefficients

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Abstract

We show the existence of strong solutions and pathwise uniqueness for two types of one-dimensional stochastic differential equations. The first type allows singular drifts:

$$X_t = X_0 + \int_0^t a(X_t) dW_t + \int_{\mathbf{R}} L_t^w(X)\,\mu(dw) \quad \text{for } t \geq 0,$$

where W is a one-dimensional Brownian motion, a is a function that is bounded between two positive constants, μ is a finite measure with $|\mu(\{w\})| \leq 1$, and L^w is the local time at w for the semimartingale X. The second type is the equation

$$dX_t = (X_t)^\alpha dW_t + dL_t,$$

where L is a continuous non-decreasing process that increases only when X is at 0, $\alpha \in (0, \frac{1}{2})$, and $X_t \geq 0$ for all t. Although this second equation does not have a unique solution, it does have a unique solution if one restricts attention to those solutions that spend zero time at 0.

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1 Introduction

In this paper, we are concerned with one-dimensional stochastic differential equations (SDEs). We will study two classes of SDEs, one with singular drifts and the other with degenerate diffusion coefficients.

In Sections 2-4, we establish both the existence of a strong solution and the pathwise uniqueness for the SDE

$$X_t = x_0 + \int_0^t a(X_s) dW_s + \int \widehat{L}_t^w(X)\mu(dw) \quad \text{for } t \geq 0, \tag{1.1}$$

where W is a one-dimensional Brownian motion, and a is a function on $\mathbf{R}$ that is bounded above and below by positive constants and such that there is a strictly increasing function f on $\mathbf{R}$ such that

$$|a(x) - a(y)|^2 \le |f(x) - f(y)|, \qquad x, y \in \mathbf{R}.$$

Here $\widehat{L}^w(X)$ is the symmetric local time of X at level w and μ is a finite measure with $|\mu\{(w\})| \le 1$ for every $w \in \mathbf{R}$. A comparison principle is also established for solutions of (1.1). These results extend early work on strong solutions and pathwise uniqueness for one-dimensional SDEs, such as that in Yamada-Watanabe (1971), Nakao (1972), Le Gall (1983), Le Gall (1984), and Barlow-Perkins (1984) as well as later work by Zhang (1994) and by Bass-Chen (2001).

In Section 5, we study strong solutions and pathwise uniqueness for the SDE

$$X_t = x_0 + \int_0^t (X_s)^\alpha dW_s + \widehat{L}_t^0(X), \qquad t \ge 0, \tag{1.2}$$

where $\alpha \in (0, 1/2)$, $x_0 \ge 0$, and $X = \{X_t, \, t \ge 0\}$ is a continuous process taking non-negative values and $\widehat{L}^0(X)$ is the symmetric local time of X at 0. Even weak uniqueness does not hold for the SDE (1.2), as there is a weak solution that spends zero time at 0, another where the process sticks at 0 upon first hitting 0, and a whole host of intermediate solutions. We show, however, that the SDE (1.2) does have a strong solution for which the amount of time spent at 0 has Lebesgue measure 0 and that pathwise uniqueness holds among the class of solutions to (1.2) that spend zero time at 0. This is a first step toward the study of strong solutions and pathwise uniqueness for solutions to SDEs such as

$$X_t = x_0 + \int_0^t a(X_s) dW_s + \widehat{L}_t^0(X) \quad \text{ for } t \ge 0,$$

and

$$X_t = x_0 + \int_0^t a(X_s) dW_s \quad \text{ for } t \ge 0 \tag{1.3}$$

with the added condition that X spends zero time in the set $\{x \in \mathbf{R} : a(x) = 0\}$, where a is a function on $\mathbf{R}$ such that $a(x)^{-2}$ is locally integrable. The comparison principle for SDE (1.1) established in Section 4 plays a key role in our approach. Engelbert and Schmidt (1985) have discussed weak uniqueness and classifications of weak solutions to the SDE (1.3).

Throughout this paper, solutions to each SDE under consideration will have continuous sample paths. For the definitions of weak solution, strong solution, pathwise uniqueness, etc.; see Revuz and Yor (1991), for example.

2 SDEs with Singular Drifts

THEOREM 2.1. *Suppose a is a measurable function on $\mathbf{R}$ that is bounded above and below by positive constants and suppose that there is a strictly increasing function f on $\mathbf{R}$ such that*

$$|a(x)-a(y)|^2 \le |f(x)-f(y)|, \qquad x,y\in\mathbf{R}. \tag{2.1}$$

For any finite signed measure μ on $\mathbf{R}$ such that $\mu(\{x\}) < \frac{1}{2}$ for each $x\in\mathbf{R}$ and every $x_0\in\mathbf{R}$, the SDE

$$X_t = x_0 + \int_0^t a(X_s)dW_s + \int L_t^w(X)\mu(dw) \quad \text{for } t\ge 0 \tag{2.2}$$

has a continuous strong solution and the continuous solution is pathwise unique. Here $L^w(X)$ is the local time of the semimartingale X at w (cf. Revuz and Yor, 1991); that is,

$$L_t^w(X) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t 1_{[w,\,w+\varepsilon)}(X_s)d\langle X\rangle_s \qquad \text{for } t\ge 0. \tag{2.3}$$

PROOF. Define

$$\pi(x) = \begin{cases} -\frac{\log(1-2x)}{2x}, & x\in(-\infty,\,0)\cup(0,\frac{1}{2}) \\ 1 & x=0. \end{cases}$$

Let $\nu(dx) := \pi(\mu(\{x\})\mu(dx)$, which is a finite signed measure. Define

$$s(x) = \int_0^x e^{-2\nu(-\infty,y]}dy.$$

Since ν is a finite measure, s' is right continuous and strictly positive. Hence s is increasing and one-to-one. Let σ denote the inverse of s, let s'_ℓ denote the left continuous version of s', i.e., the left hand derivative of s, and let σ'_ℓ denote the left hand derivative of σ. Since ν is a finite signed measure, s'_ℓ is of bounded variation. Let

$$\widetilde{a}(x) = (s'_\ell a)\circ\sigma(x)$$

and let $Y=\{Y_t,\, t\ge 0\}$ solve

$$dY_t = \widetilde{a}(Y_t)dW_t \quad \text{with} \quad Y_0 = s(x_0). \tag{2.4}$$

We remark that $\widetilde{a}$ is bounded above and below by positive constants and that since s'_ℓ has bounded variation, $\widetilde{a}$ satisfies condition (2.1), with possibly a different function f. According to Le Gall (1983), (2.4) has a strong solution and the solution is pathwise unique. Let $X = \sigma(Y)$.

We must show that X is a solution to (2.2). Clearly $X_0 = x_0$. s' is of bounded variation, hence so is $\sigma' = 1/(s' \circ \sigma)$. Therefore σ' is the difference of two nondecreasing functions, and hence σ is the difference of two convex functions. By Revuz and Yor (1991), Theorem VI.1.5], X is a semimartingale and in fact

$$X_t = \sigma(Y_0) + \int_0^t \sigma'_\ell(Y_s)dY_s + \tfrac{1}{2}\int L_t^w(Y)\sigma''(dw), \qquad t \geq 0, \qquad (2.5)$$

where $L_t^w(Y)$ is the local time for the martingale Y at level w. The stochastic integral term is

$$\int_0^t \sigma'_\ell(Y_s)\widetilde{a}(Y_s)dW_s = \int_0^t \Big(\frac{1}{s'_\ell}(s'_\ell a)\Big) \circ \sigma(Y_s)dW_s = \int_0^t a(X_s)dW_s$$

as required.

By Corollary VI.1.9 of Revuz and Yor (1991),

$$L_t^w(Y) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t 1_{[w,w+\varepsilon)}(Y_s)d\langle Y\rangle_s, \qquad t \geq 0.$$

Y_s is between w and $w+\varepsilon$ if and only if X_s is between $\sigma(w)$ and $\sigma(w+\varepsilon)$. Also

$$d\langle Y\rangle_s = (\widetilde{a})^2(Y_s)ds = (s'_\ell a)^2(X_s)ds = (s'_\ell)^2(X_s)d\langle X\rangle_s.$$

Therefore

$$L_t^w(Y) = \lim_{\delta\to 0}\sigma'(w)\frac{1}{\delta}\int_0^t 1_{[\sigma(w),\sigma(w)+\delta)}(X_s)(s'_\ell)^2(X_s)d\langle X\rangle_s, \qquad t \geq 0.$$

Note when $X_s \in [\sigma(w), \sigma(w)+\delta)$, $s'_\ell(X_s)$ is close to $s'(\sigma(w))$, as the Lebesgue measure of the amount of time that X spends at $\sigma(w)$ is zero. Applying Corollary VI.1.9 in Revuz and Yor (1991) again,

$$L_t^w(Y) = \sigma'(w)(s')^2(\sigma(w))L_t^{\sigma(w)}(X) = s'(\sigma(w))L_t^{\sigma(w)}(X).$$

If we substitute this for the last term in (2.5) we obtain

$$\tfrac{1}{2}\int s'(\sigma(w))L_t^{\sigma(w)}(X)\sigma''(dw).$$

Since $s \circ \sigma(x) = x$, $(s' \circ \sigma(x))\sigma'(x) = 1$, so

$$d\sigma'(x) = -\frac{ds'(\sigma(x))}{(s'_\ell(\sigma(x))^2)}.$$

Using the above expression and performing a change of variables, the last term in (2.5) now becomes

$$-\tfrac{1}{2}\int s'_\ell(x) L^x_t(X) \frac{s''(dx)}{(s'_\ell(x))^2}.$$

Note that $s'(x) = e^{-2\nu(\{x\})} s'_\ell(x)$ and so $s'(x) - s'_\ell(x) = (e^{-2\nu(\{x\})} - 1)s'_\ell(x)$. Hence

$$\frac{s''(dx)}{2s'_\ell(x)} = 1_{\{\nu(\{x\})\neq 0\}} \frac{e^{-2\nu(\{x\})} - 1}{2\nu(\{x\})} \nu(dx) - 1_{\{\nu(\{x\})=0\}}\nu(dx) = -\mu(dx)$$

Thus

$$-\tfrac{1}{2} s'_\ell(x) \frac{s''(dx)}{s'_\ell(x)^2} = \mu(dx). \tag{2.6}$$

This shows that the last term of (2.5) is $\int L^x_t(X)\mu(dx)$ as desired.

We now examine pathwise uniqueness. Since s' is of bounded variation, hence the difference of two nondecreasing functions, then s is the difference of two convex functions. Since X is a continuous process satisfying (2.2), then for $t \geq 0$,

$$s(X_t) = s(X_0) + \int_0^t s'_\ell(X_s) a(X_s) dW_s + \int_0^t s'_\ell(X_s) dA_s + \frac{1}{2}\int_0^t L^w_t(X) s''(dw), \tag{2.7}$$

where

$$A_t = \int L^w_t(X)\mu(dw).$$

Let $Y = s(X)$. The stochastic integral term has the form $(s'_\ell a) \circ \sigma(Y_s) dW_s = \widetilde{a}(Y_s) dW_s$. The sum of the last two terms of (2.7) is

$$\int s'_\ell(w) L^w_t(X)\mu(dw) + \tfrac{1}{2}\int L^w_t(X) s''(dw).$$

However by (2.6) $\frac{1}{2}s''(dw) + s'_\ell(w)\mu(dw) = 0$. Therefore Y solves $dY_t = \widetilde{a}(Y_t)dW_t$, and by the result of Le Gall (1983), the paths of Y are uniquely determined. Since s is one-to-one, that implies the paths of X are uniquely determined. □

Note that if a is a bounded function on $\mathbf{R}$ that is Hölder continuous of order $\frac{1}{2}$, then a satisfies condition (2.1) with $f(x) = c\,x$ for some constant $c > 0$. In many occasions, the symmetric local time $\widehat{L}^w(X)$ for a semimartingale X is of interest, where

$$\widehat{L}^w_t(X) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t 1_{[w-\varepsilon,\, w+\varepsilon)}(X_s) d\langle X\rangle_s \qquad \text{for } t \geq 0. \tag{2.8}$$

So we give a strong existence and pathwise uniqueness result for SDEs with respect to symmetric local times.

THEOREM 2.2. *(a) X is a solution of (2.2) if and only if X is a solution to*

$$X_t = x_0 + \int_0^t a(X_s)dW_s + \int_{\mathbf{R}} \widehat{L}^w_t(X)\nu(dw), \qquad t \geq 0, \tag{2.9}$$

where $\nu(dx) = \frac{1}{1-\mu(\{x\})}\mu(dx)$. In this case,

$$L^w_t(X) = \frac{1}{1-\mu(\{w\})} \widehat{L}^w_t(X) \quad \text{for } t \geq 0. \tag{2.10}$$

(b) Suppose a is a measurable function on $\mathbf{R}$ that is bounded above and below by positive constants and satisfies condition (2.1). Then for every $x_0 \in \mathbf{R}$ and for any finite signed measure ν with $|\nu(\{x\})| < 1$ on $\mathbf{R}$, the SDE (2.9) has a strong solution and the solution is pathwise unique.

PROOF. (a) Suppose that X is a continuous semimartingale. By the Tanaka formula (cf. Revuz and Yor, 1991, Theorem VI.1.2), for $w \in \mathbf{R}$,

$$(X_t - w)^+ = (x_0 - w)^+ + \int_0^t 1_{\{X_s > w\}} dX_s + \tfrac{1}{2} L^w_t(X), \tag{2.11}$$

$$(X_t - w)^- = (x_0 - w)^- - \int_0^t 1_{\{X_s \leq w\}} dX_s + \tfrac{1}{2} L^w_t(X). \tag{2.12}$$

Applying (2.11) to the semimartingale $-X$ at level $-w$,

$$\begin{aligned}(X_t - w)^- &= (-x_0 + w)^+ + \int_0^t 1_{\{-X_s > -w\}} d(-X_s) + \tfrac{1}{2} L^{-w}_t(-X) \\ &= (x_0 - w)^- - \int_0^t 1_{\{X_s < w\}} dX_s + \tfrac{1}{2} L^{-w}_t(-X).\end{aligned}$$

Comparing this with (2.12), we have

$$-\int_0^t 1_{\{X_s=w\}} dX_s + \tfrac{1}{2} L_t^w(X) = \tfrac{1}{2} L_t^{-w}(-X). \tag{2.13}$$

Note that by Revuz and Yor (1991), Corollary VI.1.6, the amount of time X spends at any $w \in \mathbf{R}$ has zero Lebesgue measure provided $d\langle X\rangle_t$ is absolutely continuous with respect to dt almost surely. In this case, we see from (2.3) and (2.8) that

$$\widehat{L}_t^w(X) = \tfrac{1}{2}\left(L_t^w(X) + L_t^{-w}(-X)\right) \qquad \text{for } t \geq 0.$$

Now assume that X is a solution to (2.2). Then $d\langle X\rangle_t = a^2(X_t)dt$. Since $t \to L_t^w(X)$ increases only when $X_t = w$, (2.13) implies that

$$\left(\tfrac{1}{2} - \mu(\{w\})\right) L_t^w(X) = \tfrac{1}{2} L_t^{-w}(-X) = \widehat{L}_t^w(X) - \tfrac{1}{2} L_t^w(X).$$

Thus

$$L_t^w(X) = \frac{1}{1-\mu(\{w\})} \widehat{L}_t^w(X).$$

and therefore X satisfies (2.9).

Conversely, if X is a solution to (2.9), then (2.13) implies that

$$-\nu(\{w\})\widehat{L}_t^w(X) + \tfrac{1}{2} L_t^w(X) = \tfrac{1}{2} L_t^{-w}(-X) = \widehat{L}_t^w(X) - \tfrac{1}{2} L_t^w(X)$$

and so

$$\widehat{L}_t^a(X) = \frac{1}{1+\nu(\{w\})} L_t^a(X).$$

Thus X satisfies (2.2) as $\frac{\nu(dw)}{1+\nu(\{w\})} = \mu(dw)$.

(b) Note that $\mu(dx) \to \nu(dx) = \frac{1}{1-\mu(\{x\})}\mu(dx)$ is a one-to-one map from the family of finite signed measures μ with $\mu(\{x\}) < 1/2$ for each x onto the family of finite signed measures ν with $|\nu(\{x\})| < 1$ for each x. Assertion (b) follows from (a) and Theorem 2.1. $\square$

REMARK 2.3. In Le Gall (1984), Le Gall proved Theorem 2.2(b) under the assumption that a is a function of bounded variation. Le Gall also suggested (without proof) that if μ had atoms of size larger than 1, then there would be no solution.

EXAMPLE 2.4. For $\beta \in (-1, 1)$, let $\nu = \beta\delta_{\{0\}}$. Then the corresponding pathwise unique solution X to (2.9) with $a(x) \equiv 1$ is a skew Brownian motion on $\mathbf{R}$ with parameter β (cf. Revuz and Yor, 1991, Exercise X.2.24).

Corollary 2.5. *Suppose $b \in L^1(\mathbf{R}, dx)$, and a is a measurable function on $\mathbf{R}$ that is bounded above and below by positive constants and satisfies condition (2.1). Then there exists a unique pathwise solution to*

$$dX_t = a(X_t)dW_t + b(X_t)dt \quad \text{with} \quad X_0 = x_0. \tag{2.14}$$

Proof. Let $\mu(dx) = (ba^{-2})(x)\,dx$. Since a is bounded below and $b \in L^1$, then μ is a finite signed measure with no atoms. By Theorem 2.1, there exists a unique pathwise solution to $dX_t = a(X_t)dW_t + dA_t$, with

$$A_t = \int L_t^x(X)\mu(dx) = \int L_t^x(X)\frac{b(x)}{a^2(x)}dx.$$

Since $d\langle X\rangle_t = a^2(X_t)dt$ and $L_t^x(X)$ is the local time for X, then

$$A_t = \int_0^t \frac{b(X_s)}{a^2(X_s)}d\langle X\rangle_s = \int_0^t b(X_s)ds$$

as desired. □

Remark 2.6. By standard stopping time arguments, we can allow μ in Theorems 1.1 and 1.2 to be a signed Radon measure and b in Corollary 2.5 to be only locally in L^1 provided solutions are permitted to have explosions in finite time. In these cases, Feller's explosion test can be applied to determine whether the solution X to (2.2) (respectively, to (2.14)) has an explosion or not. In fact, if we let $\nu(dx) = \pi(\mu(\{x\})\mu(dx)$ (respectively, $\nu(dx) = b(x)dx$), and define

$$s(x) = \int_0^x e^{-2\nu[0,t]}dt \quad \text{and} \quad v(x) = \int_0^x \frac{2(s(x)-s(y))}{s'_\ell(y)a^2(y)}dy,$$

then the solution has no explosion (i.e., is conservative) if and only if $v(\infty-) = v(-\infty+) = \infty$ (cf. Theorem 5.5.29 of Karatzas and Shreve, 1994). This remark also applies to the results in Sections 3 and 4 below.

3 Comparison Principle

Theorem 3.1. *(Comparison principle). Suppose a is a measurable function on $\mathbf{R}$ that is bounded above and below by positive constants and satisfies condition (2.1), and ν_1 and ν_2 are two finite signed measures with $|\nu_k(\{x\})| < 1$ on $\mathbf{R}$ for $k = 1, 2$. Fix $x_0 \in \mathbf{R}$ and a standard Brownian* *motion W. Let X^k be the unique strong solution of (2.9) driven by W with ν_k in place of ν. If $\nu_2 - \nu_1 \geq 0$, then $X_t^2 \geq X_t^1$ for all $t \geq 0$ a.s.*

PROOF. Let $\phi \geq 0$ be a smooth symmetric function on $\mathbf{R}$ with compact support such that $\int_{\mathbf{R}} \phi(x)dx = 1$, and set $\phi_n(x) = n\phi(nx)$ for $n \geq 1$. Define on $(-1, 1)$

$$\pi(y) = \begin{cases} \frac{1}{2y} \log \frac{1+y}{1-y}, & y \neq 0, \\ 1, & y = 0, \end{cases}$$

$$\mu_k(dy) = \pi(\nu_k(\{y\})\nu_k(dy),$$

and

$$f_n^k(x) = \int_{\mathbf{R}} \phi_n(x-y)\mu_k(dy), \qquad k = 1, 2;$$

the f_n^k are smooth functions on $\mathbf{R}$. Clearly $f_n^2(x) \geq f_n^1(x)$ on $\mathbf{R}$ and $f_n^k(x)dx$ converges weakly to $\mu_k(dx)$ on $\mathbf{R}$ as $n \to \infty$. Let $X^{k,n}$ be the unique strong solution to

$$X_t^{k,n} = x_0 + \int_0^t a(X_s^{k,n})dW_s + \int_0^t a^2(X_s^{k,n})f_n^k(X_s^{k,n})ds.$$

By the well known comparison principle for SDEs (see, e.g. Bass, 1997), we have $X_t^{2,n} \geq X_t^{1,n}$ for all $t \geq 0$ a.s On the other hand, let

$$s_n^k(x) = \int_0^x \exp\left(-2\int_0^y f_n^k(t)dt\right) dy$$

and

$$s^k(x) = \int_0^x \exp\left(-2\mu_k[0, y]\right) dy.$$

We note that

$$\lim_{n\to\infty} s_n^k(x) = s^k(x), \qquad x \in \mathbf{R},\ k = 1, 2,$$

and

$$\lim_{n\to\infty} (s_n^k(x))_l' = (s^k)_l' = -2\mu_k[0,\, x) \quad \text{a.e. on } \mathbf{R}.$$

By Ito's formula,

$$s_n^k(X_t^{k,n}) = s_n^k(x_0) + \int_0^t (s_n^k)'(X_s^{k,n})a(X_s^{k,n})dW_s, \qquad t \geq 0,$$

and

$$s^k(X_t^k) = s^k(x_0) + \int_0^t (s^k)_l'(X_s^k)a(X_s^k)dW_s, \qquad t \geq 0. \tag{3.3}$$

Note $\lim_{n\to\infty}(s^k_n)'\circ(s^k_n)^{-1}=(s^k)'_l\circ(s^k)^{-1}$ a.e. on $\mathbf{R}$, and that the SDE (3.3) for $Y^k_t=s^k(X^k_t)$ with $Y^k_0=s^k(x_0)$ has a pathwise unique solution. By the proof of Theorem B of Kaneko and Nakao (1988), we have

$$\lim_{n\to\infty}\mathbf{E}\left[\sup_{0\le t\le T}|s^k_n(X^{k,n}_t)-s^k(X^k_t)|^2\right]=0,$$

for every $T>0$. Thus there is a subsequence n_i such that

$$\lim_{i\to\infty}\sup_{0\le t\le T}|s^k_n(X^{k,n_i}_t)-s^k(X^k_t)|=0 \quad \text{a.s.}$$

Hence for each $T>0$,

$$\lim_{i\to\infty}\sup_{0\le t\le T}|X^{k,n_i}_t-X^k_t|=0 \quad \text{a.s.}$$

Therefore we have $X^2_t\ge X^1_t$ for all $t\ge 0$ a.s. □

Harrison and Shepp (1981) showed that the SDE

$$X_t=x_0+W_t+\beta\widetilde{L}^0_t(X),$$

has no solution if $|\beta|>1$, and has a unique strong solution when $|\beta|\le 1$. The solution to the above SDE is called skew Brownian if $|\beta|<1$. When $|\beta|=1$, the solution is Brownian motion reflected on the right or left side of 0 depending on whether $\beta=1$ or -1. We are going to extend this result to the SDE (2.2) in the rest of this section and next one.

Theorem 3.2. *There is no solution to the SDE (2.9) if the measure* ν has $|\nu(\{x_0\})|>1$ for some $x_0\in\mathbf{R}$.

Proof. Without loss of generality, let us assume that $x_0=0$. Suppose that there is a solution X to the SDE (2.9) for such ν. By the Tanaka formula (see Theorem VI.1.2 of Revuz and Yor, 1991),

$$X^-_t=x^-_0-\int_0^t 1_{\{X_s\le 0\}}dX_s+\tfrac{1}{2}L^0_t,$$

where L^0_t is the semimartingale local time of X; so by (2.10), $L^0_t=(1+\nu(\{0\})\widehat{L}^0_t$. Hence

$$X^-_t=x^-_0-\int_0^t 1_{\{X_s\le 0\}}a(X_s)dW_s-\int_{\{x<0\}}\widehat{L}^x_t\nu(dx)-\tfrac{1}{2}(\nu(\{0\})-1)\widehat{L}^0_t.$$

Let $T = \inf\{t > 0 : X_t = 0\}$. Then

$$\begin{aligned} X^-_{T+t} &= X^-_{T+t} - X^-_T \\ &= -\int_0^t 1_{\{X_{T+s}\le 0\}} a(X_s) d\widetilde{W}_s - \int_{\{x<0\}} \widehat{L}^x_{t+T}\nu(dx) \\ &\quad -\tfrac{1}{2}(\nu(\{0\})-1)\widehat{L}^0_{t+T}, \end{aligned} \tag{3.4}$$

where $\widetilde{W}_s = W_{s+T}$. Since $\widehat{L}^x_t$ increases only when $X_t = x$, then

$$\mathbf{P}(X_{T+t} < 0 \ \text{ infinitely often as } t \to 0+) > 0$$

if $\nu(\{0\}) > 1$. This is impossible as $X^-_t \ge 0$. Therefore $\nu(\{0\})$ cannot be larger than 1. Similarly, $\nu(\{0\})$ cannot be smaller than -1. □

THEOREM 3.3. *Suppose that ν is a finite signed measure with $|\nu(\{x\})| \le 1$ and X is a solution to the SDE (2.9) so that X has continuous paths. If $\nu(\{x_0\}) = 1$, then $\mathbf{P}(X_t \ge x_0$ for all $t \ge 0) = 1$ for $x \ge x_0$. Similarly, if $\nu(\{x_0\}) = -1$, then $\mathbf{P}(X_t \le x_0$ for all $t \ge 0) = 1$ for $x \le x_0$.*

PROOF. Without loss of generality, we may assume that $x_0 = 0$. Define $T = \inf\{t > 0 : X_t < 0\}$. Since $X_0 \ge 0$, by the continuity of the sample paths of X, $X_T = 0$ on $\{T < \infty\}$. From (3.4), we have

$$X^-_{T+t} = -\int_0^t 1_{\{X_s \le 0\}} a(X_s) d\widetilde{W}_s - \int_{\{x<0\}} (\widehat{L}^x_{t+T} - \widehat{L}^x_T)\nu(dx). \tag{3.5}$$

If $\mathbf{P}(T < \infty) > 0$, (3.5) would imply

$$\mathbf{P}\left(X^-_{T+t} < 0 \ \text{ infinitely often as } \ t \to 0+ \middle| T < \infty\right) > 0$$

unless the local martingale $t \mapsto \int_0^t 1_{\{X_s \le 0\}} a(X_s) dW_s$ is zero in a neighborhood of 0, that is, unless

$$\mathbf{P}\left(X_{T+s} \ge 0 \ \text{ for a.e. } s \in [0,\varepsilon] \text{ for some } \varepsilon > 0 \middle| T < \infty\right) = 1.$$

Since X has continuous sample paths, we have

$$\mathbf{P}\left(X_{T+s} \ge 0 \ \text{ for every } s \in [0,\varepsilon] \text{ for some } \varepsilon > 0 \middle| T < \infty\right) = 1.$$

This contradicts the definition of T. Hence $\mathbf{P}(T = \infty) = 1$, and therefore $X_s \ge 0$ for all $s \ge 0$ $\mathbf{P}$-a.s. □

4 SDEs with Reflecting Boundary

THEOREM 4.1. *Suppose a is a measurable function on $\mathbf{R}$ that is bounded above and below by positive constants, and satisfies condition (2.1). Then for any $x_0, w_0 \in \mathbf{R}$, the SDE*

$$X_t = x_0 + \int_0^t a(X_s) dW_s + \widehat{L}_t^{w_0}(X) \tag{4.1}$$

has a strong continuous solution and the solution is pathwise unique. The same conclusion holds for the SDE

$$X_t = x_0 + \int_0^t a(X_s) dW_s - \widehat{L}_t^{w_0}(X). \tag{4.2}$$

Furthermore, the unique solution of (4.1) (respectively, (4.2)) is the increasing (respectively, decreasing) limit of the strong solutions X^n to (2.9) with $\beta_n \delta_{\{w_0\}}$ (respectively, $-\beta_n \delta_{\{w_0\}}$) in place of ν for any $\beta_n \uparrow 1$ as $n \to \infty$.

PROOF. (1) (Strong existence.) It suffices to prove the theorem for the SDE (4.1). Let $0 < \beta_n \uparrow 1$ and let X^n be the unique strong solution to the SDE

$$X_t^n = x_0 + \int_0^t a(X_s^n) dW_s + \beta_n \widehat{L}_t^{w_0}(X^n). \tag{4.3}$$

By Theorem 3.1, the process $X^n = \{X_t^n, \, t \geq 0\}$ is increasing in n, a.s. For $t \geq 0$, define $X_t = \lim_{n\to\infty} X_t^n$. Note that the quadratic variation process $\int_0^\cdot a(X_s^n)^2 ds$ of X^n converges in distribution to $[X, X]$ of $X = \{X_t, \, t \geq 0\}$ by Theorem VI.6.1 of Jacod and Shiryaev (1987). Consequently,

$$\lambda^{-1} t \leq [X, X]_t \leq \lambda t \quad \text{for all } t \geq 0,$$

where $\lambda > 1$ is a constant such that $1/\lambda \leq a(x) \leq \lambda$ for a.e. $x \in \mathbf{R}$. As the function a satisfies condition (2.1), it can only have at most countably many discontinuities; we denote the set of points of discontinuity of a by F. So by the occupation time formula (cf. Corollary VI.1.6 in Revuz and Yor, 1991),

$$\mathbf{E}\left[\int_0^\infty 1_F(X_s) ds\right] = 0. \tag{4.4}$$

This implies

$$\lim_{n\to\infty} \int_0^t a(X_s^n)^2 ds = \int_0^t a(X_s)^2 ds \quad \text{for all } t \geq 0, \text{ a.s.} \tag{4.5}$$

and consequently there is a subsequence n_k so that

$$\lim_{k\to\infty}\int_0^t a(X_s^{n_k})dW_s = \int_0^t a(X_s)dW_s \qquad \text{for all } t\geq 0, \text{ a.s.}$$

Therefore

$$L_t := \lim_{k\to\infty} \beta_{n_k}\widehat{L}_t^{w_0}(X^{n_k}), \qquad t\geq 0, \tag{4.6}$$

exists and is finite a.s. Passing to the limit in (4.3) along the subsequence n_k, we have

$$X_t = x_0 + \int_0^t a(X_s)dW_s + L_t \qquad \text{for } t\geq 0. \tag{4.7}$$

It follows from Lemma 3.1 in Burdzy et al. (2004) on the uniqueness of solutions to the deterministic Skorokhod problem on $[0,\infty)$ that both X and L are continuous processes. We now show that L is the symmetric local time at w_0 for semimartingale X. By Tanaka's formula (cf. Theorem VI.1.2. of Revuz and Yor, 1991) and Theorem 2.2(a) above, we have

$$\begin{aligned}(X_t^n - w_0)^+ &= (x_0-w_0)^+ + \int_0^t 1_{\{X_s^n>w_0\}}dX_s^n + \tfrac{1}{2}L_t^{w_0}(X^n)\\ &= (x_0-w_0)^+ + \int_0^t 1_{\{X_s^n>w_0\}}a(X_s^n)dW_s + \frac{1+\beta_n}{2}\widehat{L}_t^{w_0}(X^n),\end{aligned}$$

and

$$\begin{aligned}(X_t^n - w_0)^- &= (x_0-w_0)^- - \int_0^t 1_{\{X_s^n\leq w_0\}}dX_s^n + \tfrac{1}{2}L_t^{w_0}(X^n)\\ &= (x_0-w_0)^- - \int_0^t 1_{\{X_s^n\leq w_0\}}a(X_s^n)dW_s + \frac{1-\beta_n}{2}\widehat{L}_t^{w_0}(X^n).\end{aligned}$$

Letting $n\to\infty$ in the above equations along a suitable subsequence, we get from (4.5)-(4.6) that

$$(X_t-w_0)^+ = (x_0-w_0)^+ + \int_0^t 1_{\{X_s>w_0\}}a(X_s)dW_s + L_t, \tag{4.8}$$

and

$$(X_t-w_0)^- = (x_0-w_0)^- - \int_0^t 1_{\{X_s\leq w_0\}}a(X_s)dW_s. \tag{4.9}$$

As $\widehat{L}_t^{w_0}(X^n)$ increases only when $X_t^n = w_0$, it is clear that L_t increases only when $X_t = w_0$ Now applying Tanaka's formula to X in (4.7),

$$\begin{aligned}(X_t - w_0)^+ &= (x_0 - w_0)^+ + \int_0^t 1_{\{X_s > w_0\}} dX_s + \tfrac{1}{2} L_t^{w_0}(X) \\ &= (x_0 - w_0)^+ + \int_0^t 1_{\{X_s > w_0\}} a(X_s) dW_s + \tfrac{1}{2} L_t^{w_0}(X)\end{aligned}$$

Comparing this with (4.8), we conclude $2L = L^{w_0}(X)$, the semimartingale local time at w_0 of X. Let $T = \inf\{t \geq 0 : X_t \geq w_0\}$. Equations (4.4) and (4.9) imply that $X_{T+s} \geq w_0$ for $s \geq 0$ a.s. Hence $L = \widehat{L}^{w_0}(X)$ is the symmetric local time w_0 of X. This proves that X is a strong solution to the SDE (4.1).

(2) (Pathwise uniqueness.) Suppose that X and Y are two strong solutions to the SDE (4.1). It is clear that $X_t = Y_t$ for $t \leq T = \inf\{t \geq 0 : X_t \geq w_0\}$. So without loss of generality, we assume that $w_0 = 0$ and $x_0 \geq 0$. By Theorem 3.3, we have $X_t \geq 0$ and $Y_t \geq 0$. Hence $(X, L^{w_0}(X))$ and $(Y, L^{w_0}(Y))$ are solutions (Z, L) to the following SDE with reflection:

$$\begin{aligned}Z_t &= x_0 + \int_0^t a(Z_s) dW_s + L_t, \quad \text{where } Z_t \geq 0 \text{ and} \\ &\quad t \to L_t \text{ is a continuous non-decreasing process with } L_0 = 0 \\ &\quad \text{and } L_t = \textstyle\int_0^t 1_{\{Z_s = 0\}} dL_s.\end{aligned} \tag{4.10}$$

We now show that $(X \wedge Y, L^{X \wedge Y}))$ and $(X \vee Y, L^{X \vee Y}))$ are also solutions to (4.10), with

$$\begin{aligned}L_t^{X \wedge Y} &:= \int_0^t 1_{\{X_s \leq Y_s\}} dL_s^{w_0}(X) + \int_0^t 1_{\{X_s > Y_s\}} dL_s^{w_0}(Y), \quad \text{and} \\ L_t^{X \vee Y} &:= \int_0^t 1_{\{X_s > Y_s\}} dL_s^{w_0}(X) + \int_0^t 1_{\{X_s \leq Y_s\}} dL_s^{w_0}(Y).\end{aligned}$$

According to Corollary 1.2 of Le Gall (1984), $L^0(X - Y) = 0$ and so by Tanaka's formula,

$$\begin{aligned}X_t \wedge Y_t &= X_t - (X_t - Y_t)^+ \\ &= X_t - \left(\int_0^t 1_{\{X_s > Y_s\}} d(X_s - Y_s) + \tfrac{1}{2} L_t^0(X - Y) \right) \\ &= x_0 + \int_0^t a(X_s) dW_s + L_t^{w_0}(X) - \int_0^t 1_{\{X_s > Y_s\}} (a(X_s) - a(Y_s)) dW_s \\ &\quad - \int_0^t 1_{\{X_s > Y_s\}} dL_s^{w_0}(X) + \int_0^t 1_{\{X_s > Y_s\}} dL_s^{w_0}(Y)\end{aligned}$$

$$= x_0 + \int_0^t a(X_s \wedge Y_s) dW_s$$
$$+ \left(\int_0^t 1_{\{X_s \le Y_s\}} dL_s^{w_0}(X) + \int_0^t 1_{\{X_s > Y_s\}} dL_s^{w_0}(Y) \right).$$

This shows that $(X \wedge Y, L^{X \wedge Y}))$ is indeed a solution to (4.10). Similarly one can show that $(X \vee Y, L^{X \vee Y}))$ is a solution to (4.10) as well.

We claim that weak uniqueness holds for solutions of (4.10). Certainly weak uniqueness holds for solutions of (4.10) when $a \equiv 1$, as this is just Brownian motion on $\mathbf{R}$ reflected on the right hand side of 0. As $a(x)$ is bounded above and below by positive constants, there is a one to one correspondence between the distributions of solutions to (4.10) and those to (4.10) with $a \equiv 1$ through a time-change argument (cf. Proposition IX.1.13 of Revuz and Yor, 1991). This is because if (Z_t, L_t) is a solution to (4.10), then (Z_{A_t}, L_{A_t}) is a solution to (4.10) with $a \equiv 1$, where

$$A_t = \inf \left\{ s > 0 : \int_0^s a(X_s)^2 \, ds \ge t \right\} \qquad \text{for } t \ge 0,$$

and vice versa. Thus weak uniqueness holds for solutions of (4.10). This implies that $X_t \wedge Y_t = X_t \vee Y_t$ a.s. for each $t > 0$. By the continuity of X_t and Y_t, we have $X_t = Y_t$ for all $t \ge 0$, a.s. □

COROLLARY 4.2. *Suppose a is a measurable function on $\mathbf{R}$ that is bounded above and below by positive constants and satisfies condition (2.1), and $w_1, \cdots, , w_k$ are k distinct points in $\mathbf{R}$. Then for any $x_0 \in \mathbf{R}$, the SDE*

$$X_t = x_0 + \int_0^t a(X_s) dW_s + \sum_{j=1}^k \widehat{L}_t^{w_j}(X) \qquad \textit{for } t \ge 0 \tag{4.11}$$

has a strong solution and the solution is pathwise unique. The same conclusion holds for the SDE

$$X_t = x_0 + \int_0^t a(X_s) dW_s - \sum_{j=1}^k \widehat{L}_t^{w_j}(X) \qquad \text{for } t \ge 0. \tag{4.12}$$

Furthermore, the unique solution of (4.11) (respectively, (4.12)) is the increasing (respectively, decreasing) limit of the strong solutions X^n to (2.7) with $\sum_{j=1}^k \beta_{i,n} \delta_{\{w_j\}}$ (respectively, $-\beta_{j,n}\delta_{\{w_i\}}$) in place of ν for any $\beta_{j,n} \uparrow 1$ as $n \to \infty$ with $j = 1, \cdots, k$.

The proof of this consists of dividing $\mathbf{R}$ into $k+1$ subintervals determined by the w_j and piecing together the solutions on each subinterval. We leave the details to the reader.

THEOREM 4.3. *Suppose a is a measurable function on $\mathbf{R}$ that is bounded above and below by positive constants and satisfies condition (2.1). If ν is a finite signed measure with $-1 < \nu(\{x\}) \leq 1$, then the SDE (2.9) has a strong solution and the solution is pathwise unique. The same conclusion holds for any finite signed measure ν on $\mathbf{R}$ with $-1 \leq \nu(\{x\}) < 1$.*

PROOF. The proof of this theorem is along the same lines as that of Theorems 1.1 and 1.2. However we need to single out those atoms with size 1 in absolute value and relate the SDE (2.2) to the SDE of type (4.11) or (4.12). For the reader's convenience, we spell out the details of the proof below.

Suppose ν is a finite signed measure with $-1 < \nu(\{x\}) \leq 1$ for all $x \in \mathbf{R}$. There are at most finite many points $w_1, \cdots, w_k$ with $\nu(\{w_j\}) = 1$. Define on $(-1, 1]$

$$\pi(x) = \begin{cases} \frac{1}{2x} \log\left(\frac{1+x}{1-x}\right), & x \neq 0, 1, \\ 1 & x = 0, \\ 0 & x = 1. \end{cases}$$

Note that $b(dx) := \pi(\nu(\{x\})\nu(dx)$ is a finite signed measure on $\mathbf{R}$.

Define $s(x) = \int_0^x e^{-2b(-\infty,y]}dy$. Since $b(dx)$ is a finite measure, s' is right continuous and strictly positive. Hence s is increasing and one-to-one. Let σ denote the inverse of s. Let s'_ℓ denote the left continuous version of s', i.e., the left hand derivative of s. Let σ'_ℓ denote the left hand derivative of σ. Since $b(dx)$ is a finite signed measure, s'_ℓ is of bounded variation. Let

$$\widetilde{a}(x) = (s'_\ell a) \circ \sigma(x)$$

and let Y solve

$$Y_t = s(x_0) + \int_0^t \widetilde{a}(Y_t)dW_t + \sum_{j=1}^k \widehat{L}_t^{s(w_j)}(Y). \tag{4.13}$$

We remark that $\widetilde{a}$ is bounded above and below by positive constants and that since s'_ℓ has bounded variation, $\widetilde{a}$ satisfies the condition of (2.1). By Corollary 4.2, (4.13) has a unique strong solution. Let $X = \sigma(Y)$.

We must show that X is a solution to (2.9). Clearly $X_0 = x_0$. s' is of bounded variation, hence $\sigma' = 1/(s' \circ \sigma)$ is also. Therefore σ' is the difference of two nondecreasing functions, and hence σ is the difference of two convex functions. By Revuz and Yor, 1991, Theorem VI.1.5, X is a semimartingale and in fact

$$\begin{aligned} X_t &= \sigma(Y_0) + \int_0^t \sigma'_\ell(Y_s)dY_s + \tfrac{1}{2}\int L^z_t(Y)\sigma''(dz) \\ &= x_0 + \int_0^t \sigma'_\ell(Y_s)\widetilde{a}(Y_s)dW_s + \sum_{j=1}^k \sigma'_\ell(s(w_j))\widehat{L}^{s(w_j)}_t(Y) \\ &\quad + \tfrac{1}{2}\int L^z_t(Y)\sigma''(dz), \end{aligned} \tag{4.14}$$

where $L^z(Y)$ is the local time for the martingale Y at level z. The stochastic integral term is

$$\int_0^t \sigma'_\ell(Y_s)\widetilde{a}(Y_s)dW_s = \int_0^t \Big(\frac{1}{s'_\ell}(s'_\ell a)\Big)\circ\sigma(Y_s)dW_s = \int_0^t a(X_s)dW_s$$

as required.

By Corollary VI.1.9 of Revuz and Yor (1991),

$$L^z_t(Y) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t 1_{[z,z+\varepsilon)}(Y_s)d\langle Y\rangle_s.$$

Y_s is between z and $z+\varepsilon$ if and only if X_s is between $\sigma(z)$ and $\sigma(z+\varepsilon)$. Also

$$d\langle Y\rangle_s = (\widetilde{a})^2(Y_s)ds = (s'_\ell a)^2(X_s)ds = (s'_\ell)^2(X_s)d\langle X\rangle_s.$$

Therefore

$$L^z_t(Y) = \lim_{\delta\to 0}\sigma'(z)\frac{1}{\delta}\int_0^t 1_{[\sigma(z),\sigma(z)+\delta)}(X_s)(s'_\ell)^2(X_s)d\langle X\rangle_s.$$

Note when $X_s\in[\sigma(z),\sigma(z)+\delta)$, then $s'_\ell(X_s)$ is close to $s'(\sigma(z))$, as the amount of time that X_t spends at $\sigma(z)$ has Lebesgue measure zero. Applying Corollary VI.1.9 of Revuz and Yor (1991) again,

$$L^z_t(Y) = \sigma'(z)(s')^2(\sigma(z))L^{\sigma(z)}_t(X) = s'(\sigma(z))L^{\sigma(z)}_t(X). \tag{4.15}$$

From Theorem 1.3, we see that $\widehat{L}^{s(w_j)}_t(Y) = \frac{1}{2}L^{s(w_j)}_t(Y)$ and since σ' is continuous at w_j, the second to last term in (4.14) becomes

$$\sum_{j=1}^k \sigma'_\ell(s(w_j))\widehat{L}^{s(w_j)}_t(Y) = \tfrac{1}{2}\sum_{j=1}^k \sigma'(s(w_j))L^{s(w_j)}_t(Y) = \tfrac{1}{2}\sum_{j=1}^k L^{w_j}_t(X). \tag{4.16}$$

If we substitute (4.15) for the last term in (4.14) we obtain

$$\tfrac{1}{2}\int s'(\sigma(z))L_t^{\sigma(z)}(X)\sigma''(dz).$$

Since $s\circ\sigma(x)=x$, $(s'\circ\sigma(x))\sigma'(x)=1$, so

$$d\sigma'(x)=-\frac{ds'(\sigma(x))}{(s'_\ell(\sigma(x))^2}.$$

Using the above expression and performing a change of variables, the last term in (4.14) now becomes

$$-\tfrac{1}{2}\int s'_\ell(x)L_t^x(X)\frac{s''(dx)}{(s'_\ell(x))^2}.$$

Note that $s'(x)=e^{-2b(\{x\})}s'_\ell(x)$ and so $s'(x)-s'_\ell(x)=(e^{-2b(\{x\})}-1)s'_\ell(x)$, and hence

$$\begin{aligned}\frac{s''(dx)}{2s'_\ell(x)} &= 1_{\{b(\{x\})\neq 0\}}\,\frac{e^{-2b(\{x\})}-1}{2b(\{x\})}\,b(dx)-1_{\{b(\{x\})=0\}}b(dx)\\ &= -1_{\{\nu(\{x\})\neq 1\}}\,\frac{1}{1+\nu(\{x\})}\,\nu(dx):=-\widehat{b}(dx).\end{aligned}\tag{4.17}$$

This shows that the last term of (4.14) is $\int L_t^x(X)\widehat{b}(dx)$. So X satisfies

$$X_t=x_0+\int_0^t a(X_s)dW_s+\tfrac{1}{2}\sum_{j=1}^k L_t^{w_j}+\int L_t^x(X)\widehat{b}(dx).$$

By the same argument as that for (2.10) above, one can prove that

$$L_t^w(X)=\frac{\widehat{L}_t^w(X)}{1-\widehat{b}(\{w\})}=(1+\nu(\{w\}))\widehat{L}_t^w(X)\qquad\text{for } w\notin\{w_1,\cdots,w_k\}\tag{4.18}$$

and

$$L_t^{w_j}(X)=2\widehat{L}_t^{w_j}(X)\qquad\text{for } j=1,\cdots,k.\tag{4.19}$$

Thus

$$\frac{1}{2}\sum_{j=1}^k L_t^{w_j}(X)+\int L_t^x(X)\,\widehat{b}(dx)=\int \widehat{L}_t^x(X)\nu(dx).$$

and therefore X solves SDE (2.9).

We now examine uniqueness. Suppose X is a strong solution for (2.9). Define s as above. Since s' is of bounded variation, hence the difference of two nondecreasing functions, then s is the difference of two convex functions. Since X is a continuous semimartingale, then for $t \geq 0$,

$$s(X_t) = s(X_0) + \int_0^t s'_\ell(X_s)a(X_s)dW_s + \int \widehat{L}_t^w(X)\nu(dw) + \frac{1}{2}\int_0^t L_t^w(X)s''(dw). \tag{4.20}$$

Let $Y = s(X)$. The stochastic integral term has the form $(s'_\ell a)\circ\sigma(Y_s)dW_s = \widetilde{a}(Y_s)dW_s$. By the same calculations as those in (4.15)-(4.19), the sum of the last two terms of (4.20) is $\sum_{j=1}^k \widehat{L}_t^{s(w_j)}(Y)$. Therefore Y solves $dY_t = \widetilde{a}(Y_t)dW_t + \sum_{j=1}^k d\widehat{L}_t^{s(w_j)}(Y)$, and by Corollary 4.2, the paths of Y_t are uniquely determined. Since s is one-to-one, that implies the paths of X are uniquely determined. □

By the same argument as in the proof for strong existence in Theorem 4.1, one has

COROLLARY 4.4. *Suppose that ν is a finite signed measure having $-1 < \nu(\{x\}) \leq 1$ (respectively, $-1 \leq \nu(\{x\}) < 1$) for all $x \in \mathbf{R}$ with w_j, $1 \leq j \leq k$, being those values of x where $\nu(\{x\}) = 1$ (respectively, $\nu(\{x\}) = -1$). Under the condition of Theorem 4.3, the unique strong solution of the SDE (2.9) is the increasing (respectively, decreasing) limit of solutions X^n to (2.9) with $\nu - \sum_{j=1}^k \varepsilon_{j,n}\delta_{\{w_j\}}$ (respectively, with $\nu + \sum_{j=1}^k \varepsilon_{j,n}\delta_{\{w_i\}}$) in place of ν for any $\varepsilon_{j,n} \downarrow 0$ as $n \to \infty$ with $j = 1, \cdots, k$.*

THEOREM 4.5. *Suppose a is a measurable function on $\mathbf{R}$ that is bounded above and below by positive constants and satisfies condition (2.1). Then for any finite signed measure ν with $|\nu(\{x\})| \leq 1$, the SDE (2.9) has a strong solution and the solution is pathwise unique. Furthermore, suppose that $\{w_1, \cdots, , w_k\} = \{x \in \mathbf{R} : \nu(\{x\}) = 1\}$ and $\{z_1, \cdots, z_l\} = \{x \in \mathbf{R} : \nu(\{x\}) = -1\}$. Let $\{\varepsilon_{i,n}\}_{n\geq 1}$ and $\{\gamma_{j,n}\}_{n\geq 1}$ be any $k+l$ sequences of positive numbers taking values in $(0, 1)$ such that $\varepsilon_{i,n} \downarrow 0$ and $\gamma_{j,n} \downarrow 0$ as $n \to \infty$ for each $1 \leq i \leq k$ and $1 \leq j \leq l$. Let $X^{n,m}$ be the unique solutions to the SDE (2.9) with*

$$\nu_{n,m} = \nu - \sum_{i=1}^k \varepsilon_{i,n}\delta_{\{w_i\}} + \sum_{j=1}^l \gamma_{j,m}\delta_{\{w_i\}}$$

in place of ν there. Then

$$X_t = \lim_{m\to\infty}\lim_{n\to\infty} X_t^{n,m} = \lim_{n\to\infty}\lim_{m\to\infty} X_t^{n,m} \quad \textit{for all } t \geq 0$$

almost surely.

The proof for this consists of dividing $\mathbf{R}$ into $k+l+1$ subintervals determined by the w_i and z_j, and piecing together the solutions on each subintervals. The convergence result follows from Theorem 3.1 by an argument similar to that of the proof of strong existence in Theorem 4.1. We leave the details to the reader.

Similarly to the proof of Theorem 3.1 and the strong existence proof in Theorem 4.1, we have

THEOREM 4.6. *Suppose a is a measurable function on $\mathbf{R}$ that is bounded above and below by positive constants and satisfies condition (2.1), and ν_1 and ν_2 are two finite signed measures with $|\nu_k(\{x\})| \leq 1$ on $\mathbf{R}$ for $k=1,2$. Fix $x_0 \in \mathbf{R}$ and a standard Brownian motion W. Let X^k be the unique strong solution of (2.2) driven by W with ν_k in place of ν. If $\nu_2 - \nu_1 \geq 0$, then* $X_t^2 \geq X_t^1$ for all $t \geq 0$ a.s.

5 SDEs with a Degenerate Diffusion Coefficient

It is shown by Barlow (1982) that

$$dX_t = a(X_t)dW_t$$

may not have any strong solution if $a \in C^\alpha$ with $\alpha \in (0,\, 1/2)$ even if a is bounded between two positive constants. In this section we will consider a particular SDE on $[0,\infty)$ with reflection at 0,

$$X_t = x_0 + \int_0^t a(X_s)dW_s + \widehat{L}_t^0(X), \qquad t \geq 0, \tag{5.1}$$

such that X is a continuous process that stays in $[0,\infty)$, $\widehat{L}^0(X)$ is the symmetric local time of X at level 0, and $a(x) = x^\alpha$, $\alpha \in (0,1/2)$. This is equivalent to considering the SDE on $[0,\infty)$ with reflection at 0,

$$X_t = x_0 + \int_0^t a(X_s)dW_s + L_t, \qquad t \geq 0, \tag{5.2}$$

such that X is a continuous process that stays in $[0,\infty)$, L is a non-decreasing continuous process that increases only when $X_t = 0$, and $a(x) = x^\alpha$. This is because by Theorem 3.3 and the definition of symmetric local time, a solution to the SDE (5.1) is a solution to the SDE (5.2). Conversely, if X is a solution to the SDE (5.2), by using the Tanaka formula, one can identify L in (5.2) as the symmetric local time of X at 0 (cf. the argument following (4.8)).

We will show that there is a strong solution to the SDE (5.1) for which the time spent at 0 has zero Lebesgue measure and that this solution is the pathwise unique solution to (5.2) among the class of solutions that spend zero time at 0. This is a first step toward understanding strong solutions to SDEs with degenerate diffusion coefficients. The comparison principle established in Theorem 4.6 plays an important part in our approach. Throughout this section, W is a one-dimensional Brownian motion.

THEOREM 5.1. *For every $\alpha \in (0, 1/2)$ and every $x_0 \geq 0$, there is a continuous strong solution $X = \{X_t, t \geq 0\}$ to*

$$X_t = x_0 + \int_0^t X_s^\alpha dW_s + \widehat{L}_t^0(X) \quad \text{for } t \geq 0 \tag{5.3}$$

such that $X_t \geq 0$ for all $t \geq 0$ and X spends zero Lebesgue amount of time at 0. Here $\widehat{L}^0(X)$ is the symmetric local time of X at level 0.

PROOF. Let $\gamma = \frac{\alpha}{2(1-\alpha)}$. Define

$$D_n(y) = \begin{cases} -\gamma/y & \text{for } y \geq \gamma/n \\ -n & \text{for } y \leq \gamma/n. \end{cases}$$

Let $W = \{W_t, t \geq 0\}$ be a given Brownian motion on a complete probability filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions, and let $y_0 := (1 + 2\gamma)^{1-\alpha} x_0 \geq 0$. By Remark 2.6 and Theorem 4.5, there is a unique strong solution Y^n to

$$dY_t^n = dW_t + D_n(Y_t^n)dt + d\widehat{L}_t^n \quad \text{with } Y_0^n = y_0, \tag{5.4}$$

such that Y^n is conservative, $Y_t^n \geq 0$ for all $t \geq 0$, and $\widehat{L}^n$ is the symmetric local time of Y^n at 0. Since D_n is decreasing in n, by the comparison principle in Theorem 4.6, the processes Y^n are decreasing in n.

Let s_n be defined on $\mathbf{R}$ by

$$\frac{s_n''(x)}{s_n'(x)} = -2D_n(x), \quad s_n(0) = 0, \quad s_n(1) = 1.$$

In other words,

$$s_n(x) = c_n \int_1^x \exp\left(-2\int_1^t D_n(s)ds\right) dt + 1.$$

A calculation shows that

$$s_n(x) = \begin{cases} \frac{c_n}{1+2\gamma}(x^{1+2\gamma} - 1) + 1 & \text{for } x \geq \gamma/n \\ \frac{c_n \gamma^{2\gamma}}{2n^{1+2\gamma}} \left(e^{2n(x-\frac{\gamma}{n})} - 1 \right) + \frac{c_n}{1+2\gamma} \left((\frac{\gamma}{n})^{1+2\gamma} - 1 \right) + 1 & \text{for } x \leq \gamma/n, \end{cases}$$

where

$$c_n = \left(\frac{\gamma^{2\gamma}}{2n^{1+2\gamma}} \left(1 - e^{-2\gamma}\right) + \frac{1}{1+2\gamma} \left(1 - (\tfrac{\gamma}{n})^{1+2\gamma} \right) \right)^{-1}.$$

Clearly $\lim_{n\to\infty} c_n = 1 + 2\gamma = \frac{1}{1-\alpha}$ and so

$$\lim_{n\to\infty} s_n(x) = (x^+)^{1+2\gamma} := s(x).$$

The convergence is uniform on compact subintervals of $\mathbf{R}$.

Define $X^n = s_n(Y^n)$. Recall that Y^n decreases with n. So X^n converges to some process $\widetilde{X}$. The function s_n is C^2 except at one point, and Y^n spends zero time at any point. So an easy approximation argument allows us to apply Ito's formula with the function s_n, and we obtain

$$dX_t^n = s_n' \circ s_n^{-1}(X_t^n) dW_t + s_n' \circ s_n^{-1}(X_t^n) d\widehat{L}_t^n. \tag{5.5}$$

Note that

$$s_n'(x) = \begin{cases} c_n x^{2\gamma} & \text{for } x \geq \gamma/n \\ \frac{c_n \gamma^{2\gamma}}{n^{2\gamma}} e^{2n(x-\frac{\gamma}{n})} & \text{for } x \leq \gamma/n. \end{cases}$$

Hence $s_n'(x)$ converges uniformly on compact subintervals of $\mathbf{R}$ to $(1 + 2\gamma)(x^+)^{2\gamma}$. It follows that

$$\lim_{n\to\infty} s_n' \circ s_n^{-1}(x) = (1 + 2\gamma) x^{2\gamma/(1+2\gamma)} = (1 + 2\gamma) x^\alpha$$

uniformly on compact subintervals of $[0, \infty)$. By truncating the processes at level $K > 0$, the same argument as that in (4.5)-(4.6) shows that there is a subsequence n_k such that process $\int_0^\cdot s_{n_k}' \circ s_{n_k}^{-1}(X_s^{n_k}) dW_s$ converges to the process $\int_0^\cdot (1 + 2\gamma) X_s^\alpha dW_s$ uniformly on compact subsets of $[0, \infty)$. Hence

$$\widetilde{L}_t := \lim_{k\to\infty} \int_0^t s_{n_k}'(Y_s^{n_k}) d\widehat{L}_s^n$$

exists and is finite a.s. Passing to the limit in (5.5), we have

$$\widetilde{X}_t = y_0 + (1 + 2\gamma) \int_0^t \widetilde{X}_s^\alpha dW_t + \widetilde{L}_t \quad \text{for } t \geq 0. \tag{5.6}$$

Since $t \mapsto \int_0^t s_n'(Y_s^n) d\widetilde{L}_s^n$ increases only when $Y_t^n = 0$, or equivalently, only when $X_t^n = 0$, we conclude that $t \mapsto \widetilde{L}_t$ increases only when $\widetilde{X}_t = 0$. Since $t \mapsto \int_0^t X_s^\alpha dW_s$ is continuous, by Lemma 3.1 in Burdzy et al. (2004) on the uniqueness of the deterministic Skorokhod decomposition, both $\widetilde{X}_t$ and $\widetilde{L}_t$ are continuous in $t \geq 0$.

We now show that $\widetilde{X}$ spends zero time at 0. Note that $s_n(x) \in (-\frac{c_n \gamma^{2\gamma}}{2n^{1+2\gamma}} e^{-2\gamma}, \infty)$, Let

$$\begin{aligned} a_n(x) &:= s_n' \circ s_n^{-1}(x) \\ &= \begin{cases} c_n \Big(\frac{1+2\gamma}{c_n}(x-1) + 1 \Big)^{2\gamma/(1+2\gamma)} & \text{for } x \geq c_n(\frac{\gamma}{n})^{2\gamma} \\ 2n(x + \frac{c_n \gamma^{2\gamma}}{2n^{1+2\gamma}} e^{-2\gamma}) & \text{for } -\frac{c_n \gamma^{2\gamma}}{2n^{1+2\gamma}} e^{-2\gamma} < x \leq c_n(\frac{\gamma}{n})^{2\gamma}. \end{cases} \end{aligned}$$

A direct calculation shows that

$$\lim_{\varepsilon \to 0} \left[\sup_{n \geq 1} \int_0^\varepsilon \frac{1}{a_n(x)^2} \, dx \right] = 0.$$

Let $T_K^n = \inf\{t \geq 0 : X_t^n > K\}$. Since

$$X_t^n = y_0 + \int_0^t a_n(X_s^n) dW_s + \widehat{L}_t^n \quad \text{for } t \geq 0,$$

which spends zero time at 0 and $\widehat{L}^n$ increases only when $X_t^n = 0$, we see that X^n is on natural scale and by the same argument as that for Theorem IV.3.2 and Section IV.4 in Bass (1997), its speed measure is $a_n(x)^{-1} dx$ on $\mathbf{R}_+$. Let $\tau_{(0,K)}^n = \inf\{t \geq 0 : X_t^n \notin (0,K)\}$. Since the solution to (5.4) is unique for each y_0, then $(Y^n, \mathbf{P}^y)$ is a strong Markov process, where $\mathbf{P}^y$ denotes the probability when we start Y^n from the point y. Since s_n is one to one, it follows that $(X^n, \mathbf{P}^x)$ is also a strong Markov process, where here $\mathbf{P}^x$ denotes the probability when we start X^n from x. Applying the strong Markov property at the time $\tau_{(0K)}^n$,

$$\mathbf{E}^{y_0} \left[\int_0^{T_K^n} 1_{[0,\varepsilon)}(X_s^n) \, ds \right] \leq \mathbf{E}^{y_0} \left[\int_0^{\tau_{(0,K)}^n} 1_{[0,\varepsilon)}(X_s^n) \, ds \right] + \mathbf{E}^0 \left[\int_0^{T_k^n} 1_{[0,\varepsilon)}(X_s^n) \, ds \right]$$

By Corollary IV.2.4 from Bass (1997), when $y_0 > 0$,

$$\mathbf{E}^{y_0} \left[\int_0^{\tau_{(0,K)}^n} 1_{[0,\varepsilon)}(X_s^n) \, ds \right] = \int_0^\varepsilon G_{0,K}(y_0, y) a_n(y)^{-2} dy,$$

where

$$G_{0,K}(y_0,y)=\begin{cases}\frac{2y_0(K-y)}{K} & \text{for} \quad 0<y_0<y<K\\ \frac{2y(K-y_0)}{K} & \text{for} \quad 0<y<y_0<K.\end{cases}$$

By (VI.4.4) of Bass (1997),

$$\mathbf{E}^0\left[\int_0^{T_k} 1_{[0,\varepsilon)}(X_s^n)\,ds\right]=\int_0^{\varepsilon} 2(K-y)a_n(y)^{-2}dy.$$

Thus there is a constant $c_K>0$ independent of $n\geq 1$ such that

$$\mathbf{E}^{y_0}\left[\int_0^{T_K^n} 1_{[0,\varepsilon)}(X_s^n)\,ds\right]\leq c_K\int_0^{\varepsilon} a_n(y)^{-2}dy,$$

and therefore

$$\lim_{\varepsilon\to\infty}\sup_{n\geq 1}\mathbf{E}^{y_0}\left[\int_0^{T_K^n} 1_{[0,\varepsilon)}(X_s^n)\,ds\right]=0.$$

We deduce, by letting $n\to\infty$, and then $\varepsilon\to 0$, that the amount of time that $\widetilde{X}$ spends at 0 up to time T_K has zero Lebesgue measure. We then let $K\to\infty$.

Define $X_t:=(1+2\gamma)^{\alpha-1}\widetilde{X}_t$ and $L_t:=(1+2\gamma)^{\alpha-1}\widetilde{L}_t$. Then

$$X_t=x_0+\int_0^t X_s^\alpha dW_s+L_t \quad \text{for } t\geq 0, \tag{5.7}$$

$X_t\geq 0$ for all $t\geq 0$ and X spends zero time at 0. Applying Tanaka's formula to X and using the fact that X spends zero time at 0, we have

$$\begin{aligned} X_t &= (X_t)^+ = x_0+\int_0^t 1_{\{X_s>0\}}dX_s+\tfrac{1}{2}L_t^0(X)\\ &= x_0+\int_0^t X_s^\alpha dW_s+\tfrac{1}{2}L_t^0(X).\end{aligned}$$

Comparing this with (5.7) yields $L_t=\frac{1}{2}L_t^0(X)$. Since $X_t\geq 0$ for all $t\geq 0$, $\widehat{L}_t^0(X)=\frac{1}{2}L_t^0(X)$. Hence

$$X_t=x_0+\int_0^t X_s^\alpha dW_s+\widehat{L}_t^0(X).$$

This proves the theorem. □

THEOREM 5.2. *Weak existence for the SDE (5.3) and weak uniqueness for the SDE (5.7) hold among the class of solutions for which the time spent at 0 has zero Lebesgue measure.*

PROOF. Let $Z = x_0 + W + L$ be a reflecting Brownian motion on $[0, \infty)$ starting from x_0. It is well-known that Z has the same distribution as $|x_0 + W|$. Since $\alpha \in (0, 1/2)$, by a result of Engelbert and Schmidt (see Proposition 3.6.27 in Karatzas and Shreve, 1994), $\int_0^t \frac{1}{|x_0+W_s|^{2\alpha}} ds < \infty$ for every $t > 0$. Thus we have $A_t := \int_0^t \frac{1}{Z_s^{2\alpha}} ds < \infty$, a.s. for every $t > 0$. Define $\tau_t = \inf\{s : A_s \geq t\}$. Then $X_t := Z_{\tau_t}$ spends zero time at 0 and has the decomposition

$$X_t = x_0 + \int_0^t X_s^\alpha dB_s + \widetilde{L}_t \quad \text{for } t \geq 0,$$

where $B = \{B_t, t \geq 0\}$ with

$$B_t := \int_0^t Z_{\tau_s}^{-\alpha} dW_{\tau_s} = \int_0^{\tau_t} Z_s^{-\alpha} dW_s$$

is a Brownian motion and $\widetilde{L}_t := L_{\tau_t}$ is a non-decreasing process that increases only when $X_t = 0$. By an argument similar to that of the proof of Theorem 5.1, it can be shown that $\widetilde{L}$ is the symmetric local time of X at level 0. Therefore, we conclude that X is a weak solution to the SDE (5.3) that spends zero Lebesgue amount of time at 0. (In fact, as mentioned at the beginning of this section, SDE (5.1) is equivalent to SDE (5.2).)

Conversely, suppose

$$X_t = x_0 + \int_0^t (X_s)^\alpha dW_s + L_t \quad \text{for } t \geq 0,$$

with $X_t \geq 0$, X spends zero time at 0, and L_t is a continuous non-decreasing function that increases only when X_t is at 0. Then

$$C_t = \int_0^t X_s^{2\alpha} ds < \infty$$

a.s. and is strictly increasing for $t > 0$. Let

$$\sigma_t := \inf\{s \geq 0 : C_s \geq 0\},$$

which is continuous in t. Then

$$Z_{\sigma_t} = x_0 + \int_0^{\sigma_t} X_s^\alpha dW_s + L_{\sigma_t}, \qquad t \geq 0.$$

Clearly $\widehat{W}_t = \int_0^{\sigma_t} X_s^\alpha dW_s$ is a Brownian motion since it is a continuous local martingale with $\langle W \rangle_t = t$ and $t \to L_{\sigma_t}$ is non-decreasing and increases only when $Z_{\sigma_t} = 0$. Hence $X_t = Z_{\sigma_t}$ is a reflecting Brownian motion on $[0, \infty)$ starting from x_0. Thus we have established that there is a one-to-one correspondence between weak solutions to (5.7) that spend zero time at 0 and weak solutions to the SDE for reflecting Brownian motion on $[0, \infty)$. The conclusion of the theorem follows since weak existence and weak uniqueness hold for reflecting Brownian motion on $[0, \infty)$. □

THEOREM 5.3. *Pathwise uniqueness holds among the class of solutions of (5.2) for which the time spent at 0 has Lebesgue measure 0.*

PROOF. Suppose (X, W, L) is a weak solution to (5.7) such that X spends zero time at 0. Since $d\langle X \rangle_t$ is absolutely continuous with respect to dt, we see that

$$W_t = \int_0^t (X_s)^{-\alpha} 1_{\{X_s \neq 0\}} dX_s.$$

Since $L_t = X_t - x_0 - \int_0^t (X_s)^\alpha dW_s$, then W and L are measurable with respect to the filtration generated by X. Now if (X^i, W^i, L^i), $i = 1, 2$, are two weak solutions to (5.7), then the law of X^1 is equal to the law of X^2, and with the above that tells us that the joint law of (X^1, W^1, L^1) is equal to the joint law of (X^2, W^2, L^2).

Let W be a given Brownian motion and let X' be the strong solution to (5.3) given in Theorem 5.1. So there exists a Borel measurable function H from $C([0, \infty)$ to $C([0, \infty)$ such that $X' = H(W)$. If X is a pathwise solution to (5.7) driven by W, then by weak uniqueness the law of X equals the law of X', and by the preceding paragraph the joint law of (X, W) is equal to the joint law of (X', W). Since $\mathbf{P}(X' \neq H(W)) = 0$, it follows that $\mathbf{P}(X \neq H(W)) = 0$, and we therefore have $X = H(W) = X'$ almost surely. □

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