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Two-step Regression Quantiles

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Abstract

We propose a new version of the regression α-quantile in the linear regression model, ordering the residuals with respect to an initial R-estimate of the slope parameter. In this way we obtain a consistent estimator of $(\beta_0 + F^{-1}(\alpha), \beta_1, \ldots, \beta_p)'$, asymptotically equivalent to the regression α-quantile of Koenker and Bassett. The result is extended to the extreme regression quantiles. Similarly we construct a version of the autoregression quantile in the linear AR(p) model. We also propose an estimate of the extreme error in the linear regression and autoregression models, using the initial R-estimate of the slope. A simulation experiment illustrates a very small difference between the original regression quantiles and their new versions, and a very good approximation of the extreme errors.

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1 Introduction

Consider the linear regression model

$$\mathbf{Y} = \beta_0 \mathbf{1}_n + \mathbf{X}\boldsymbol{\beta} + \mathbf{E} \tag{1.1}$$

with observations $\mathbf{Y} = (Y_1, \ldots, Y_n)'$, i.i.d. errors $\mathbf{E} = (E_1, \ldots, E_n)'$ with an unknown distribution function F, and unknown parameter $\boldsymbol{\beta}^* = (\beta_0, \beta_1, \ldots, \beta_p)'$. The $n \times p$ matrix $\mathbf{X} = \mathbf{X}_n$ is known and $\mathbf{1}_n = (1, \ldots, 1)' \in \mathbb{R}^n$.

A natural idea for extending quantiles and L-estimates to model (1.1) is to order residuals with respect to an initial estimate of $\boldsymbol{\beta}^*$. Such are the estimates proposed by Bickel (1973) and by Ruppert and Carroll (1980), among others. Bickel's two-step L-estimator has good asymptotic properties, but it is not invariant to a reparametrization separating the quantile functions of the components. The asymptotic distribution of two-step estimator of Ruppert and Carroll, constructed as a regression analog to the trimmed mean, heavily depends on the initial estimator. Neither of these estimating procedures leads to a direct extension of the sample quantiles.

Welsh (1987a, 1987b) constructed two-step regression analogs of the trimmed mean and of more general L-estimators; his estimators are based on the known or estimated shape of the pertaining influence function, and cover a two-step version of the regression α-quantile of Koenker and Bassett (1978) as a special case. Because the influence function of the α-regression quantile contains density quantile function $f(F^{-1}(\alpha))$, Welsh estimated it consistently using the empirical quantile function of residuals of $Y_1, \ldots, Y_n$ from an initial estimate of $\boldsymbol{\beta}^*$. Because he did not provide any numerical evidence, while it is of interest, we have simulated (Table 1.1) the Welsh estimate of $f(F^{-1}(\alpha))$ for $\alpha = 0.01,\ 0.1,\ 0.5$ and for regression matrices of orders 20×3 and 500×3, both with the first column of 1's, while the 2nd and 3rd columns are independent samples from the uniform $R(-10, 10)$ distribution. The true parameter value is $\boldsymbol{\beta} = (5, -1, 2)'$, and errors are simulated 1000-times from the standard normal, Cauchy and logistic distributions. The true values of $f(F^{-1}(\alpha))$ are also tabulated for a comparison. The values do not show a high stability of the estimate of $f(F^{-1}(\alpha))$.

Table 1.1. The Welsh estimates of $f(F^{-1}(\alpha))$ for $\alpha = 0.01,\ 0.1,\ 0.5$.

Distribution	Normal			Cauchy			Logistic		
α	0.01	0.1	0.5	0.01	0.1	0.5	0.01	0.1	0.5
$n = 20$	0.296	0.382	0.507	0.076	0.136	0.301	0.165	0.219	0.305
$n = 500$	0.200	0.286	0.374	0.011	0.081	0.252	0.099	0.162	0.230
true value	0.027	0.175	0.399	0.000	0.030	0.318	0.010	0.090	0.250

Furthermore, the values of Welsh' two-step linear model trimmed means with 10% trimming are compared to 0.5th regression quantiles and two-step regression quantiles in Figures 5.4 and 5.5.

Another version of the two-step α-regression quantile, very well approximating $\widehat{\boldsymbol{\beta}}^*(\alpha)$ even for moderate sample sizes, is constructed in the present paper. We use a suitably chosen R-estimator $\widehat{\boldsymbol{\beta}}_{nR}$ (rank-estimator) as the initial estimate of $\boldsymbol{\beta}$, that itself is very close to the slope component of the α-regression quantile, and then order the residuals. Because the ranks are invariant to the shift, the R-estimator automatically estimates only the slope parameter in model (1.1). The $[n\alpha]$-order statistic $\tilde{\beta}_{n0}$ of the residuals $Y_i - \mathbf{x}_i'\widehat{\boldsymbol{\beta}}_{nR},\ i = 1, \ldots, n$ very closely approximates $E_{n:[n\alpha]} + \beta_0$, where

$E_{n:1} \leq \ldots \leq E_{n:n}$ are the order statistics of errors $E_1, \ldots, E_n$. The whole vector $(\tilde{\beta}_{n0}, \widehat{\boldsymbol{\beta}}_{nR}')'$ very closely approximates the regression quantile $\widehat{\boldsymbol{\beta}}_n^*(\alpha)$, while it avoids an estimation of $f(F^{-1}(\alpha))$. In this way we can also estimate the extreme errors $E_{n:1}, E_{n:n}$, which is important in many applications.

In the following text, $\widehat{\boldsymbol{\beta}}_n^*(\alpha) = (\hat{\beta}_{n0}(\alpha), \hat{\beta}_{n1}(\alpha), \ldots, \hat{\beta}_{np}(\alpha))'$ denotes the *regression α-quantile*, which is a solution of the minimization problem

$$\sum_{i=1}^{n} \rho_\alpha(Y_i - b_0 - \mathbf{x}_i'\mathbf{b}) := \min \tag{1.2}$$

with respect to $(b_0, b_1, \ldots, b_p)' \in I\!R^{p+1}$, where $\mathbf{x}_i'$ is the i-th row of $\mathbf{X}$ and $\rho_\alpha(x) = |x|\{\alpha I[x > 0] + (1-\alpha)I[x < 0]\},\ x \in I\!R^1$. The population counterpart of $\widehat{\boldsymbol{\beta}}_n^*(\alpha)$ is the vector $\boldsymbol{\beta}^*(\alpha) = (\beta_0 + F^{-1}(\alpha), \beta_1, \ldots, \beta_p)'$, and the stochastic order of $\|\widehat{\boldsymbol{\beta}}_n^*(\alpha) - \boldsymbol{\beta}^*(\alpha)\|$ is $\mathcal{O}_p(n^{-\frac{1}{2}})$ under general conditions on $\{\mathbf{X}_n\}$ and on F. A genuine analog of the α-trimmed mean is the *trimmed least squares estimator,* ordering and consequently trimming the residuals with respect to the αth and the $(1-\alpha)$th regression quantiles; the resulting estimator is the least squares estimator from the untrimmed observations.

If derivative $f(\cdot)$ of F exists in a neighbourhood of the quantile $F^{-1}(\alpha)$, and if the matrix $\mathbf{D}_n^* = n^{-1}\sum_{i-1}^n \mathbf{x}_i^*\mathbf{x}_i^{*\prime}$ is positive definite starting with some n (where we denote $\mathbf{x}_i^* = (1, x_{i1}, \ldots, x_{in})'$ for the sake of brevity), then $n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_n^*(\alpha) - \boldsymbol{\beta}^*(\alpha))$ admits the asymptotic representation

$$n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_n^*(\alpha) - \boldsymbol{\beta}^*(\alpha)) = \frac{1}{n^{\frac{1}{2}} f(F^{-1}(\alpha))}(\mathbf{D}_n^*)^{-1}\sum_{i=1}^{n} \mathbf{x}_i^*(\alpha - I[E_i < F^{-1}(\alpha)]) + o_p(1) \tag{1.3}$$

as $n \to \infty$. We refer to Gutenbrunner and Jurečková (1992) for the detailed conditions and proofs of (1.3) and similar representations. The intercept part of the representation (1.3) can be rewritten as

$$\hat{\beta}_{n0}(\alpha) = F^{-1}(\alpha) + \beta_0 + (nf(F^{-1}(\alpha))^{-1}\sum_{i=1}^{n}(\alpha - I[E_i < F^{-1}(\alpha)]) + o_p(n^{-\frac{1}{2}}) \tag{1.4}$$

as $n \to \infty$. It means that the first component of the regression quantile represents $F^{-1}(\alpha)$, up to an unknown intercept β_0, and (1.4) is in fact the Bahadur representation of the sample α-quantile applied to $\hat{\beta}_{n0}(\alpha) - \beta_0$. The representation (1.3) further implies that $\hat{\beta}_{n1}(\alpha), \ldots, \hat{\beta}_{np}(\alpha)$ are consistent estimates of the slope components $\beta_1 \ldots, \beta_p$, and that they are asymptotically independent of the intercept component $\hat{\beta}_{n0}(\alpha)$. Hence, the inference about quantile $F^{-1}(\alpha)$ can be based only on the first component $\hat{\beta}_{n0}(\alpha)$ of the regression quantile, considering the other components as nuisance. An example is the kernel estimate of the density quantile function $f(F^{-1}(\alpha))$ of the errors in model (1.1), based on $\hat{\beta}_{n0}(\alpha)$, constructed in Dodge and Jurečková (1995). Another example is the difference $\hat{\beta}_{n0}(1-\alpha) - \hat{\beta}_{n0}(\alpha)$ that provides a studentizing scale statistic for model (1.1), scale equivariant and invariant to the slope parameters. The first component of the regression interquartile range was used in Jurečková, Picek and Sen (2003) in a construction of the goodness-of-fit test in the presence of nuisance regression and scale parameters.

We obtain another estimate of $F^{-1}(\alpha)+\beta_0$, asymptotically equivalent to $\hat{\beta}_{n0}(\alpha)$, starting with an appropriate R-estimate (rank-estimate) of the slope parameters, and then calculating the $[n\alpha]$-order statistic of the corresponding residuals. This further leads to the two-step version $\tilde{\boldsymbol{\beta}}_n^*(\alpha)$ of the α-regression quantile in the model (1.1), asymptotically equivalent to $\widehat{\boldsymbol{\beta}}_n^*(\alpha)$, as $n \to \infty$. The score-generating function of the R-estimate is related to the function ρ_α in (1.2); hence, unlike in other two-step estimators mentioned above, the score function of the initial estimate depends on α.

More precisely, if $\widehat{\boldsymbol{\beta}}_{nR}$ is the initial R-estimator of the slope parameter $(\beta_1, \ldots, \beta_p)'$ and $\tilde{\beta}_{n0}$ is the $[n\alpha]$-th order statistic of the residuals $Y_i - \mathbf{x}_i'\widehat{\boldsymbol{\beta}}_{nR}$, $i = 1, \ldots, n$, then $\tilde{\beta}_{n0}$ is an analog of $E_{n:[n\alpha]} + \beta_0$ where $E_{n:1} \leq \ldots \leq E_{n:n}$ are the order statistics of unobservable errors $E_1, \ldots, E_n$. Moreover, $\widehat{\boldsymbol{\beta}}_{nR}$ and $\tilde{\beta}_{n0}$ are asymptotically independent, and the whole $(p+1)$-dimensional vector $\left(\tilde{\beta}_{n0}, \widehat{\boldsymbol{\beta}}_{nR}'\right)'$ is asymptotically equivalent to the α-regression quantile $\widehat{\boldsymbol{\beta}}_n^*(\alpha)$ in (1.2).

A similar approach applies also to the AR(p) model. This two-step form of regression quantile makes the structure of the regression quantile more transparent; it facilitates the applications, namely when the quantile $F^{-1}(\alpha)$ is our main interest. It also facilitates the study of asymptotic properties of L-estimators in the linear regression and autoregression models. Furthermore, an analogous two-step approach with preliminary R-estimate of the slopes enables to estimate the extreme of unobservable errors $E_1, \ldots, E_n$, and in turn to make an inference on the tail index of F.

The simulation study illustrates that the two-step form of the regression quantile is numerically very close, for various distributions of errors, to the original regression quantile.

Section 2 describes the construction of the two-step regression quantile. Section 3 constructs the two-step autoregression quantiles, extending the approach to the linear autoregressive model. The estimation of the extremes of errors $E_1,\ldots,E_n$ and the two-step extreme regression quantiles are described in Section 4. The simulation study is contained in Section 5.

2 Linear Regression Model

Consider the model (1.1) and assume that the distribution function $F(x)$ is increasing on the set $\{x : 0 < F(x) < 1\}$. For any fixed $\alpha \in (0,1)$, denote $E_{i\alpha} = E_i - F^{-1}(\alpha),\ i = 1,\ldots,n$. Then $E_{1\alpha},\ldots,E_{n\alpha}$ are i.i.d. random variables with distribution function $F_\alpha(x) = F(x + F^{-1}(\alpha)),\ x \in I\!R$, and $F_\alpha^{-1}(u) = F^{-1}(u) - F^{-1}(\alpha),\ 0 < u < 1$, so that $F_\alpha^{-1}(\alpha) = 0$. Rewrite the model (1.1) in the following way:

$$Y_{ni} = \beta_0(\alpha) + \mathbf{x}'_{ni}\boldsymbol{\beta} + E_{i\alpha},\ i = 1,\ldots,n \tag{2.1}$$

with $\beta_0(\alpha) = \beta_0 + F^{-1}(\alpha)$. We shall omit the subscript n whenever it does not cause a confusion. The α-regression quantile for the reparametrized model (2.1) is then a solution of the minimization

$$\sum_{i=1}^{n} \rho_\alpha(Y_i - b_0(\alpha) - \mathbf{x}'_i\mathbf{b}) := \min, \quad b_0(\alpha) \in I\!R^1,\ \mathbf{b} \in I\!R^p. \tag{2.2}$$

Notice that the solution $(\hat\beta_{n0}(\alpha), \widehat{\boldsymbol{\beta}}'(\alpha))'$ of (2.2) is an M-estimator of the parameter $(\beta_0(\alpha), \boldsymbol{\beta}')'$.

Our main results will be proved under the following conditions on F and on $\mathbf{X}_n$:

(A1) F has a continuous density f that is positive on the support of F and has finite Fisher's information, i.e.

$$0 < \int \left(\frac{f'(x)}{f(x)}\right)^2 dF(x) < \infty.$$

(A2) (*Generalized Noether condition.*) $\lim_{n\to\infty} \max_{1\le i\le n} \mathbf{x}'_{ni} \left(\sum_{k=1}^n \mathbf{x}_{nk}\mathbf{x}'_{nk}\right)^{-1} \mathbf{x}_{ni} = 0.$

(A3) $\lim_{n\to\infty} n^{-1} \sum_{i=1}^n \mathbf{x}^*_{ni}\mathbf{x}^{*\prime}_{ni} = \mathbf{D}^*$, where $\mathbf{x}^*_{ni} = (1, x_{n,i1},\ldots,\mathbf{x}_{n,ip})',\ i = 1,\ldots,n$, and $\mathbf{D}^*$ is a positive definite $(p+1)\times(p+1)$ matrix.

(A4) $\sum_{i=1}^n x_{n,ij} = 0,\ j = 1,\ldots,p, \quad n > n_0.$

Remark 2.1. (i) Condition **(A1)** guarantees that the quantiles are well defined and together with **(A2)**, it guarantees that the R-estimators are

asymptotically normally distributed (see, e.g., Heiler and Willers, 1988).
(ii) Conditions **(A3)** and **(A4)**: The ranks, and hence also the R-estimators, are invariant with respect to the shift of observations by a constant. If $\sum_{i=1}^{n} \mathbf{x}_{ni} \neq \mathbf{0}$, then the R-estimator of the slope components $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_n)'$ has the representation (compare with (2.11))

$$n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_{nR}(\alpha)-\boldsymbol{\beta})=\frac{1}{n^{\frac{1}{2}}f(F^{-1}(\alpha))}\bar{\mathbf{D}}_n^{-1}\sum_{i=1}^{n}(\mathbf{x}_i-\bar{\mathbf{x}}_n)(\alpha-I[E_i<F^{-1}(\alpha)])+o_p(1), \tag{2.3}$$

where $\bar{\mathbf{D}}_n = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_{ni}-\bar{\mathbf{x}}_n)(\mathbf{x}_{ni}-\bar{\mathbf{x}}_n)'$, $\bar{\mathbf{x}}_n = n^{-1}\sum_{i=1}^{n}\mathbf{x}_{ni}$. By condition **(A3)**,

$$\lim_{n\to\infty}\bar{\mathbf{x}}_n = \mathbf{d}_1^*,$$

where $\mathbf{d}_1^*$ is the first column of the limiting matrix $\mathbf{D}^*$. Then, to obtain **(A4)**, we can reparametrize model (1.1) in the following way:

$$Y_i = \breve{\beta}_0 + \breve{\mathbf{x}}_i'\boldsymbol{\beta} + E_i, \quad i=1,\ldots,n \tag{2.4}$$

where $\breve{\mathbf{x}}_i = \mathbf{x}_i - \bar{\mathbf{x}},\ i=1,\ldots,n$ and $\breve{\beta}_0 = \beta_0 + \mathbf{d}_1^{*\prime}\boldsymbol{\beta}$.

Denote by ψ_α the right-hand derivative of ρ_α, i.e. $\psi_\alpha(x) = \alpha - I[x < 0],\ x \in I\!R$. Using Lemma A.2 in Ruppert and Carroll (1980), we can show that, as $n\to\infty$,

$$\begin{aligned} n^{-\frac{1}{2}}\textstyle\sum_{i=1}^{n}\left(\alpha - I\left[Y_i < \hat{\beta}_{n0}(\alpha)+\mathbf{x}_i'\widehat{\boldsymbol{\beta}}(\alpha)\right]\right) &\to 0 \quad \text{a.s.} \\ \mathbf{D}_n^{-\frac{1}{2}}\textstyle\sum_{i=1}^{n}\mathbf{x}_i\left(\alpha - I\left[Y_i < \hat{\beta}_{n0}(\alpha)+\mathbf{x}_i'\widehat{\boldsymbol{\beta}}(\alpha)\right]\right) &\to 0 \quad \text{a.s.} \end{aligned} \tag{2.5}$$

Then the R-estimator of $\boldsymbol{\beta}$, generated by the score function

$$\varphi_\alpha(u) = \psi_\alpha(F_\alpha^{-1}(u)) = \alpha - I[u<\alpha],\ 0<u<1 \tag{2.6}$$

is asymptotically equivalent to $\widehat{\boldsymbol{\beta}}_n(\alpha)$ (see Jurečková, 1977). This R-estimator is constructed from the rank test statistics for regression in the manner analogous to Hodges and Lehmann (1963). Denote by $R_{ni}(\mathbf{Y}-\mathbf{X}\mathbf{b})$ the rank of $Y_i - \mathbf{x}_i'\mathbf{b}$ among $(Y_1-\mathbf{x}_1'\mathbf{b},\ldots,Y_n-\mathbf{x}_n'\mathbf{b})$, $\mathbf{b}\in I\!R^p,\ i=1,\ldots,n$. Then $R_{ni}(\mathbf{Y}-\mathbf{X}\mathbf{b})$ is also the rank of $Y_i - b_0(\alpha) - \mathbf{x}_i'\mathbf{b}$ among $(Y_1-b_0(\alpha)-\mathbf{x}_1'\mathbf{b},\ldots,$ $Y_n-b_0(\alpha)-\mathbf{x}_n'\mathbf{b})$ because the ranks are translation invariant. Consider the vector $\mathbf{S}_n(\mathbf{b}) = (S_{n1}(\mathbf{b}),\ldots,S_{np}(\mathbf{b}))'$ of *linear rank statistics*, where

$$S_{nj}(\mathbf{b}) = \sum_{i=1}^{n} x_{ij}\varphi_\alpha\left(\frac{R_{ni}(\mathbf{Y}-\mathbf{X}\mathbf{b})}{n+1}\right), \quad \mathbf{b}\in I\!R^p, \quad j=1,\ldots,p. \tag{2.7}$$

Because $\mathbb{E}\mathbf{S}_n(\mathbf{0}) = \mathbf{0}$, the R-estimator of $\boldsymbol{\beta}$ can be defined in several alternative ways, all identifying $\widehat{\boldsymbol{\beta}}_{nR}$ with $\mathbf{b}$ leading $\mathbf{S}_n(\mathbf{b})$ as close to $\mathbf{0}$ as possible. These possible definitions are asymptotically equivalent. Either,

$$\widehat{\boldsymbol{\beta}}_{nR} = \operatorname{argmin}_{\mathbf{b}\in R^p} \|\mathbf{S}_n(\mathbf{b})\|_1 \tag{2.8}$$

(Jurečková, 1971), where $\|\mathbf{S}\|_1 = \sum_{j=1}^p |S_j|$ is the L_1 norm of $\mathbf{S}$; or,

$$\widehat{\boldsymbol{\beta}}_{nR} = \operatorname{argmin}_{\mathbf{b}\in R^p} \mathcal{D}_n(\mathbf{b}), \tag{2.9}$$

where

$$\mathcal{D}_n(\mathbf{b}) = \sum_{i=1}^n (Y_i - \mathbf{x}_i'\mathbf{b})\varphi_\alpha\left(\frac{R_{ni}(\mathbf{Y}-\mathbf{X}\mathbf{b})}{n+1}\right) \tag{2.10}$$

is Jaeckel's measure of *rank dispersion* (Jaeckel, 1972). Koul (1971) proposed $\widehat{\boldsymbol{\beta}}_{nR}$ as a minimizer of a quadratic form of $\mathbf{S}_n(\mathbf{b})$. Notice that Jaeckel's criterion $\mathcal{D}_n(\mathbf{b})$ is convex in $\mathbf{b}$, and $\mathbf{S}_n(\mathbf{b})$ is a gradient of $\mathcal{D}_n(\mathbf{b})$. In any case, $\widehat{\boldsymbol{\beta}}_{nR}$ estimates only the slope parameter, and there is no need to estimate the intercept for its computation.

The solutions of (2.8) and (2.9) are generally not unique. If $\mathcal{B}_n$ is the set of all solutions, we can e.g., take the center of gravity of $\mathcal{B}_n$, or the expectation of a random vector uniformly distributed over $\mathcal{B}_n$, as the estimator of $\boldsymbol{\beta}$; however, the asymptotic representations and distributions apply to any point of $\mathcal{B}_n$ (see Jurečková, 1971).

Under the conditions **(A1)–(A4)**, the R-estimators (2.8) and (2.9) admit the following asymptotic ($n \to \infty$) representation, whose detailed proof can be found in Jurečková and Sen (1996).

$$n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_{nR} - \boldsymbol{\beta}) = \frac{1}{n^{\frac{1}{2}} f(F^{-1}(\alpha))}\mathbf{D}^{-1}\sum_{i=1}^n \mathbf{x}_i\left(\alpha - I[E_i < F^{-1}(\alpha)]\right) + o_p(n^{-1/4}), \tag{2.11}$$

where $\mathbf{D} = \lim_{n\to\infty}\mathbf{D}_n$, $\mathbf{D}_n = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_{ni}\mathbf{x}_{ni}'$. Then (2.11) coincides with the representation for the slope components $\widehat{\boldsymbol{\beta}}_n(\alpha) \in \mathbb{R}^p$ of the α-regression quantile (it is proved, e.g., in Gutenbrunner and Jurečková, 1992). Hence, under **(A1)–(A4)**,

$$n^{\frac{1}{2}}\|\widehat{\boldsymbol{\beta}}_{nR} - \widehat{\boldsymbol{\beta}}_n(\alpha)\| = o_p(1) \quad \text{as } n \to \infty. \tag{2.12}$$

Having estimated $\boldsymbol{\beta}$ by R-estimate $\widehat{\boldsymbol{\beta}}_{nR}$, consider the minimization problem

$$\sum_{i=1}^n \rho_\alpha(Y_i - b - \mathbf{x}_i'\widehat{\boldsymbol{\beta}}_{nR}) := \min \quad \text{with respect to} \quad b \in \mathbb{R}^1. \tag{2.13}$$

Its solution, denoted as $\tilde{\beta}_{n0}$, is the $[n\alpha]$-th order statistic of the residuals $Y_i - \mathbf{x}_i'\widehat{\boldsymbol{\beta}}_{nR},\ i = 1, \ldots, n$. Recalling Lemma A2 in Ruppert and Carroll (1980), we conclude that

$$a_n^{-1} \sum_{i=1}^{n} \Big(\alpha - I[Y_i < \tilde{\beta}_{n0} + \mathbf{x}_i'\widehat{\boldsymbol{\beta}}_{nR})]\Big) = o_p(1) \quad \text{as} \quad n \to \infty \tag{2.14}$$

for any sequence $\{a_n\},\ 0 < a_n \uparrow \infty$. The following theorem shows that $\tilde{\beta}_{n0}$ is a consistent estimate of $\beta_0 + F^{-1}(\alpha)$, asymptotically normally distributed, and the variance of its asymptotic distribution coincides with that of the sample α-quantile in the location model.

Theorem 2.1 *Under the conditions* **(A1)**–**(A4)**, *for any* $\alpha \in (0, 1)$,

$$\tilde{\beta}_{n0} \xrightarrow{p} F^{-1}(\alpha) + \beta_0 \ \textit{and}$$

$$\tilde{\beta}_{n0} - F^{-1}(\alpha) - \beta_0 = \frac{1}{nf(F^{-1}(\alpha))} \sum_{i=1}^{n} \big(\alpha - I[E_i < F^{-1}(\alpha)]\big) + o_p(n^{-\frac{1}{2}}), \tag{2.15}$$

as $n \to \infty$. *Hence,* $n^{\frac{1}{2}}(\tilde{\beta}_{n0} - F^{-1}(\alpha) - \beta_0)$ *is asymptotically normally distributed with zero expectation and with the variance*

$$\frac{\alpha(1-\alpha)}{(f(F^{-1}(\alpha)))^2}. \tag{2.16}$$

Proof. $\tilde{\beta}_{n0}$ is a solution of the minimization (2.13) with $\widehat{\boldsymbol{\beta}}_{nR}$ being the R-estimate of $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$, generated by the score function φ_α of (2.6). Under the conditions **(A1)**–**(A4)**, $\widehat{\boldsymbol{\beta}}_{nR} - \boldsymbol{\beta}$ admits the asymptotic representation (2.11), hence $n^{\frac{1}{2}} \|\widehat{\boldsymbol{\beta}}_{nR} - \boldsymbol{\beta}\| = \mathcal{O}_p(1)$ as $n \to \infty$. Moreover, by (2.14),

$$n^{-\frac{1}{2}} \sum_{i=1}^{n} \Big(\alpha - I[Y_i < \tilde{\beta}_{n0} + \mathbf{x}_i'\widehat{\boldsymbol{\beta}}_{nR})]\Big) = o_p(1) \tag{2.17}$$

as $n \to \infty$. Using the identity $\sum_{i=1}^{n} \mathbf{x}_i = \mathbf{0}$ (see **(A4)**), we conclude from Jurečková and Sen (1996), Section 4.7, that

$$\sup_{|b| \le C, \|\mathbf{b}\| \le C} \left\{ \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{n} \Big(I\Big[E_i < F^{-1}(\alpha) + \frac{b}{\sqrt{n}} + \frac{\mathbf{x}_i'\mathbf{b}}{\sqrt{n}}\Big] - I\Big[E_i < F^{-1}(\alpha) + \frac{b}{\sqrt{n}}\Big]\Big) \right| \right\} = o_p(1). \tag{2.18}$$

Replacing $\mathbf{b}$ by $n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_{nR}-\boldsymbol{\beta})=\mathcal{O}_p(1)$ into (2.18), we obtain

$$\sup_{|b|\le C}\left\{\frac{1}{\sqrt{n}}\left|\sum_{i=1}^{n}\left(I\left[Y_i-\beta_0<F^{-1}(\alpha)+\frac{b}{\sqrt{n}}+\mathbf{x}_i'\widehat{\boldsymbol{\beta}}_{nR}\right]\right.\right.\right.$$
$$\left.\left.\left.-I\left[E_i<F^{-1}(\alpha)+\frac{b}{\sqrt{n}}\right]\right)\right|\right\}=o_p(1). \quad (2.19)$$

Following the lines of analogous proofs in Jurečková (1977) or Jurečková and Sen (1996), Section 4.7, we can show that $n^{\frac{1}{2}}(\tilde{\beta}_{n0}-F^{-1}(\alpha)-\beta_0)=\mathcal{O}_p(1)$, as $n\to\infty$. Hence, replacing b by $n^{\frac{1}{2}}(\tilde{\beta}_{n0}-F^{-1}(\alpha)-\beta_0)$ in (2.19) and using (2.17), we obtain that

$$n^{-\frac{1}{2}}\sum_{i=1}^{n}\Big(\alpha-I[E_{i\alpha}<\tilde{\beta}_{n0}-F^{-1}(\alpha)-\beta_0]\Big)=o_p(1),$$

and this, together with the Bahadur representation of the sample quantiles (see Bahadur, 1966), implies that $\tilde{\beta}_{n0}-\beta_0$ estimates $F^{-1}(\alpha)$, and hence admits the representation (2.15). □

Let $\mathbf{D}_n^*=n^{-1}\sum_{i=1}^{n}\mathbf{x}_i^*\mathbf{x}_i^{*\prime}$. Then $\mathbf{D}_n^*\to\mathbf{D}^*$ by condition **(A3)** and, by condition **(A4)**,

$$\mathbf{D}^*=\begin{bmatrix}1 & \mathbf{0}'\\ \mathbf{0} & \mathbf{D}\end{bmatrix}. \quad (2.20)$$

The following theorem gives the asymptotic representation and asymptotic distribution of the sequence of random vectors

$$\Big(\tilde{\beta}_{n0}-\beta_0-F^{-1}(\alpha),(\widehat{\boldsymbol{\beta}}_{nR}-\boldsymbol{\beta})'\Big)'\in\mathbb{R}^{p+1}. \quad (2.21)$$

This asymptotic representation and asymptotic distribution are the same as those of the Koenker-Bassett α-regression quantile of the model (1.1). Thus, we shall call the vector

$$\tilde{\boldsymbol{\beta}}_n(\alpha)=\Big(\tilde{\beta}_{n0},\widehat{\boldsymbol{\beta}}_{nR}'\Big)'$$

the *two-step α-regression quantile.*

Theorem 2.2 *Under the conditions* **(A1)**–**(A4)**, *the random vector $\tilde{\boldsymbol{\beta}}_n(\alpha)$ admits the asymptotic representation*

$$\tilde{\boldsymbol{\beta}}_n(\alpha)-F^{-1}(\alpha)\mathbf{e}_1-(\beta_0,\beta_1\ldots,\beta_p)'$$
$$=\big(nf(F^{-1}(\alpha))\big)^{-1}(\mathbf{D}^*)^{-1}\sum_{i=1}^{n}\mathbf{x}_i^*\big(\alpha-I[E_i<F^{-1}(\alpha)]\big)+o_p(n^{-\frac{1}{2}}) \quad (2.22)$$

as $n \to \infty$, and the sequence

$$n^{\frac{1}{2}}\left(\tilde{\boldsymbol{\beta}}_n(\alpha) - F^{-1}(\alpha)\mathbf{e}_1 - (\beta_0, \beta_1 \ldots, \beta_n)'\right)$$

has asymptotic distribution

$$\mathcal{N}_{p+1}\left(\mathbf{0}, \frac{\alpha(1-\alpha)}{(f(F^{-1}(\alpha))^2}(\mathbf{D}^*)^{-1}\right). \tag{2.23}$$

Proof. The theorem follows from (2.11) and from Theorem 2.1. □

Corollary 2.1 *The asymptotic equivalence of two-step regression quantile to the Koenker-Bassett regression quantile remains valid under sequences of distributions of vector* $\mathbf{E} = (E_1, \ldots, E_n)$*, contiguous to the sequence* $\{\prod_{i=1}^n F(z_i)\}_{n=1}^{\infty}$. *Namely,*

$$n^{\frac{1}{2}}\|\widehat{\boldsymbol{\beta}}_n^*(\alpha) - \tilde{\boldsymbol{\beta}}_n(\alpha)\| \xrightarrow{p} 0 \quad as \quad n \to \infty \tag{2.24}$$

provided the distribution function of $(E_1, \ldots, E_n)'$ *is*

either **(i)**

$$\prod_{i=1}^n F(z_i - \Delta d_{ni}) \text{ where } \Delta > 0, \sum_{i=1}^n d_{ni} = 0, \sum_{i=1}^n d_{ni}^2 \to d^2 > 0 \text{ as } n \to \infty$$

(local regression alternative);

or **(ii)**

$$\prod_{i=1}^n F\big(z_i \mathrm{e}^{-\Delta d_{ni}}\big) \text{ where } \Delta > 0, \sum_{i=1}^n d_{ni} = 0, \sum_{i=1}^n d_{ni}^2 \to d^2 > 0 \text{ as } n \to \infty$$

(local heteroscedasticity).

Proof. The asymptotic representations of $\widehat{\boldsymbol{\beta}}_n^*(\alpha)$ and of $\tilde{\boldsymbol{\beta}}_n(\alpha)$ are valid, and hence coincide, also under contiguous alternatives. The sequences of distributions in (i) and (ii) are contiguous to $\{\prod_{i=1}^n F(z_i)\}_{n=1}^{\infty}$, as follows from LeCam's lemmas (see, e.g., Hájek and Šidák, 1967, Chapter 5). □

3 Linear Autoregression Model

Consider the AR(p) model, in which the observation X_t satisfies

$$X_t = \theta_1 X_{t-1} + \ldots + \theta_p X_{t-p} + \varepsilon_t, \quad t = 0, \pm 1, \pm 2, \ldots, \tag{3.1}$$

where $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_p)' \in I\!R^p$ is an unknown parameter and ε_t, $t = 0, \pm 1, \pm 2, \ldots$ are independent identically distributed *(i.i.d.)* random variables with distribution function F. Denote

$$\begin{aligned} \mathbf{Y}_{t-1} &:= (X_{t-1}, \ldots, X_{t-p})' \in I\!R^p, \\ & \qquad\qquad\qquad\qquad\qquad t = 1, \ldots, n \\ \mathbf{Y}^*_{t-1} &:= (1, X_{t-1}, \ldots, X_{t-p})' \in I\!R^{p+1}, \end{aligned} \tag{3.2}$$

and let $\mathcal{X}_n$ denote the $n \times p$ matrix whose t-th row is $\mathbf{Y}'_{t-1}$, and similarly let $\mathcal{X}^*_n$ denote the $n \times (p+1)$ matrix whose t-th row is $(\mathbf{Y}^*_{t-1})'$, $1 \le t \le n$.

The distribution function F is unknown; however, we assume that

(B1) $I\!E\varepsilon = 0, \qquad I\!E\varepsilon^2 < \infty$ and that

(B2) all roots of the equation $x^p - \theta_1 x^{p-1} - \ldots - \theta_p = 0$ are inside the unit circle.

Let $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}^*$ denote the matrices:

$$\boldsymbol{\Sigma}_n = n^{-1} \sum_{t=1}^{n} \mathbf{Y}_{t-1} \mathbf{Y}'_{t-1}, \qquad \boldsymbol{\Sigma}^*_n = n^{-1} \sum_{t=1}^{n} \mathbf{Y}^*_{t-1} (\mathbf{Y}^*_{t-1})'; \tag{3.3}$$

then there exist positive definite matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}^*$ of respective orders $p \times p$ and $(p+1) \times (p+1)$, such that

$$\boldsymbol{\Sigma}_n \xrightarrow{p} \boldsymbol{\Sigma} \text{ and } \boldsymbol{\Sigma}^*_n \xrightarrow{p} \boldsymbol{\Sigma}^* \quad \text{as} \quad n \to \infty. \tag{3.4}$$

The α*-autoregression quantile* $\widehat{\boldsymbol{\theta}}^*_n(\alpha)$ for the model (3.1) was first considered by Koul and Saleh (1995), and later it was studied by Hallin and Jurečková (1999). Analogously as in model (1.1), $\widehat{\boldsymbol{\theta}}^*_n(\alpha)$ is defined as a solution of the minimization problem

$$\sum_{t=1}^{n} \rho_\alpha(X_t - b_0 - \mathbf{b}'\mathbf{Y}_{t-1}) := \min, \quad (b_0, \mathbf{b}')' \in I\!R^{p+1} \tag{3.5}$$

Under **(A1)**, **(B1)**, **(B2)**, Koul and Saleh (1995) proved the following asymptotic representation for the α-autoregression quantile $\widehat{\boldsymbol{\theta}}_n^*(\alpha)$:

$$\widehat{\boldsymbol{\theta}}_n^*(\alpha) - (F^{-1}(\alpha), \theta_1, \ldots, \theta_p)' = \tag{3.6}$$
$$n^{-1}(\boldsymbol{\Sigma}^*)^{-1}(f(F^{-1}(\alpha)))^{-1} \sum_{t=1}^{n} \mathbf{Y}_{t-1}^* \left(\alpha - I[\varepsilon_t \le F^{-1}(\alpha)]\right) + o_p(n^{-\frac{1}{2}})$$

as $n \to \infty$, and the convergence is uniform over each subinterval $[\alpha^*, 1-\alpha^*] \subset (0,1)$.

Similarly as in the linear regression model, consider the R-estimator $\widehat{\boldsymbol{\theta}}_{nR}$ of $\boldsymbol{\theta}$ in model (3.1), generated by the score generating function φ_α defined in (2.8). Again, the R-estimator can be defined in several alternative ways. Let $R_{tb} = R_t(X_t - \mathbf{Y}_{t-1}'\mathbf{b})$ denote the rank of $X_t - \mathbf{Y}_{t-1}'\mathbf{b}$ among $X_1 - \mathbf{Y}_0'\mathbf{b}, \ldots, X_n - \mathbf{Y}_{n-1}'\mathbf{b},\ \mathbf{b} \in I\!R^p$. One version of R-estimator of $\boldsymbol{\theta}$ is defined with the aid of linear rank statistics

$$\begin{aligned} S_j(\mathbf{b}) &= n^{-1} \sum_{t=1}^{n} X_{t-j} \varphi_\alpha \left(\frac{R_{tb}}{n+1}\right),\ j = 1, \ldots, p \\ \mathbf{S}(\mathbf{b}) &= (S_1(\mathbf{b}), \ldots, S_p(\mathbf{b}))' \end{aligned} \tag{3.7}$$

as a solution of the minimization problem

$$\|\mathbf{S}(\mathbf{b})\| := \min, \quad \mathbf{b} \in I\!R^p. \tag{3.8}$$

This is a special case of R-estimator considered by Denby and Martin (1979) and later by Koul and Ossiander (1994). Another possibility is to minimize an analog of Jaeckel's (1972) measure of dispersion of rank residuals,

$$\widehat{\boldsymbol{\theta}} = \arg\min\{\mathcal{J}(\mathbf{b}) :\ \mathbf{b} \in I\!R^p\}, \tag{3.9}$$

where

$$\mathcal{J}(\mathbf{b}) := \sum_{t=1}^{n} \varphi_\alpha \left(\frac{R_{tb}}{n+1}\right) (X_t - \mathbf{Y}_{t-1}'\mathbf{b}). \tag{3.10}$$

Both forms (3.8) and (3.9) lead to asymptotically equivalent estimators.

It is shown in Koul and Ossiander (1994) that, under **(A1)**, **(B1)** and **(B2)**,

$$n^{-1} \sum_{t=1}^{n} (\mathbf{Y}_{t-1} - \overline{\mathbf{Y}})(\mathbf{Y}_{t-1} - \overline{\mathbf{Y}})' \xrightarrow{p} \tilde{\boldsymbol{\Sigma}} \quad \text{as} \ \ n \to \infty, \tag{3.11}$$

where $\tilde{\boldsymbol{\Sigma}}$ is a positive definite $p \times p$ matrix, and

$$\widehat{\boldsymbol{\theta}}_{nR} - \boldsymbol{\theta} = \frac{\tilde{\boldsymbol{\Sigma}}^{-1}(f(F^{-1}(\alpha)))^{-1}}{n} \sum_{t=1}^{n} (\mathbf{Y}_{t-1} - \overline{\mathbf{Y}})(\alpha - I[\varepsilon_t \leq F^{-1}(\alpha)]) + o_p\Big(\frac{1}{\sqrt{n}}\Big). \tag{3.12}$$

Hence, $n^{\frac{1}{2}}(\widehat{\boldsymbol{\theta}}_{nR} - \boldsymbol{\theta})$ has asymptotic distribution

$$\mathcal{N}_p\left(\mathbf{0}, \frac{\alpha(1-\alpha)}{(f(F^{-1}(\alpha))^2} \tilde{\boldsymbol{\Sigma}}^{-1}\right). \tag{3.13}$$

Let $\tilde{\theta}_{n0}$ be the solution of the minimization problem

$$\sum_{t=1}^{n} \rho_\alpha(Y_t - b - \mathbf{Y}'_{t-1}\widehat{\boldsymbol{\theta}}_{nR}) := \min, \quad b \in I\!R^1 \tag{3.14}$$

where $\widehat{\boldsymbol{\theta}}_{nR}$ is the R-estimate of $\boldsymbol{\theta}$. In other words, $\tilde{\theta}_{n0}$ is the $[n\alpha]$-th order statistic of the residuals $X_t - \mathbf{Y}'_{t-1}\widehat{\boldsymbol{\theta}}_{nR}$, $t = 1, \ldots, n$. Then, inserting $h(\mathbf{Y}_{t-1}) \equiv 1$ in Lemma 1.1 of Koul and Ossiander (1994), we obtain

$$\sup_{|b| \leq C} \Big\{ n^{-\frac{1}{2}} \Big| \sum_{t=1}^{n} \Big(I[X_t - \mathbf{Y}'_{t-1}\widehat{\boldsymbol{\theta}}_{nR} < F^{-1}(\alpha) + n^{-\frac{1}{2}} b] - I[\varepsilon_t < F^{-1}(\alpha) + n^{-\frac{1}{2}} b] \Big) \Big| \Big\} = o_p(1). \tag{3.15}$$

The following theorem shows that $\tilde{\theta}_{n0}$ is a consistent estimate of $F^{-1}(\alpha)$, which is asymptotically normally distributed. Moreover, the random vector $\left(\tilde{\theta}_{n0}, \widehat{\boldsymbol{\theta}}'_{nR}\right)'$ is a consistent and asymptotically normally distributed estimator of $(F^{-1}(\alpha), \theta_1, \ldots \theta_p)'$ in the AR(p) model (3.1). The first component of this estimator is asymptotically independent of the other components. We shall call this estimator the *two-step autoregression quantile*.

THEOREM 3.1 *(i) Under the conditions* **(A1)** *and* **(B1)**–**(B2)**, *for any* $\alpha \in (0, 1)$,

$$\tilde{\theta}_{n0} \xrightarrow{p} F^{-1}(\alpha) \text{ and}$$
$$\tilde{\theta}_{n0} - F^{-1}(\alpha) = \big(nf(F^{-1}(\alpha))\big)^{-1} \sum_{t=1}^{n} \big(\alpha - I[\varepsilon_t < F^{-1}(\alpha)]\big) + o_p(n^{-\frac{1}{2}}), \tag{3.16}$$

as $n \to \infty$. *Hence,* $n^{\frac{1}{2}}(\tilde{\theta}_{n0} - F^{-1}(\alpha))$ *is asymptotically normally distributed with zero expectation and with the variance*

$$\frac{\alpha(1-\alpha)}{(f(F^{-1}(\alpha)))^2}. \tag{3.17}$$

(ii) Let $\tilde{\mathcal{X}}_n^*$ *denote the* $n \times (p+1)$ *matrix whose* t*-th row is* $(1, (\mathbf{Y}_{t-1} - \overline{\mathbf{Y}}_{t-1})')$, $1 \le t \le n$ *and let* $\tilde{\mathbf{\Sigma}}_n^* = n^{-1}\tilde{\mathcal{X}}_n^*(\tilde{\mathcal{X}}_n^*)'$. *Then*

$$\tilde{\mathbf{\Sigma}}_n^* \xrightarrow{p} \tilde{\mathbf{\Sigma}}^* = \begin{bmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & \tilde{\mathbf{\Sigma}} \end{bmatrix}, \tag{3.18}$$

where $\tilde{\mathbf{\Sigma}}^*$ *is positive definite, and the random vector* $(\tilde{\theta}_{n0}, \widehat{\boldsymbol{\theta}}'_{nR})'$ *admits the asymptotic representation*

$$\begin{aligned} &(\tilde{\theta}_{n0}, \widehat{\boldsymbol{\theta}}'_{nR})' - (F^{-1}(\alpha), \boldsymbol{\theta})' \\ &= \big(nf(F^{-1}(\alpha)\big)^{-1} (\tilde{\mathbf{\Sigma}}^*)^{-1} \sum_{t=1}^{n} \mathbf{Y}_{t-1}^* \left(\alpha - I[\varepsilon_t < F^{-1}(\alpha)]\right) + o_p(n^{-\frac{1}{2}}) \end{aligned} \tag{3.19}$$

as $n \to \infty$. *The sequence*

$$n^{\frac{1}{2}} \begin{pmatrix} \tilde{\theta}_{n0} - F^{-1}(\alpha) \\ \widehat{\boldsymbol{\theta}}_{nR} - \boldsymbol{\theta} \end{pmatrix}$$

has asymptotic distribution

$$\mathcal{N}_{p+1}\left(\mathbf{0}, \frac{\alpha(1-\alpha)}{(f(F^{-1}(\alpha))^2}(\tilde{\mathbf{\Sigma}}^*)^{-1}\right). \tag{3.20}$$

4 Extreme Errors and Extreme Regression Quantiles

In this section we construct an estimator of the extreme error $E_{n:n}$ in model (1.1), and a two-step version of the extreme regression quantile, starting with a suitable R-estimator of $\boldsymbol{\beta}$.

Estimates of extreme errors provide a tool for inference on the tails of the distribution of the errors. For instance, the Pareto index of the distribution of errors can be estimated even in the presence of the nuisance regression or autoregression, using the Hill (or other) estimate based on estimated higher order quantiles. A hypothesis on the value of the Pareto index can also be tested. This, in turn, enables to detect an eventual upward trend of the white noise in the environmental or other time series.

We propose an estimate of $E_{n:n}$ in model (1.1), based on observations $Y_1, \ldots, Y_n$, and an estimate of $\varepsilon_{n:n}$ in model (3.1), based on observations $X_1, \ldots, X_n$; the regression and autoregression parameters are taken as nuisance. Our method can also be used for simultaneous estimation of several higher order statistics of $E_1, \ldots, E_n$.

We shall construct an estimate of $E_{n:n}$ in the linear regression model (1.1) under the following restriction on the matrix $\mathbf{X}$:

$$\max_{1 \le i \le n} \|\mathbf{x}_i\| = \mathcal{O}\left(n^{\frac{1}{2}-\delta}\right) \quad \text{as} \quad n \to \infty, \; 0 < \delta < \tfrac{1}{2}. \tag{4.1}$$

Let $\hat{E}_{n:n}$ be the maximum of the residuals,

$$\hat{E}_{n:n} = \max\{Y_1 - \mathbf{x}_1'\widehat{\boldsymbol{\beta}}_{nR}, \ldots, Y_n - \mathbf{x}_n'\widehat{\boldsymbol{\beta}}_{nR}\} \tag{4.2}$$

calculated with respect to an appropriate R-estimate $\widehat{\boldsymbol{\beta}}_{nR}$ of $\boldsymbol{\beta}$. Then $\hat{E}_{n:n}$ is a consistent estimate of $E_{n:n}+\beta_0$. Its rate of convergence and the asymptotic distribution are given in the following theorem:

THEOREM 4.1 *Assume that the distribution function F of errors in model (1.1) satisfies the conditions* **A1**–**A4**, $\mathbf{X}$ *satisfies (4.1) and*

$$F(x) < 1 \quad \text{for} \quad x \in \mathbb{R}. \tag{4.3}$$

Denote

$$\xi_n = F^{-1}\left(1 - \frac{1}{n}\right). \tag{4.4}$$

Let $\widehat{\boldsymbol{\beta}}_{nR}$ be an R-estimate of $\boldsymbol{\beta}$, generated by a fixed nondecreasing and square integrable score function $\varphi : (0,1) \mapsto \mathbb{R}$, independent of n, in the same manner as in (2.7) - (2.9).

(i) Then, for estimator $\hat{E}_{n:n}$ defined in (4.2), we have

$$|\hat{E}_{n:n} - E_{n:n} - \beta_0| = \mathcal{O}_p(n^{-\delta}) \quad \text{as} \quad n \to \infty. \tag{4.5}$$

(ii) If F belongs to the domain of attraction of the Gumbel distribution, then

$$\mathbb{P}\left\{nf(\xi_n)(\hat{E}_{n:n} - \beta_0 - \xi_n) \le t\right\} \to \exp\left\{-\mathrm{e}^{-t}\right\} \quad \text{as} \quad n \to \infty, \; t \in \mathbb{R}. \tag{4.6}$$

(iii) If F belongs to the domain of attraction of the Fréchet distribution and $1 - F(x) = x^{-m}L(x), \; m > 0$, with a slowly varying function L, then

$$\mathbb{P}\left\{\xi_n^{-1}(\hat{E}_{n:n} - \beta_0) \le t\right\} \to \exp\left\{-t^{-m}\right\} \quad \text{as} \quad n \to \infty, \; t > 0. \tag{4.7}$$

Proof. Let $D_1, \ldots, D_n$ denote the antiranks of $E_1, \ldots, E_n$, *i.e.* the indices satisfying $E_{n:k} = E_{D_k},\ k = 1, \ldots, n$. Then, for some k between 1 and n, the following holds.

$$\begin{aligned}
\hat{E}_{n:n} - E_{n:n} - \beta_0 &= \max_{1 \le i \le n} \left\{ E_i + \beta_0 - \mathbf{x}_i'(\widehat{\boldsymbol{\beta}}_{nR} - \boldsymbol{\beta}) \right\} - E_{n:n} - \beta_0 \\
&= E_{n:n-k} - \mathbf{x}'_{D_{n-k}}(\widehat{\boldsymbol{\beta}}_{nR} - \boldsymbol{\beta}) - E_{n:n} \qquad (4.8) \\
&\le -\mathbf{x}'_{D_{n-k}}(\widehat{\boldsymbol{\beta}}_{nR} - \boldsymbol{\beta}) \\
&\le \max_{1 \le i \le n} \|\mathbf{x}_i\| \cdot \|\widehat{\boldsymbol{\beta}}_{nR} - \boldsymbol{\beta}\| = \mathcal{O}_p(n^{-\delta}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E_{n:n} + \beta_0 - \hat{E}_{n:n} &\le E_{n:n} - \Big(E_{n:n} - \mathbf{x}'_{D_n}(\widehat{\boldsymbol{\beta}}_{nR} - \boldsymbol{\beta}) \Big) \\
&\le \max_{1 \le i \le n} \|\mathbf{x}_i\| \cdot \|\widehat{\boldsymbol{\beta}}_{nR} - \boldsymbol{\beta}\| = \mathcal{O}_p(n^{-\delta}). \qquad (4.9)
\end{aligned}$$

(4.8) and (4.9) imply (4.5). Moreover, (4.5) together with the well-known asymptotic behaviour of the extremes of the i.i.d. observations imply (4.6) and (4.7). □

Remark 4.1. The situation is slightly different with the Weibull extreme distribution with distribution function $\Psi_m(t) = \exp\{-(-t)^m\}$ for $t \le 0$ and $\Psi_m(t) = 0$ otherwise, and this case needs a special study. Distributions that belong to its domain of attraction have a finite right endpoint, say e_F, and $\mathbb{P}\left(c_n^{-1}(E_{n:n} - e_F) \le t\right) \to \Psi_m(t)$ for a suitable c_n. Some of these distributions do not have finite Fisher information (e.g., $R(0,1)$), and e_F is not typically known in regression models.

Analogously, using the preliminary rank estimate $\widehat{\boldsymbol{\theta}}_{nR}$ of $\boldsymbol{\theta}$ defined in (3.8), we get a consistent estimator of $\varepsilon_{n:n} = \max\{\varepsilon_1, \ldots, \varepsilon_n\}$ for AR(p) model (3.1) in the form

$$\hat{\varepsilon}_{n:n} = \max \left\{ X_1 - \mathbf{Y}_0' \widehat{\boldsymbol{\theta}}_{nR}, \ldots, X_n - \mathbf{Y}_{n-1}' \widehat{\boldsymbol{\theta}}_{nR} \right\}.$$

The *extreme regression quantile* of model (1.1) can also be approximated with a two-step version, starting in a suitable R-estimate of the slope. With an appropriate choice of the R-estimator, the two-step version even coincides with the original extreme regression quantile.

Following Portnoy and Jurečková (1999), define the maximal regression quantile $\widehat{\boldsymbol{\beta}}^*(1)$ as a solution of the following minimization problem:

$$\min_{b_0 \in \boldsymbol{R}, \mathbf{b} \in \boldsymbol{R}^p} \sum_{i=1}^{m} (Y_i - b_0 - \mathbf{x}_i' \mathbf{b})^+. \tag{4.10}$$

Alternatively, this can be described as any solution to the linear program:

$$\min_{b \in \boldsymbol{R}, \mathbf{b} \in \boldsymbol{R}^p} b_0 + \sum_{i=1}^{n} \mathbf{x}_i' \mathbf{b} \qquad \text{s.t. } Y_i \le b_0 + \mathbf{x}_i' \mathbf{b}, \ \ i = 1, \dots, n. \tag{4.11}$$

If $\sum_{i=1}^{n} x_{ij} = 0, \ j = 1, \dots, p$, then we minimize only b_0 subject to the restriction. Let $\widehat{\boldsymbol{\beta}}_{nR}^{+}$ be the R-estimate of $\boldsymbol{\beta}$ generated by the score function (2.6) with $\alpha = 1 - \frac{1}{n}$, i.e.

$$\varphi_{1-\frac{1}{n}}(u) = I[u \ge 1 - \tfrac{1}{n}] - \tfrac{1}{n}, \ 0 < u < 1. \tag{4.12}$$

Then the Jaeckel measure of the rank dispersion (2.10) takes on the form

$$\begin{aligned} \mathcal{D}_n(\mathbf{b}) &= \sum_{i=1}^{n} (Y_i - \mathbf{x}_i' \mathbf{b}) \varphi_{1-\frac{1}{n}} \left(\frac{R_{ni}(\mathbf{b})}{n+1} \right) \\ &= \sum_{i=1}^{n} (Y_i - \mathbf{x}_i' \mathbf{b}) I[R_{ni}(\mathbf{b}) = n] - \bar{Y}_n, \\ &= \max_{1 \le i \le n} \{ Y_i - \mathbf{x}_i' \mathbf{b} \} - \bar{Y}_n = (Y_i - \mathbf{x}_i' \mathbf{b})_{n:n} - \bar{Y}_n, \end{aligned} \tag{4.13}$$

where $\bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$. Hence, $\widehat{\boldsymbol{\beta}}_{nR}^{+}$ minimizes $(Y_i - \mathbf{x}_i' \mathbf{b})_{n:n}$ with respect to $\mathbf{b} \in I\!\!R^p$. Moreover, define $\hat{\beta}_0^+$ as

$$\hat{\beta}_0^+ = \max\{ Y_i - \mathbf{x}_i' \widehat{\boldsymbol{\beta}}_{nR}^{+}, \ 1 \le i \le n \} \tag{4.14}$$

and call $\left(\hat{\beta}_0^+, \widehat{\boldsymbol{\beta}}_{nR}^{+\prime} \right)'$ the two-step maximal regression quantile. It follows from its definition that

$$\hat{\beta}_0^+ + \mathbf{x}_i' \widehat{\boldsymbol{\beta}}_{nR}^{+} \ge Y_i, \ i = 1, \dots, n \tag{4.15}$$

while for some i_0 the inequality reduces to an equality. Hence, $\hat{\beta}_0^+$ is minimized subject to the restriction (4.15), and this coincides with the definition (4.11) of the extreme regression quantile.

5 Numerical Illustration

The following simulation study shows a very good approximation of the regression quantiles by the proposed two-step versions, even for moderate sample sizes. Because the calculation of the two-step version is also quite fast (see the remarks below), we can propose to consider the two-step regression quantiles for use as an alternative to the original regression quantiles. The estimation of the extreme errors in the linear model, apparently the first in the literature, also gives very good results.

5.1 Two-step regression quantiles. The simulation study of the model

$$Y_i = \beta_0 + \mathbf{x}_i'\boldsymbol{\beta} + E_i, \quad i = 1, \ldots, n$$

is made with the following parameters:

- sample sizes: $n = 20,\ 150,\ 500$;
- $\beta_0 = 5$;
- $\boldsymbol{\beta} = (\beta_1, \beta_2) = (-1, 2)$;
- $\alpha = 0.25,\ 0.5$;

The regression matrix $\mathbf{X}$ of order (n×2) is generated in such a way that $(x_{11}, \ldots, x_{n1})$ and $(x_{12}, \ldots, x_{n2})$ are either two independent samples from the uniform distributions $R(0, 10)$ and $R(-5, 15)$, respectively, or the samples are standardized so that $\sum_{i=1}^{n} x_{ij} = 0,\ j = 1, 2$. The R-estimator $\widehat{\boldsymbol{\beta}}_R$ is computed by minimizing Jaeckel's objective function (2.10) with the score-generating function (2.6). This task has to be carried out by a numerical method. If we use the statistical computing environments such as S-plus or R, it is possible to utilize a standard minimization routine. The most general minimization routine in S-plus is `nlminb`, where the underlying algorithm is a quasi-Newton optimizer. This function is based on the Fortran functions (see Gay, 1983, 1984) from NETLIB (see Dongarra and Grosse, 1987). An important problem is also the choice of the starting point because we use the nonlinear minimizer. It seems from our simulation experiment that the resulting solution to the minimization is not too sensitive to the initial point but may take too long to converge for a poorly chosen initial iterate. Thus, it is advisable to use another appropriate estimator as the starting value.

1000 replications of the model were simulated for each case, and the two-step α-regression quantiles (denoted as 2RQ(α)) and the ordinary α-regression quantiles (RQ(α)) were computed. For the sake of comparison, the results were sorted and the empirical quantile functions of 2RQ(α) and of RQ(α) were plotted, see Figures 5.1 – 5.3. Among them, Figures 5.1 and

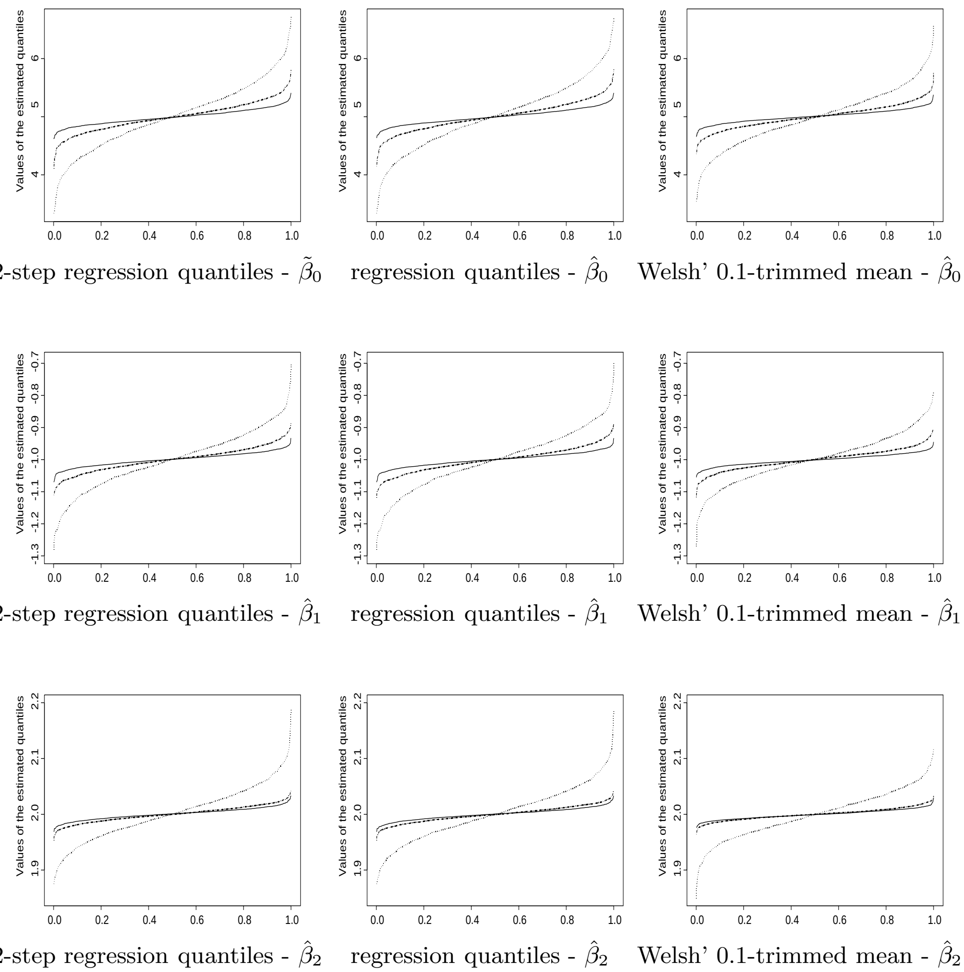


Figure 5.1. Empirical quantile functions of components of 1000 values of 0.5-two-step regression quantile, of 0.5-regression quantile and of Welsh' 0.1-trimmed mean for standard normal errors and for the uniform matrix, $n = 20$ (dotted), $n = 150$ (dashed), $n = 500$ (solid).

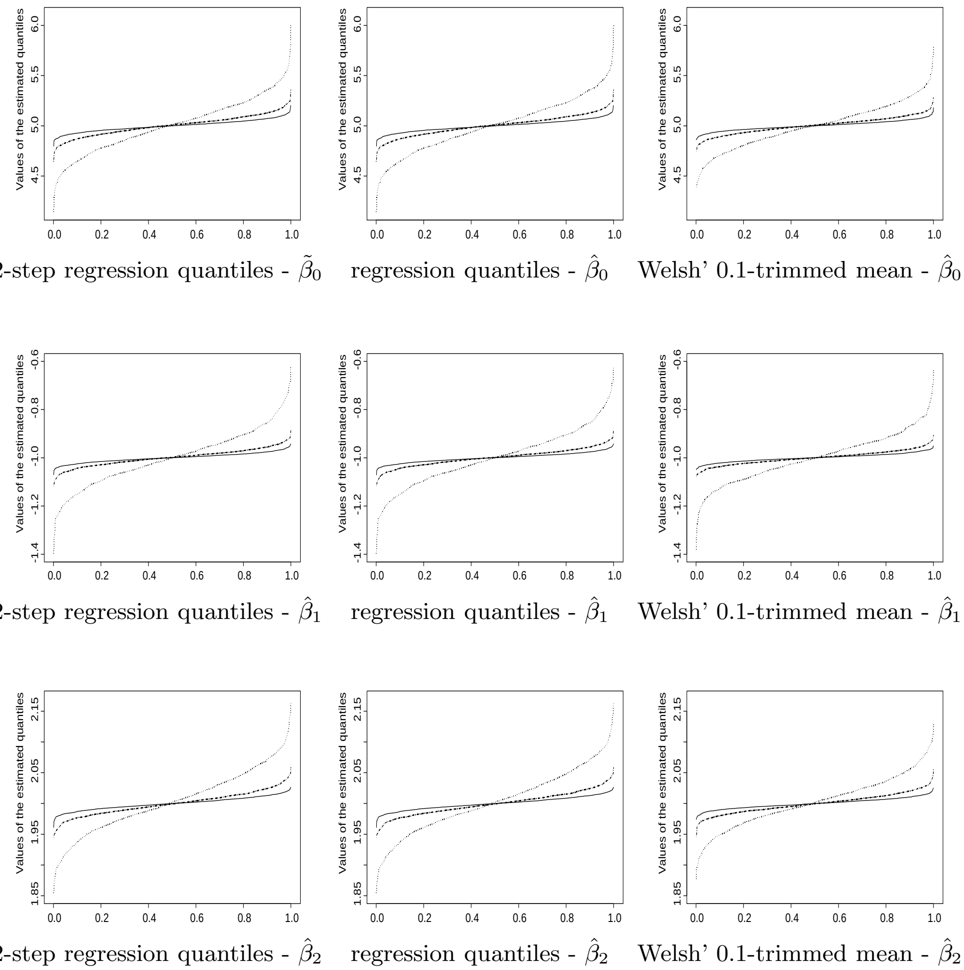
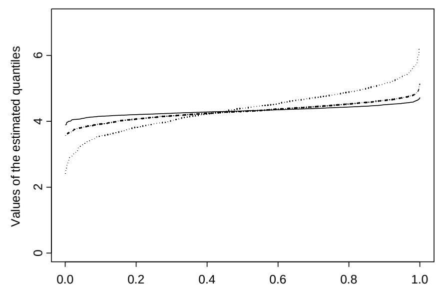
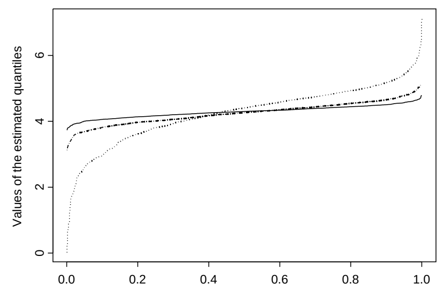


Figure 5.2. Empirical quantile functions of components of 1000 values of 0.5-two-step regression quantile, of 0.5-regression quantile and of Welsh' 0.1-trimmed mean for standard normal errors and for $\sum_{i=1}^{n} x_{ij} = 0$, $n = 20$ (dotted), $n = 150$ (dashed), $n = 500$ (solid).

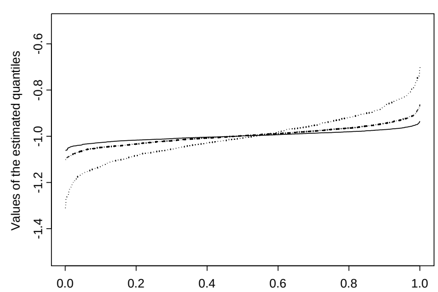
5.2, illustrating the 0.5 regression and two-step regression quantiles under $\mathcal{N}(0,1)$ distribution of errors, also contain the empirical quantile function of 1000 Welsh' (1987a) α–trimmed means ($\alpha = 0.1$); the comparison is very good, though intuitively Welsh' 0.1–trimmed mean would be more favourable for the normal distribution.



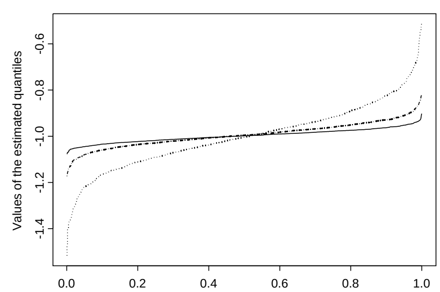
0.25-two-step regression quantiles - $\tilde{\beta}_0$,
the normal errors



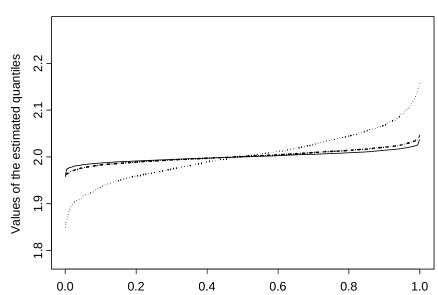
0.25-two-regression quantiles - $\hat{\beta}_0$,
the Laplace errors



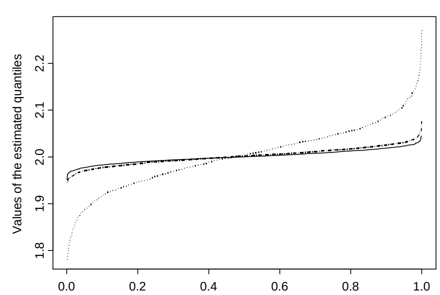
0.25-two-step regression quantiles - $\hat{\beta}_1$,
the normal errors



0.25-two-regression quantiles - $\hat{\beta}_1$,
the Laplace errors



0.25-two-step regression quantiles - $\hat{\beta}_2$
the normal errors



0.25-two-regression quantiles - $\hat{\beta}_2$,
the Laplace errors

Figure 5.3. Empirical quantile functions of components of 1000 values of 0.25-two-step regression quantile for the normal(left) and the Laplace (right) errors and uniform matrix, $n = 20$ (dotted), $n = 150$ (dashed), $n = 500$ (solid).

Figure 5.3, calculated for the Laplace and normal errors, illustrates that the results are very similar under different models.

The simulation study indicates:

(i) Two-step regression quantiles and ordinary regression quantiles practically coincide even for moderate sample sizes.

(ii) As we have verified on a considerably larger simulation experiment, the properties of the two-step regression quantiles are very weakly affected by the chosen α, by the form of the matrix, and by the distribution of the error terms.

(iii) The influence of the reparametrazation $\sum_{i=1}^{n} x_{ij} = 0,\ j = 2, \dots, p$ is small.

(iv) The computation of the two-step regression quantiles is quite fast. One simulation experiment took between 1-5 minutes. (We used S-plus version 4.5 running on 666 MHz Pentium III with 128 MB memory.)

5.2 Extreme errors. The numerical study is based on the model

$$Y_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + e_i, \quad i = 1, \dots, n. \tag{5.1}$$

with the following parameter values:

- sample sizes: $n = 20,\ 150,\ 500$;
- $\boldsymbol{\beta} = (\beta_1, \beta_2) = (-1, 2)$;
- distributions of errors: standard normal, Laplace and Cauchy;
- criterion for R-estimates: the Jaeckel objective function;
- the columns $(x_{11}, \dots, x_{n1})'$ and $(x_{12}, \dots, x_{n2})'$ of the $(n \times 2)$ simulated matrix come from two independent samples standardized with the respective sample averages.

For each of the above combinations, we proceeded as follows:

(1) The errors $E_1, \dots,\ E_n$ were generated and the maximum $E_{n:n} = \max\{E_1, \dots, E_n\}$ was found;

(2) the R-estimate $\widehat{\boldsymbol{\beta}}_{nR}(\alpha)$, generated by φ_α of (2.6), was calculated for $\alpha = 0.25,\ 0.50,\ 0.75$;

(3) the maximal residual

$$\hat{E}_{n:n}(\alpha) = \max\{Y_1 - \mathbf{x}_1' \widehat{\boldsymbol{\beta}}_{nR}(\alpha), \dots, Y_n - \mathbf{x}_n' \widehat{\boldsymbol{\beta}}_{nR}(\alpha)\}$$

was calculated;

(4) the steps (1), (2) and (3) were repeated 1 000 times;

(6) selected sample quantiles of estimates $(\hat{E}^1_{n:n}(\alpha), \ldots, \hat{E}^{1000}_{n:n}(\alpha))$ and of real extremes $(E^1_{n:n}, \ldots, E^{1000}_{n:n})$ were computed.

The selected sample quantiles for various distributions of errors are summarized in Tables 5.1–5.3. Figure 5.4 further illustrates the quality of the approximations.

Table 5.1. Sample quantiles of 1000 extreme errors and of their estimators for model (5.1)
(errors simulated from the standard normal distribution)

n	real extreme or approx., $\alpha =$	min	5%	20%	35%	50%	65%	80%	95%	max
20	real extreme	0.612	1.063	1.429	1.639	1.825	2.035	2.277	2.822	3.842
	$\alpha = 0.25$	0.598	1.068	1.452	1.701	1.903	2.128	2.436	2.982	4.009
	$\alpha = 0.50$	0.540	1.017	1.382	1.640	1.855	2.046	2.342	2.916	5.017
	$\alpha = 0.75$	0.373	0.950	1.330	1.555	1.746	1.957	2.218	2.851	4.509
150	real extreme	1.612	2.041	2.305	2.447	2.587	2.739	2.952	3.451	4.465
	$\alpha = 0.25$	1.787	2.066	2.321	2.470	2.602	2.782	2.983	3.445	4.390
	$\alpha = 0.50$	1.668	2.056	2.303	2.447	2.580	2.747	2.969	3.446	4.422
	$\alpha = 0.75$	1.638	2.047	2.291	2.448	2.579	2.730	2.959	3.442	4.383
500	real extreme	2.167	2.518	2.729	2.849	2.980	3.145	3.337	3.726	4.908
	$\alpha = 0.25$	2.057	2.513	2.728	2.867	2.993	3.154	3.332	3.752	4.921
	$\alpha = 0.50$	2.164	2.501	2.729	2.855	2.996	3.151	3.332	3.728	4.962
	$\alpha = 0.75$	2.041	2.503	2.728	2.863	2.991	3.137	3.323	3.715	5.032

Table 5.2. Sample quantiles of 1000 extreme errors and of their estimators for model (5.1)
(errors simulated from the Laplace distribution)

n	real extreme or approx., $\alpha =$	min	5%	20%	35%	50%	65%	80%	95%	max
20	real extreme	0.582	1.346	1.936	2.337	2.761	3.227	3.904	5.387	9.027
	$\alpha = 0.25$	0.567	1.420	1.999	2.442	2.883	3.318	4.084	5.366	9.193
	$\alpha = 0.50$	0.424	1.350	1.894	2.323	2.755	3.210	3.914	5.333	9.011
	$\alpha = 0.75$	0.466	1.233	1.778	2.140	2.571	3.106	3.725	5.189	9.213
150	real extreme	2.347	3.224	3.841	4.326	4.698	5.185	5.847	7.293	12.599
	$\alpha = 0.25$	2.385	3.225	3.842	4.319	4.705	5.211	5.832	7.389	12.487
	$\alpha = 0.50$	2.331	3.229	3.846	4.297	4.701	5.164	5.838	7.317	12.529
	$\alpha = 0.75$	2.299	3.220	3.831	4.287	4.647	5.147	5.847	7.394	12.665
500	real extreme	3.710	4.433	5.102	5.499	5.861	6.317	6.881	8.231	9.490
	$\alpha = 0.25$	3.694	4.374	5.076	5.478	5.866	6.332	6.921	8.243	9.453
	$\alpha = 0.50$	3.709	4.414	5.063	5.469	5.881	6.289	6.936	8.224	9.535
	$\alpha = 0.75$	3.726	4.437	5.108	5.461	5.863	6.295	6.890	8.226	9.551

Table 5.3. Sample quantiles of 1000 extreme errors and of their estimators for model (5.1)
(errors simulated from the Cauchy distribution)

n	real extr. or approx.	min	5%	20%	35%	50%	65%	80%	95%	max
20	real extr.	1.00	2.01	4.01	6.31	9.59	14.47	26.59	103.6	2997.8
	$\alpha = 0.25$	0.77	2.35	4.19	6.53	9.58	14.50	26.32	104.0	2996.1
	$\alpha = 0.50$	0.85	1.95	3.87	6.18	9.42	14.38	25.74	104.1	2997.5
	$\alpha = 0.75$	0.50	1.81	3.60	5.88	9.06	13.63	25.73	105.8	2998.1
150	real extr.	5.30	14.71	29.47	44.37	64.92	106.06	209.22	950.3	105946.2
	$\alpha = 0.25$	5.71	14.83	29.56	44.11	65.04	105.84	209.19	949.5	105946.0
	$\alpha = 0.50$	5.39	14.76	29.59	44.43	65.09	105.91	209.09	950.1	105946.2
	$\alpha = 0.75$	5.17	14.66	29.54	44.42	64.82	106.09	209.05	950.0	105946.1

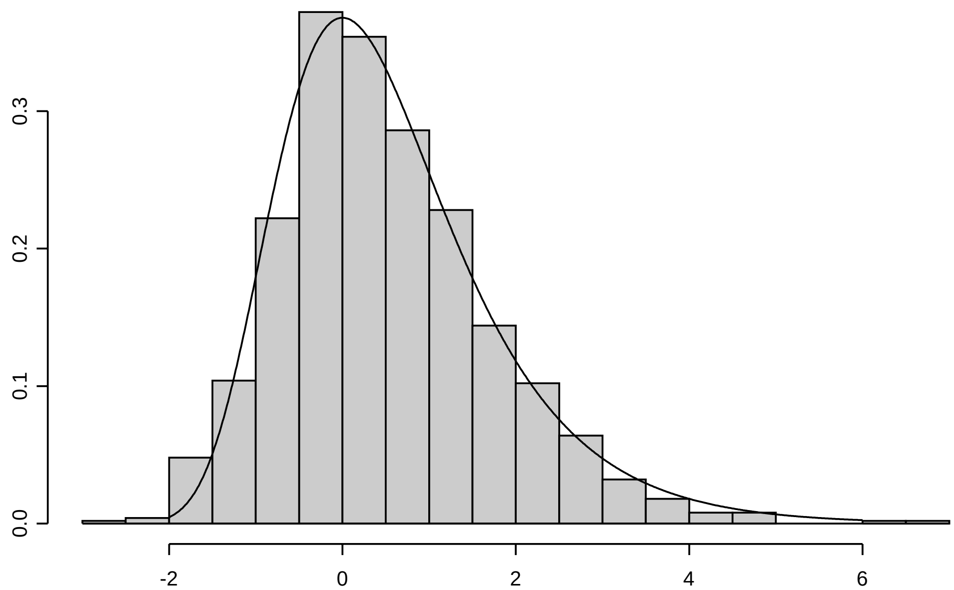


Figure 5.4. Histogram of $nf(\xi_n)(\hat{E}_{n:n} - \xi_n)$, $n = 500$, plotted against the Gumbel density; standard normal errors ($\hat{E}_{n:n}$ is based on residuals with respect to $\widehat{\boldsymbol{\beta}}_{nR}$, $\alpha = 0.75$)

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References

BAHADUR, R.R. (1966). A note on quantiles in large samples. *Ann. Math. Statist.*, **37**, 557-580.

BICKEL, P.J. (1973). On some analogues to linear combinations of order statistics in the linear model. *Ann. Statist.*, **1**, 597-616.

DENBY, L. and MARTIN, D. (1979). Robust estimation of the first order autoregressive parameter. *J. Amer. Statist. Assoc.*, **74**, 140-146.

DODGE, Y. and JUREČKOVÁ, J. (1995). Estimation of the quantile density function based on regression quantiles. *Statist. Probab. Lett.*, **23**, 73-78.

DONGARRA, J.J. and GROSSE, E. (1987). Distribution of mathematical software via electronic mail. *Communications of the ACM*, **30**, 403-407.

GAY, D.M. (1983). Algorithm 611. Subroutines for Unconstrained Minimization using a Model/Trust-Region Approach. *ACM Transactions on Mathematical Software*, **9**, 503-524.

GAY, D.M. (1984). *A trust region approach to linearly constrained optimization. Numerical Analysis.* Proceedings, Dundee 1983. Springer: Berlin.

GUTENBRUNNER, C. and JUREČKOVÁ, J. (1992). Regression rank scores and regression quantiles. *Ann. Statist.*, **20**, 305-330.

HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests.* Academic Press, New York.

HALLIN, M. and JUREČKOVÁ, J. (1999). Optimal tests for autoregressive models based on autoregression rank scores. *Ann. Statist.*, **27**, 1385-1414.

HEILER, S. and WILLERS, R. (1988). Asymptotic normality of R-estimates in the linear model. *Statistics*, **19**, 173-184.

HODGES, J.L. and LEHMANN, E.L. (1963). Estimation of location based on rank tests. *Ann. Math. Statist.*, **34**, 598-611.

JAECKEL, L.A. (1972). Estimating regression coefficients by minimizing the dispersion of the residuals. *Ann. Math. Statist.*, **43**, 1449-1459.

JUREČKOVÁ, J. (1971). Nonparametric estimate of regression coefficients. *Ann. Math. Statist.*, **42**, 1328-1338.

JUREČKOVÁ, J. (1977). Asymptotic relation of M-estimates and R-estimates in the linear regression model. *Ann. Statist.*, **5**, 464-472.

JUREČKOVÁ, J. (1995). Affine- and scale-equivariant M-estimators in linear models. *Prob. Math. Statist.*, **15**, 397-407.

JUREČKOVÁ, J., PICEK, J. and SEN, P.K. (2003). Goodness-of-fit test with nuisance regression and scale. *Metrika*, **58**, 235-258.

JUREČKOVÁ, J. and SEN, P.K. (1984). On adaptive scale-equivariant M-estimators in linear models. *Statist. & Decisions, Suppl.*, **1**, 31-46.

JUREČKOVÁ, J. and SEN, P.K. (1996). *Robust Statistical Procedures : Asymptotics and Inter-Relations.* J. Wiley, New York.

KOENKER, R. and BASSETT, G. (1978). Regression quantiles. *Econometrica*, **46**, 33-50.

Koul, H.L. (1971). Asymptotic behavior of a class of confidence regions in multiple linear regression. *Ann. Math. Statist.*, **42**, 42-57.

Koul, H.L. and Ossiander, M. (1994). Weak convergence of randomly weighted dependent residual empiricals with application to autoregression. *Ann. Statist.*, **22**, 540-562.

Koul, H.L. and Md. E. Saleh, A.K. (1995). Autoregression quantiles and related rank-scores processes. *Ann. Statist.*, **23**, 670-689.

Portnoy, S. and Jurečková, J. (1999). On extreme regression quantiles. *Extremes*, **2**, 227-243.

Ruppert, D. and Carroll, R.J. (1980). Trimmed least squares estimation in the linear model. *J. Amer. Statist. Assoc.*, **75**, 828-838.

Welsh, A.H. (1987a). The trimmed mean in the linear model. *Ann. Statist.*, **15**, 20-36.

Welsh, A.H. (1987b). One-step *L*-estimators for the linear model. *Ann. Statist.*, **15**, 626-641.

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