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Chover's Law of the Iterated Logarithm for Weighted Sums with Application

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Abstract

Consider weighted sums of independent variables with a common distribution function in the domain of attraction of a stable law. Peng and Qi (2003) established the Chover-type law of the iterated logarithm. This paper extends the law to a larger class of weighted sums. As an application, it is shown that the Chover-type law of the iterated logarithm for products of sums holds when the underlying distribution is in the domain of attraction of a stable law with an exponent larger than one.

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1 Introduction

Assume $\{X, X_n, n \geq 1\}$ is a sequence of independent random variables with a common distribution function F. Set $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$. The partial sum S_n has been well studied in the past century. For example, Hartman-Wintner's law of the iterated logarithm (LIL) states that

$$\limsup_{n\to\infty} \frac{S_n}{(2n\log\log n)^{1/2}} = 1 \quad \text{a.s.}$$

if and only if $E(X) = 0$ and $E(X^2) = 1$.

Consider a weighted sum defined by

$$S_n(h) = \sum_{i=1}^{n} h(\frac{k}{n})X_i,$$

where h is a well-defined function over $[0,1]$. An analogous LIL proved by Li and Tomkins (1996) states that for certain type of functions h,

$$\limsup_{n\to\infty} \frac{S_n(h)}{(2n\log\log n)^{1/2}} = \int_0^1 h(x)^2 dx \quad \text{a.s.}$$

if and only if $E(X)=0$ and $E(X^2)=1$. For more references on the study of the LIL for the weighted sums, we refer to Gaposkin (1965), Tomkins (1976), Lai and Wei (1982), Stadtmüller (1984).

The aforementioned results demonstrate that the classic LIL for the partial sums and weighted sums is no longer valid if $E(X^2)=\infty$. Study reveals that the law of the logarithm and its analogue for weighted sums $\sum_{i=1}^n a_{n,i}X_i$ and sums S_n may hold only if X is the domain of partial attraction of the normal distribution as well as some additional regularity conditions. We offer Li and Tomkins (2003) for weighted sums and Heyde (1969) for partial sums.

In this paper let G_α be a stable distribution function with characteristic exponent α (α-stable law) with $0<\alpha<2$. Suppose that F is in the domain of attraction of the stable law G_α, that is, there exist sequences of constants $\{A_n, n\ge 1\}$ and $\{B_n, n\ge 1\}$ such that $B_n>0$ and for all $x\in(-\infty,\infty)$

$$\lim_{n\to\infty} P\left\{\frac{1}{B_n}(S_n-A_n)\le x\right\} = G_\alpha(x). \tag{1.1}$$

Write $G(x)=P(|X|>x)$ for $x>0$ and define

$$B(x)=\inf\left\{y : G(y)\le \frac{1}{x}\right\}, \quad x>0, \tag{1.2}$$

and $\mu_n(c)=n\int \frac{x}{1+x^2}dF(cx)$ for $c>0$.

From Loève (1977), if (1.1) holds for some A_n and B_n, one can take $B_n=B(n)$ and $A_n=\mu_n(B(n))$, and (1.1) still holds with some α-stable limit distribution function. In particular, if $\alpha>1$ then $E(|X|)<\infty$, and one can always select $B_n=B(n)$ and $A_n=nE(X)$ in (1.1). If $\alpha<1$ then one can set $A_n=0$ in (1.1).

Without loss of generality, in this paper we assume that condition (1.1) holds with $B_n=B(n)$ defined in (1.2) and some constants A_n with $A_0=0$. That is, A_n can be any sequence of constants satisfying (1.1).

When X has a symmetric stable distribution function F characterized by

$$E(\exp(itX)) = \exp(-|t|^\alpha), \text{ for } t \in R \tag{1.3}$$

Chover (1966) established that

$$\limsup_{n\to\infty} \left| \frac{S_n}{n^{1/\alpha}} \right|^{\frac{1}{\log\log n}} = e^{1/\alpha} \quad a.s.$$

which is known as Chover's law of the iterated logarithm (LIL). Under general assumption (1.1) Qi and Cheng (1996) proved that

$$\limsup_{n\to\infty} \left| \frac{1}{B(n)} (S_n - A_n) \right|^{\frac{1}{\log\log n}} = e^{1/\alpha} \quad a.s.. \tag{1.4}$$

Vasudeva (1984) also obtained similar result under assumption (1.1) with $A_n \equiv 0$. For further results on this topic one can see Chen (1993) and Chen and Chen (2003).

Some recent papers have been devoted to the study of Chover's LIL for weighted sums. We refer to Chen and Huang (2000), Chen (2002), Chen and Liu (2003) and Peng and Qi (2003). One of the results in Peng and Qi (2003) states that

$$\limsup_{n\to\infty} \left(\frac{|\sum_{k=1}^n a_{n,k} X_k - C_n|}{B(n)} \right)^{1/\log\log n} = e^{1/\alpha} \text{ a.s.},$$

where $C_n = \sum_{k=1}^n a_{n,k}(A_k - A_{k-1})$, provided that the array of weights $\{a_{n,k}\}$ satisfies conditions

C1. There exist two increasing sequences $\{n(k), k \geq 1\}$ and $\{m(k), k \geq 1\}$ such that

$$\sup_{k\geq 1}(n(k+1) - n(k)) < \infty \text{ and } \liminf_{k\to\infty} |a_{n(k),m(k)}| > 0;$$

C2. $\sup_{n\geq 1}(\sum_{k=1}^{n-1} |a_{n,k} - a_{n,k+1}| + |a_{n,n}|) < \infty$.

Notice that the condition C2 requires that coefficients $a_{n,k}$ are uniformly bounded. This rules out many weighted sums which have unbounded weights. In this paper we relax this condition. Throughout the paper we define

$$D_n = \max\left(1, \sum_{k=1}^{n-1} |a_{n,k} - a_{n,k+1}| \sqrt{\frac{k}{n}} + |a_{n,n}|\right) \tag{1.5}$$

In this section we prove the Chover's LIL under assumption C1 and some additional condition imposed on D_n. We shall apply our result to establish the Chover-type LIL for products of sums in next section. Our main theorems are as follows.

THEOREM 1.1. *Assume* C1 *holds. Let* $\{p_n\}$ *be a sequence of non-decreasing numbers with* $p_n \geq 1$. *Set* $C_n = \sum_{k=1}^{n} a_{n,k}(A_k - A_{k-1})$. *Under* (1.1) *we have that*

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n} a_{n,k}X_k - C_n|}{D_n B(np_n)} = 0 \ a.s. \tag{1.6}$$

if $\sum_{n=1}^{\infty} \frac{1}{np_n} < \infty$, *and*

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n} a_{n,k}X_k - C_n|}{B(np_n)} = \infty \ a.s. \tag{1.7}$$

if $\sum_{n=1}^{\infty} \frac{1}{np_n} = \infty$ *and* $\log D_n = o(\log n)$.

Set $\lg_0(x) = x$ and $\lg_k(x) = \log\max(\lg_{k-1}(x), e)$ for $k \geq 1$.

THEOREM 1.2. *Assume* C1 *holds and for some integer* $r_0 \geq 1$

$$\lim_{n\to\infty} \frac{\log D_n}{\lg_{r_0+1}(n)} = 0. \tag{1.8}$$

Under (1.1) *we have for each integer* r *with* $1 \leq r \leq r_0$,

$$\limsup_{n\to\infty} \left(\frac{|\sum_{k=1}^{n} a_{n,k}X_k - C_n|}{B(\prod_{i=0}^{r-1} \lg_i(n))} \right)^{1/\lg_{r+1}(n)} = e^{1/\alpha} \ a.s., \tag{1.9}$$

where $C_n = \sum_{k=1}^{n} a_{n,k}(A_k - A_{k-1})$. *In particular, if* (1.8) *holds we have the Chover's LIL*

$$\limsup_{n\to\infty} \left(\frac{|\sum_{k=1}^{n} a_{n,k}X_k - C_n|}{B(n)} \right)^{1/\log\log n} = e^{1/\alpha} \ a.s.. \tag{1.10}$$

If D_n *is bounded, then* (1.9) *holds for all integer* $r \geq 1$.

REMARK 1.1. Restriction condition (1.8) seems necessary. We present an example here to show that when (1.8) does not hold for some r_0, (1.9) may fail for $r = r_0$. Let X have a symmetric stable distribution, i.e., (1.3) holds. Let $a_{n,k} = 1$ for $1 \leq k < n$ and $a_{n,n} = 1 + a_n$, where $0 < a_n \to \infty$. Then we have $\sum_{k=1}^{n} a_{n,k}X_k = S_n + a_nX_n$, $C_n = 0$, and $D_n = \left(1 + \sqrt{\frac{n-1}{n}}\right) a_n + 1$

for $n \ge 1$. Assume (1.8) fails for some $r_0 \ge 1$ but the limit on the left-hand side exists, that is,

$$\log a_n \sim \log D_n \sim \delta \operatorname{lg}_{r_0+1}(n) \quad \text{as } n \to \infty$$

for some $\delta > 0$. Since (1.3) implies that for some constant $C > 0$, $P(|X| > x) \sim Cx^{-\alpha}$, we have $B(x) \sim C^{1/\alpha} x^{1/\alpha}$. For every $\varepsilon \in (0,1)$, $\sum_{n=2}^{\infty} P(|X| > B((\operatorname{lg}_{r_0}(n))^{1+\varepsilon} \prod_{j=0}^{r_0-1} \operatorname{lg}_j(n))) < \infty$, and $\sum_{n=2}^{\infty} P(|X| > B((\operatorname{lg}_{r_0}(n))^{1-\varepsilon/2} \prod_{j=0}^{r_0-1} \operatorname{lg}_j(n))) = \infty$. From Lemma 3.3 of Peng and Qi (2003),

$$\limsup_{n\to\infty} \left| \frac{S_n}{B((\operatorname{lg}_{r_0}(n))^{1+\varepsilon} \prod_{j=0}^{r_0-1} \operatorname{lg}_j(n))} \right| = 0 \quad \text{a.s.}$$

and from Borel-Cantelli Lemma,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{a_n |X_n|}{B((\operatorname{lg}_{r_0}(n))^{1+\alpha\delta-\varepsilon} \prod_{j=0}^{r_0-1} \operatorname{lg}_j(n))} \\ &= \limsup_{n\to\infty} \frac{|X_n|}{B((\operatorname{lg}_{r_0}(n))^{1-\varepsilon/2} \prod_{j=0}^{r_0-1} \operatorname{lg}_j(n))} \frac{a_n}{(\operatorname{lg}_{r_0}(n))^{\delta-\varepsilon/2\alpha}} \\ &= \infty \quad \text{a.s..} \end{aligned}$$

Therefore, whenever $0 < \varepsilon < \min(1, \alpha\delta/2)$,

$$\begin{aligned} &\limsup_{n\to\infty} \left| \frac{\sum_{k=1}^{n} a_{n,k} X_k}{B((\operatorname{lg}_{r_0}(n))^{1+\alpha\delta-\varepsilon} \prod_{j=0}^{r_0-1} \operatorname{lg}_j(n))} \right| \\ &\quad = \limsup_{n\to\infty} \frac{a_n |X_n|}{B((\operatorname{lg}_{r_0}(n))^{1+\alpha\delta-\varepsilon} \prod_{j=0}^{r_0-1} \operatorname{lg}_j(n))} = \infty \text{ a.s.,} \end{aligned}$$

from which we conclude that with probability one

$$\begin{aligned} &\limsup_{n\to\infty} \left| \frac{\sum_{k=1}^{n} a_{n,k} X_k}{B(\prod_{j=0}^{r_0-1} \operatorname{lg}_j(n))} \right|^{1/\operatorname{lg}_{r_0+1}(n)} \\ &= \limsup_{n\to\infty} \left| \frac{\sum_{k=1}^{n} a_{n,k} X_k}{B((\operatorname{lg}_{r_0}(n))^{1+\alpha\delta-\varepsilon} \prod_{j=0}^{r_0-1} \operatorname{lg}_j(n))} \right|^{1/\operatorname{lg}_{r_0+1}(n)} \\ &\qquad \times \left\{ (\operatorname{lg}_{r_0}(n))^{1/\alpha+\delta-\varepsilon/\alpha} \right\}^{1/\operatorname{lg}_{r_0+1}(n)} \\ &\ge e^{1/\alpha+\delta-\varepsilon/\alpha}, \end{aligned}$$

i.e.,

$$\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^{n}a_{n,k}X_k}{B(\prod_{j=0}^{r_0-1}\lg_j(n))}\right|^{1/\lg_{r_0+1}(n)}\geq e^{1/\alpha+\delta}\ \text{a.s.}.$$

So (1.9) fails for $r=r_0$.

As a consequence of Theorem 1.2 we have

THEOREM 1.3. *Assume* $h(x)=h_1(x)-h_2(x)$, *where* h_1 *and* h_2 *are two nonnegative monotone functions well-defined on* $(0,1)$, *such that*

(*a*) $\int_0^1 \frac{h_i(x)}{\sqrt{x}}dx<\infty$ *for* $i=1,2$;

(*b*) *for some* $x_0\in(0,1)$ $h(x)$ *is continuous at* $x=x_0$ *and* $h(x_0)\neq 0$;

(*c*) *there exists an integer* $r_0\geq 1$ *such that*

$$\lim_{n\to\infty}\frac{\log\max(1,|h(\frac{n}{n+1})|)}{\lg_{r_0+1}(n)}=0.$$

Then under (1.1) *we have for any integer* $1\leq r\leq r_0$,

$$\limsup_{n\to\infty}\left(\frac{|\sum_{k=1}^{n}h(\frac{k}{n+1})X_k-C_n|}{B(\prod_{i=0}^{r-1}\lg_i(n))}\right)^{1/\lg_{r+1}(n)}=e^{1/\alpha}\ a.s.,$$

where $C_n=\sum_{k=1}^{n}h(\frac{k}{n})(A_k-A_{k-1})$. *In particular, for* $r=1$ *we get the Chover's LIL:*

$$\limsup_{n\to\infty}\left(\frac{|\sum_{k=1}^{n}h(\frac{k}{n+1})X_k-C_n|}{B(n)}\right)^{1/\log\log n}=e^{1/\alpha}\ a.s..$$

REMARK 1.2. Theorem 1.3 is an extension of Theorem 2.1 in Peng and Qi (2003) where $h(x)$ is assumed to be a bounded variation function over $[0,1]$. In the present paper $h(x)$ is a function of bounded variation over $[\delta,1-\delta]$ for any $\delta\in(0,1/2)$, but may not be bounded near $x=0$ and $x=1$ so that the Chover's LIL can be applied to a larger class of weighted sums.

2 Chover's LIL for Products of Partial Sums

In this section we assume that $\{X,X_n,n\geq 1\}$ is a sequence of positive and independent random variables with a common distribution function $F(x)$ in the domain of attraction of G_α $(1<\alpha<2)$, that is, (1.1) holds.

As in section 1, set $S_n = \sum_{j=1}^n X_j$ for $n \geq 1$. We will consider asymptotic property for products of the partial sums S_n in this section.

The study of limit distribution for the products of sums was initiated first by Arnold and Villaseñor (1998) in order to obtain the limit distribution of sums of record values. It has been furthered by Rempala and Wesolowski (2002), Qi (2003) and Lu and Qi (2004). It is proved in Qi (2003) that for $x \in (-\infty, \infty)$

$$\lim_{n\to\infty} P\left\{ \frac{\mu}{B(n)} \log \frac{\prod_{j=1}^n S_j}{n!\mu^n} \leq x \right\} = G_\alpha(\Gamma(\alpha+1)^{-1/\alpha} x),$$

where $\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x} dx$ and $\mu = EX$.

As an application of the theorems in Section 1, the paper presents a series test to determine the strong limiting behavior of $\log\left\{\prod_{j=1}^n S_j/(n!\mu^n)\right\}$, and then shows Chover's law of the iterated logarithm.

THEOREM 2.1. *Let* $\{p_n, n \geq 1\}$ *be a nondecreasing sequence of positive numbers with* $\lim_{n\to\infty} p_n = \infty$. *If* (1.1) *holds with* $\alpha \in (1,2)$, *then*

$$\limsup_{n\to\infty} \frac{1}{B(np_n)} \left| \log \frac{\prod_{j=1}^n S_j}{n!\mu^n} \right| = \begin{cases} 0 \\ +\infty \end{cases} \Leftrightarrow \sum_{n=1}^\infty \frac{1}{np_n} \begin{cases} < +\infty, \\ = +\infty. \end{cases}$$

We conclude from this theorem that

THEOREM 2.2. *If* (1.1) *holds with* $\alpha \in (1,2)$, *then for each* $r \geq 1$ *we have*

$$\limsup_{n\to\infty} \left| \frac{1}{B(\prod_{i=0}^{r-1} \lg_i(n))} \log \frac{\prod_{j=1}^n S_j}{n!\mu^n} \right|^{1/\lg_{r+1}(n)} = e^{1/\alpha} \quad a.s..$$

In particular, if $r = 1$ *we obtain the Chover's LIL for products of sums*

$$\limsup_{n\to\infty} \left| \frac{1}{B(n)} \log \frac{\prod_{j=1}^n S_j}{n!\mu^n} \right|^{1/\log\log n} = e^{1/\alpha} \quad a.s..$$

REMARK 2.1. As pointed out by a referee, it would be interesting to see whether the Chover's LIL for the product of the partial sums holds for $\alpha \in (0,1]$. We cannot solve the problem here due to the difficulty in approaching logarithm of the product by weighted sums. Meanwhile, it is not clear to us how one can normalize the product and whether a limiting distribution exists when $\alpha < 1$ or $\alpha = 1$ but $E(X) = \infty$.

3 Proofs

Before we proceed to the proofs we rephrase Lemma 3 of Chow and Lai (1973) for convenience.

LEMMA 3.1. *Let $\{Y_n\}$ and $\{Z_n\}$ are two sequences of random variables such that $Y_n + Z_n$ converges to zero almost surely. Assume that $\{\mathcal{F}_n\}$ is a monotone increasing sequence of σ-fields. For each $n \geq 1$, $Y_1, \cdots, Y_n$ are adapted to $\mathcal{F}_n$, and Z_n and $\mathcal{F}_n$ are independent. If $Z_n \xrightarrow{p} 0$, then both Y_n and Z_n converge to zero almost surely.*

PROOF OF THEOREM 1.1.

Since $G(x) = P(|X| > x)$ is a regularly varying function with index $-\alpha$ at infinity, $B(x)$ is a regularly varying function with index $1/\alpha$ from Bingham et al. (1987). Therefore $B(x)$ has the following representation:

$$B(x) = c(x)x^{1/\alpha} \exp\left\{\int_1^x \frac{d(u)}{u} du\right\},$$

where $\lim_{x\to\infty} c(x) =: c \in (0, \infty)$ and $\lim_{x\to\infty} d(x) = 0$. Moreover,

$$xG(B(x)) \sim 1 \text{ as } x \to \infty. \tag{3.1}$$

Set $b(x) = cx^{1/\alpha} \exp\left\{\int_1^x \frac{d(u)}{u} du\right\}$. Then

$$B(x) \sim b(x) \quad \text{as } x \to \infty. \tag{3.2}$$

Note that

$$\frac{b(kp_k)}{\sqrt{k}} = cp_k^{1/2} \exp\left\{\int_1^{kp_k} \frac{d(u) + (2-\alpha)/(2\alpha)}{u} du\right\}.$$

Since $d(u) + (2-\alpha)/(2\alpha) > (2-\alpha)/(4\alpha) > 0$ for all large u and p_k is non-decreasing in k, we conclude that

$$\frac{\sqrt{n}}{b(np_n)} = O(n^{-(2-\alpha)/(4\alpha)}) \to 0. \tag{3.3}$$

We can also see that there exists an integer i_0 such that $\frac{b(kp_k)}{\sqrt{k}}$ is increasing for all $k \geq i_0$ since the integrand in the representation of $\frac{b(kp_k)}{\sqrt{k}}$ is ultimately positive.

First assume that $\sum_{n=1}^{\infty}(np_n)^{-1} < \infty$. We will show (1.6).

We define a non-decreasing sequence of $\{q_n\}$ as follows: first set $q_1 = p_1$; if for some $n \geq 0$, $q_1, \cdots, q_{2^n}$ are well-defined, then for $2^n < j \leq 2^{n+1}$ define q_j by

$$q_j = \begin{cases} p_j & \text{if } p_j < 2q_{2^n} \\ 2q_{2^n} & \text{if } p_j \geq 2q_{2^n}. \end{cases}$$

It is easily verified that $0 \leq q_n \leq p_n$ for all $n \geq 1$, $\sup(q_{2n}/q_n) < \infty$ and $\sum_{n=1}^{\infty}(nq_n)^{-1} < \infty$. Then from Lemma 3.3 in Peng and Qi (2003) we have with probability one

$$\limsup_{n\to\infty} \frac{|S_n - A_n|}{b(nq_n)} = 0,$$

which implies

$$\limsup_{n\to\infty} \frac{|S_n - A_n|}{b(np_n)} = \limsup_{n\to\infty} \frac{|S_n - A_n|}{B(np_n)} = 0 \text{ a.s..}$$

Let $s_i = \sup_{k\geq i} \frac{|S_k - A_k|}{b(kp_k)}$. Then $s_i \to 0$ with probability one as $i \to \infty$. In the meantime, $|S_k - A_k| \leq s_i b(kp_k)$ for all $k \geq i$. Thus,

$$\begin{aligned}
&\left|\sum_{k=1}^{n} a_{n,k}X_k - C_n\right| \\
&= \left|\sum_{k=1}^{n} a_{n,k}[X_k - (A_k - A_{k-1})]\right| \\
&= \left|\sum_{k=1}^{n-1}(a_{n,k} - a_{n,k+1})(S_k - A_k) + a_{n,n}(S_n - A_n)\right| \\
&\leq \sum_{k=1}^{n-1} |a_{n,k} - a_{n,k+1}|\sqrt{\frac{k}{n}}\frac{\sqrt{n}(S_k - A_k)}{\sqrt{k}} + |a_{n,n}(S_n - A_n)| \\
&\leq D_n\sqrt{n} \max_{1\leq k\leq n} \frac{|S_k - A_k|}{\sqrt{k}} \\
&\leq D_n\sqrt{n} \max_{1\leq k<i} \frac{|S_k - A_k|}{\sqrt{k}} + D_n\sqrt{n} \max_{i\leq k\leq n} \frac{|S_k - A_k|}{\sqrt{k}} \\
&\leq D_n\sqrt{n} \max_{1\leq k<i} \frac{|S_k - A_k|}{\sqrt{k}} + D_n\sqrt{n}s_i \max_{i\leq k\leq n} \frac{b(kp_k)}{\sqrt{k}}
\end{aligned}$$

for all $1 \le i \le n$. Therefore, for any fixed integer $i \ge i_0$ we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{D_n b(np_n)} \left| \sum_{k=1}^{n} a_{n,k} X_k - C_n \right| \\
&\le \limsup_{n\to\infty} \frac{\sqrt{n}}{b(np_n)} \max_{1\le k<i} \frac{|S_k - A_k|}{\sqrt{k}} + \limsup_{n\to\infty} \frac{\sqrt{n} s_i}{b(np_n)} \max_{i\le k\le n} \frac{b(kp_k)}{\sqrt{k}} \\
&\le \limsup_{n\to\infty} \frac{\sqrt{n} s_i}{b(np_n)} \frac{b(np_n)}{\sqrt{n}} \\
&= s_i,
\end{aligned}$$

which tends to zero as $i \to \infty$. By virtue of (3.2), (1.6) is proved.

Assume now that $\sum_{n=1}^{\infty} (np_n)^{-1} = \infty$ and $\log D_n = o(\log n)$. We will show (1.7). Notice that for each $1 \le k \le n$,

$$a_{n,k} \le \sum_{i=k}^{n-1} |a_{n,i} - a_{n,i+1}| + |a_{n,n}| \le \sqrt{n} \sum_{i=k}^{n-1} |a_{n,i} - a_{n,i+1}| \sqrt{\frac{i}{n}} + |a_{n,n}| \le D_n \sqrt{n}.$$

Thus, for any fixed integer $j \ge 1$,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{B(np_n)} \left| \sum_{k=1}^{j} a_{n,k} X_k \right| &\le \max_{1\le k\le j} |X_k| \limsup_{n\to\infty} \frac{1}{B(np_n)} \sum_{k=1}^{j} |a_{n,k}| \\
&\le \max_{1\le k\le j} |X_k| \limsup_{n\to\infty} \frac{j D_n \sqrt{n}}{B(np_n)} = 0
\end{aligned}$$

from (3.2) and (3.3) and the assumption that $\log D_n = o(\log n)$. This implies that the limit on the left-hand side of (1.7) is independent of $\{X_i, 1 \le i \le j\}$ for every $j \ge 1$. From Kolmogorov's zero-one law, there exists a non-random constant d such that

$$\limsup_{n\to\infty} \frac{1}{B(np_n)} \left| \sum_{k=1}^{n} a_{n,k} X_k - C_n \right| = d \text{ a.s.} \tag{3.4}$$

We shall prove that $d = \infty$. Otherwise, if $d < \infty$, we will show that this results in some contradiction. We give an outline of the proof.

First, there exists a sequence of non-decreasing constants $\{r_n\}$ with $\lim_{n\to\infty} r_n = \infty$ such that

$$\sum_{n=1}^{\infty} \frac{1}{np_n r_n} = \infty. \tag{3.5}$$

If (3.4) holds for some $d < \infty$, then

$$\limsup_{n\to\infty} \frac{1}{B(np_n r_n)} \left| \sum_{k=1}^{n} a_{n,k} X_k - C_n \right| = 0$$

with probability one.

Second, let $\{X', X'_n, n \geq 1\}$ be an independent copy of $\{X, X_n, n \geq 1\}$ and set $X^s = X - X'$ and $X_n^s = X_n - X'_n$ for $n \geq 1$. The above equation gives

$$\limsup_{n\to\infty} \frac{1}{B(np_n r_n)} \left| \sum_{k=1}^{n} a_{n,k} X'_k - C_n \right| = 0 \text{ a.s.}$$

and thus

$$\limsup_{n\to\infty} \frac{1}{B(np_n r_n)} \left| \sum_{k=1}^{n} a_{n,k} X_k^s \right| = 0 \text{ a.s..}$$

Set

$$Y_j = \frac{1}{B(n(j) p_{n(j)} r_{n(j)})} \sum_{k=1}^{m(j)} a_{n(j),k} X_k^s$$

and

$$Z_j = \frac{1}{B(n(j) p_{n(j)} r_{n(j)})} \sum_{k=m(j)+1}^{n(j)} a_{n(j),k} X_k^s.$$

Then $Y_j + Z_j$ converges to zero with probability one. Since Y_j and Z_j are independent symmetric random variables, it follows from Levy inequality that for any $\delta > 0$, $P(|Z_j| > \delta) \leq 2P(|Y_j + Z_j| > \delta) \to 0$ as $j \to \infty$, i.e., $Z_j \xrightarrow{p} 0$. Denote $\mathcal{F}_j$ as the σ field generated by random variables X_i, $1 \leq i \leq m(j)$. By Lemma 3.1, we have

$$Y_j = \frac{1}{B(n(j) p_{n(j)} r_{n(j)})} \sum_{k=1}^{m(j)} a_{n(j),k} X_k^s \to 0 \text{ a.s.}$$

Similarly, one can show

$$\frac{1}{B(n(j) p_{n(j)} r_{n(j)})} \sum_{k=1}^{m(j)-1} a_{n(j),k} X_k^s \to 0 \text{ a.s.}$$

Combination of these two equations yields

$$\frac{a_{n(j),m(j)} X_{m(j)}^s}{B(n(j) p_{n(j)} r_{n(j)})} \to 0 \text{ almost surely,}$$

which implies

$$\frac{X^s_{m(j)}}{B(n(j)p_{n(j)}r_{n(j)})} \to 0 \text{ a.s.}$$

in view of condition C1. By using Borel-Cantelli lemma, we have

$$\sum_{j=1}^{\infty} P(|X^s| > B(n(j)p_{n(j)}r_{n(j)})) < \infty.$$

From the following inequality (cf., Lemma 10.1.1 in Chow and Teicher, 1997)

$$P(|X - m(X)| > x) \le 2P(|X^s| > x)$$

where $m(X)$ denotes the median of X, we have

$$\sum_{j=1}^{\infty} P(|X - m(X)| > B(n(j)p_{n(j)}r_{n(j)})) < \infty.$$

If $x > |m(X)|$, then

$$G(x + |m(X)|) \le P(|X - m(X)| > x) \le G(x - |m(X)|),$$

and $G(x \pm |m(X)|) \sim G(x)$ as $x \to \infty$ since $G(x)$ is regularly varying. Therefore, if $j \to \infty$, we have

$$P(|X - m(X)| > B(n(j)p_{n(j)}r_{n(j)})) \sim G(B(n(j)p_{n(j)}r_{n(j)})) \sim \frac{1}{n(j)p_{n(j)}r_{n(j)}},$$

which implies

$$\sum_{j=1}^{\infty} \frac{1}{n(j)p_{n(j)}r_{n(j)}} < \infty.$$

From C1 we conclude that

$$\sum_{n=n(1)}^{\infty} \frac{1}{np_nr_n} = \sum_{j=1}^{\infty} \sum_{i=n(j)}^{n(j+1)} \frac{1}{ip_ir_i} \le \sum_{j=1}^{\infty} \frac{1}{n(j)p_{n(j)}r_{n(j)}} \sup_j (n(j+1) - n(j)) < \infty,$$

contradicting (3.5). This completes the proof. □

Proof of Theorem 1.2. We need to prove (1.9) only. Let "i.o." denote "infinitely often". It suffices to show that for every $\epsilon \in (0,1)$

$$P\left(\left(\frac{|\sum_{k=1}^{n} a_{n,k}X_k - C_n|}{B(\prod_{i=0}^{r-1} \lg_i(n))}\right)^{1/\lg_{r+1}(n)} > e^{(1+\epsilon)/\alpha} \text{ i.o.}\right) = 0$$

and

$$P\left(\left(\frac{|\sum_{k=1}^{n} a_{n,k}X_k - C_n|}{B(\prod_{i=0}^{r-1}\lg_i(n))}\right)^{1/\lg_{r+1}(n)} > e^{(1-\epsilon)/\alpha} \text{ i.o.}\right) = 1,$$

or equivalently

$$P\left(\left|\sum_{k=1}^{n} a_{n,k}X_k - C_n\right| > B\left(\prod_{i=0}^{r-1}\lg_i(n)\right)e^{((1+\epsilon)/\alpha)\lg_{r+1}(n)} \text{ i.o.}\right) = 0 \quad (3.6)$$

and

$$P\left(\left|\sum_{k=1}^{n} a_{n,k}X_k - C_n\right| > B\left(\prod_{i=0}^{r-1}\lg_i(n)\right)e^{((1-\epsilon)/\alpha)\lg_{r+1}(n)} \text{ i.o.}\right) = 1. \quad (3.7)$$

In view of Theorem 1.1, the proof of (3.7) follows from the same argument as that in the proof of Theorem 2.1 in Peng and Qi (2003). So we only need to show (3.6).

Now we take $p_n = [\lg_r(n)]^{\epsilon_1}\prod_{i=1}^{r}\lg_i(n)$ for $\epsilon_1 = \epsilon/3$, that is, $np_n = [\lg_r(n)]^{1+\epsilon_1}\prod_{i=0}^{r-1}\lg_i(n)$ for all large n. Since

$$\sum_{n=1}^{\infty}\frac{1}{[\lg_r(n)]^{1+\epsilon_1}\prod_{i=0}^{r-1}\lg_i(n)} < \infty,$$

we have from (1.6) that

$$P\left(\left|\sum_{k=1}^{n} a_{n,k}X_k - C_n\right| > D_nB\left([\lg_r(n)]^{1+\epsilon_1}\prod_{i=0}^{r-1}\lg_i(n)\right) \text{ i.o.}\right) = 0.$$

Hence, we only need to prove that for all large n

$$D_nB\left([\lg_r(n)]^{1+\epsilon_1}\prod_{i=0}^{r-1}\lg_i(n)\right) < B\left(\prod_{i=0}^{r-1}\lg_i(n)\right)e^{((1+\epsilon)/\alpha)\lg_{r+1}(n)}. \quad (3.8)$$

In fact, it has been proved in the proof of Theorem 2.1 in Peng and Qi (2003) that for all large n

$$\frac{B([\lg_r(n)]^{1+\epsilon_1}\prod_{i=0}^{r-1}\lg_i(n))}{B(\prod_{i=0}^{r-1}\lg_i(n))} < e^{((1+2\epsilon_1)/\alpha)\lg_{r+1}(n)}.$$

Since we can conclude from (1.8) that $D_n < e^{(\epsilon_1/\alpha)\lg_{r+1}(n)}$ for all large n, (3.8) follows. This completes the proof. □

Proof of Theorem 1.3. The theorem is a straightforward application of Theorem 1.3 with $a_{n,k} = h(\frac{k}{n+1})$. Verification of conditions C1 and (1.8) is easy, and thus the detail is omitted. □

Proof of Theorem 2.1. It is proved in Qi (2003) that

$$\left(\prod_{j=1}^{n} \frac{\log(1+\frac{1}{j})}{\frac{1}{j}}\right)^{\mu/B(n)} \to 1.$$

See the second equation in the proof of Theorem 2.2 in Qi (2003). Also by combining both the equation above equation (2.10) and the last equation in the proof of Theorem 2.2 in Qi (2003) we have

$$\left(\prod_{j=1}^{n} \frac{\log(1+\frac{1}{j})S_j}{\mu}\right)^{\mu/B(n)} = \exp\left\{\sum_{j=1}^{n} \frac{\log(\frac{n+1}{j})}{B(n)}(X_j-\mu)+o(1)\right\},$$

where $o(1)$ is an error term that converges to zero almost surely. Therefore, as $n \to \infty$

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{\mu/B(n)} = \exp\left\{\sum_{k=1}^{n} \frac{\log(\frac{n+1}{k})}{B(n)}(X_k-\mu)+o(1)\right\}.$$

By taking logarithm on both sides and applying the strong law of large numbers we have

$$\log \frac{\prod_{k=1}^{n} S_k}{n!\mu^n} = \frac{1}{\mu}\sum_{k=1}^{n} \log(\frac{n+1}{k})(X_k-\mu)+o(B(n)). \tag{3.9}$$

Since the assumption $p_n \to \infty$ implies $B(n)/B(np_n) \to \infty$, we have

$$\frac{1}{B(np_n)} \log \frac{\prod_{k=1}^{n} S_k}{n!\mu^n} = \frac{1}{B(np_n)\mu}\sum_{k=1}^{n} \log(\frac{n+1}{k})(X_k-\mu)+o(1).$$

Set $a_{n,k} = \frac{1}{\mu}\log\frac{n+1}{k}$. It is easy to verify that C1 holds and that D_n is bounded. The theorem follows from Theorem 1.1. □

Proof of Theorem 2.2. With representation (3.9) the theorem is a direct consequence of Theorem 1.2 or Theorem 1.3 □

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