

## A Single Vacation Model G/M/1/K with $N$ Threshold Policy

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### Abstract

This paper studies a G/M/1/K queueing system, where the server applies an  $N$  policy and takes a single vacation when the system is empty. We provide a recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining inter-arrival time, to develop the steady-state probability distributions of the number of customers in the system. The method is illustrated analytically for exponential inter-arrival time distribution. Hereby, we establish the distributions of the number of customers in the queue at pre-arrival epochs and at arbitrary epochs as well as the distributions of the waiting time and the busy period.

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### 1 Introduction

Consider the finite capacity G/M/1 queueing system with  $N$  policy and a single vacation. The server leaves for a vacation of random length whenever the system becomes empty. Returning from the vacation, the server inspects the system and decides whether to remain idle in the system or to begin serving exhaustively, based on whether the number of customers is less than some predetermined threshold or not, respectively.

The input queue uses non-Poisson process, and as a result, modeling and mathematical analysis is difficult. Chatterjee and Mukherjee (1990) used the embedded Markov chain to examine the ordinary GI/M/1 queueing system with exponential vacation. Tian et al. (1989) proposed a similar analysis for the ordinary GI/M/1 queueing system where the server takes exponential vacations. Karaesmen and Gupta (1996) used the embedded Markov chain to investigate the ordinary GI/M/1 queueing system, with finite capacity and server vacation. They proposed heuristic procedures to compute the blocking probability, and provided an explicit solution for special arrival processes. Laxmi and Gupta (1999) used both the supplementary variable technique and the embedded Markov chain method to study the ordinary G/M/1 queueing system with finite capacity, in which the customers are served in batches.

The supplementary variable technique was first introduced by Cox (1955), and in this technique, a non-Markovian process in continuous time is made Markovian by the inclusion of one or more supplementary variables. The supplementary variable technique has been used by many authors to solve a good number of queueing problems, and their solutions seem to be pretty good, and the mathematics involved becomes less cumbersome (see Keilson and Kooharian, 1960, Hokstad, 1975, Chaudhry and Templeton, 1983, Gupta and Rao, 1994, 1996, Takagi, 1991, Laxmi and Gupta, 1999, Ke and Wang, 2002). Even in the steady-state case, many problems are more readily treated by the supplementary variable technique than by the embedded Markov chain technique. (see Chaudhry and Templeton, 1983, Chapter 2). In order to obtain the steady-state probability distributions easily, we study the single vacation G/M/1 queueing system using supplementary variable technique rather than other techniques. The supplementary variable technique used to solve our model is pretty good and gets the exact solution in a computable form.

The  $N$ -policy without vacation was first introduced by Yadin and Naor (1963). Heyman (1968) considered the M/G/1 queueing system with  $N$ -policy. The extensions of this model can be found in Hersh and Brosh (1980), Kimura (1981), Tijms (1986), Teghem (1987), Gakis et al. (1995), Artalejo (1998), Ke and Wang (2002), and others. Recently, an infinite buffer G/M/1 queueing system with  $N$ -policy was studied via the embedded Markov chain by Zhang and Tian (2004). The system characteristics are obtained by Zhang and Tian through geometric-matrix forms.

Queueing systems with server vacations have attracted much attention from numerous researchers since Levy and Yechiali (1975). Server vacations are useful for the system in which the server wants to utilize his idle time for different purposes. Some excellent surveys of queueing systems with server vacations were done by Doshi (1986) and Takagi (1991). Queueing systems with  $N$ -policy and multiple vacations including some applications were at first studied by Lee and Srinivasan (1989) and Kella (1989). They respectively dealt with the batch arrival  $M^{[x]}/G/1$  and the single-unit arrival  $M/G/1$  queueing systems, examined the system performance and obtained the optimal threshold under a stationary cost function. Lee et al. (1994, 1995) analyzed in detail Lee and Srinivasan's system with a single vacation and multiple vacations, respectively. They provided the probabilistic interpretation of the single (multiple) vacation system with a single threshold policy, and their results confirmed the stochastic decomposition property given by Fuhrmann and Cooper (1985). Recently, Krishna Reddy et al. (1998) examined Lees' system by combining both vacations and startup of a reliable server. Ke (2003) investigated Lee and Srinivasan's system by considering vacations and startup of an un-reliable server. However, so far very few authors have studied the control policy of the  $G/M/1$  queue model with single vacation. This motivates us to develop the  $N$ -policy for the finite capacity  $G/M/1$  queueing system, where the server is characterized by single vacation.

In the system studied here, we assume that the times elapsed between successive arrivals are independent and identically distributed random variables with general distribution  $A(u)$  ( $u \geq 0$ ), probability density function  $a(u)$  ( $u \geq 0$ ) and mean inter-arrival time  $a_1$ . As soon as the system becomes empty, the server goes into exactly one vacation of random length  $V$  with  $\Pr[V \leq t] = 1 - e^{-\eta t}$ . If the server returns from the vacation and finds the system size is less than  $N$ , the server remains in the system but does not begin to serve until there are exactly  $N$  customers. If upon return from vacation, the server finds  $N$  or more customers queueing for service, the server immediately begins serving the waiting customers until the system is empty. The service times for the successive customers are independent random variables and follow a common exponential distribution with mean  $1/\mu$ . The service process is independent of the arrival process. We assume that arriving customers form a single waiting line based on the order of their arrivals, i.e., a 'first-come, first-served' discipline is followed. As only one customer can be served at a time, customers have to wait in the queue when they enter the service facility and find that the server is busy.

In this paper, we study the finite capacity G/M/1 queueing system under  $N$ -policy with a single vacation. The purpose of this paper is fourfold. First, we provide a recursive method by using the supplementary variable technique, and then treat the supplementary variable as the remaining inter-arrival time to obtain the steady-state probability distributions for the number of customers in the G/M/1/K queueing system with  $N$ -policy and a single exponential vacation. Second, we illustrate a recursive method by presenting one simple example for exponential inter-arrival time. Third, we derive some system characteristics such as the blocking probability, the waiting time distribution in the queue and the distribution of the busy period. Finally, we report numerical results for system characteristics on specified parameters and discuss the sensitivity of the system parameters.

*1.1. Notations.* The following notation and probabilities are used throughout the paper.

$N$  – threshold level

$K$  – system capacity ( $N < K$ )

$A(u)$  – distribution function of inter-arrival time  $A$

$a(u)$  – probability density function of inter-arrival time  $A$

$a^*(s)$  – Laplace-Stieltjes transform (LST) of inter-arrival time  $A$

$a^{*(l)}(s)$  –  $l$ th order derivative of  $a^*(s)$  with respect to  $s$

$P_n$  – steady-state probability of  $n$  customers in the system when the server is on vacation, where  $n = 0, 1, 2, \dots, K$

$Q_n$  – steady-state probability of  $n$  customers in the system when the server is dormant in the system, where  $n = 0, 1, 2, \dots, N - 1$

$R_n$  – steady-state probability of  $n$  customers in the system when the server is working, where  $n = 1, 2, \dots, K$

$P_n^*(s)$  – LST of  $P_n(u)$

$Q_n^*(s)$  – LST of  $Q_n(u)$

$R_n^*(s)$  – LST of  $R_n(u)$

$P_n^{*(l)}(s)$  –  $l$ th order derivative of  $P_n^*(s)$  with respect to  $s$

$Q_n^{*(l)}(s)$ –  $l$ th order derivative of  $Q_n^*(s)$  with respect to  $s$

$R_n^{*(l)}(s)$ –  $l$ th order derivative of  $Q_n^*(s)$  with respect to  $s$

$a_1$ – mean inter-arrival time

$\rho$ – offered load, where  $\rho = (a_1\mu)^{-1}$

$\lambda'$ – effective arrival rate

## 2 Formulation and Steady-state Results

We first establish the mathematical equations that govern the system, by using the remaining inter-arrival time as the supplementary variable. Next, we develop a recursive method to derive the steady-state probability distributions of the number of customers in the system.

The state of the system at time  $t$  is given by

$X(t) \equiv$  number of customers in the system, and

$U(t) \equiv$  remaining inter – arrival time for the customer who is arriving.

Let us define

$$P_n(u, t)du = \Pr\{X(t) = n, u < U(t) \leq u + du, \text{ the server is on vacation}\}, \\ u \geq 0, \quad n = 0, 1, \dots, K,$$

$$Q_n(u, t)du = \Pr\{X(t) = n, u < U(t) \leq u + du, \text{ the server is dormant in} \\ \text{the system}\}, u \geq 0, \quad n = 0, 1, \dots, N - 1, \text{ and}$$

$$R_n(u, t)du = \Pr\{X(t) = n, u < U(t) \leq u + du, \text{ the server is working}\}, \\ u \geq 0, \quad n = 1, 2, \dots, K.$$

From the defined probabilities, we can obtain the steady-state equations as follows:

$$-\frac{d}{du}P_0(u) = \mu R_1(u) - \eta P_0(u), \quad (2.1)$$

$$-\frac{d}{du}P_n(u) = a(u)P_{n-1}(0) - \eta P_n(u), \quad 1 \leq n \leq K - 1, \quad (2.2)$$

$$-\frac{d}{du}P_K(u) = a(u)[P_{K-1}(0) + P_K(0)] - \eta P_K(u), \quad n = K, \quad (2.3)$$

$$-\frac{d}{du}Q_0(u) = \eta P_0(u), \quad (2.4)$$

$$-\frac{d}{du}Q_n(u) = \eta P_n(u) + a(u)Q_{n-1}(u), \quad 1 \leq n \leq N - 1, \quad (2.5)$$

$$-\frac{d}{du}R_1(u) = \mu R_2(u) - \mu R_1(u), \tag{2.6}$$

$$-\frac{d}{du}R_n(u) = a(u)R_{n-1}(0) + \mu R_{n+1}(u) - \mu R_n(u), \quad 2 \leq n \leq N - 1, \tag{2.7}$$

$$-\frac{d}{du}R_N(u) = \eta P_N(u) + a(u)R_{N-1}(0) + \mu R_{N+1}(u) + a(u)Q_{N-1}(0) - \mu R_N(u), \quad n = N, \tag{2.8}$$

$$-\frac{d}{du}R_n(u) = \eta P_n(u) + a(u)R_{n-1}(0) + \mu R_{n+1}(u) - \mu R_n(u), \quad N + 1 \leq n \leq K - 1, \tag{2.9}$$

$$-\frac{d}{du}R_K(u) = \eta P_K(u) + a(u)[R_{K-1}(0) + R_K(0)] - \mu R_K(u), \quad n = K. \tag{2.10}$$

We introduce the following Laplace-Stieltjes transform for any probability function  $S_n(u)$  like  $P_n(u)$ ,  $Q_n(u)$  and  $R_n(u)$  :

$$S_n^*(s) = \int_0^\infty e^{-su} S_n(u) du,$$

$$S_n = S_n^*(0) = \int_0^\infty S_n(u) du,$$

and

$$\int_0^\infty e^{-su} \frac{\partial}{\partial u} S_n(u) du = sS_n^*(s) - S_n(0).$$

Further, we define

$$P_0(u) = a(u)P_0. \tag{2.11}$$

Taking the LST on both sides of (2.11), it yields

$$P_0^*(s) = a^*(s)P_0. \tag{2.12}$$

Similarly, proceeding in the usual manner with (2.1)-(2.10), we get

$$(\eta - s)P_0^*(s) = \mu R_1^*(s) - P_0(0), \tag{2.13}$$

$$(\eta - s)P_n^*(s) = a^*(s)P_{n-1}(0) - P_n(0), \quad 1 \leq n \leq K - 1, \tag{2.14}$$

$$(\eta - s)P_K^*(s) = a^*(s)[P_{K-1}(0) + P_K(0)] - P_K(0), \quad n = K, \tag{2.15}$$

$$-sQ_0^*(s) = \eta P_0^*(s) - Q_0(0), \tag{2.16}$$

$$-sQ_n^*(s) = \eta P_n(s) + a^*(s)Q_{n-1}(0) - Q_n(0), \quad 1 \leq n \leq N - 1, \tag{2.17}$$

$$(\mu - s)R_1^*(s) = \mu R_2^*(s) - R_1(0), \tag{2.18}$$

$$(\mu - s)R_n^*(s) = a^*(s)R_{n-1}(0) + \mu R_{n+1}^*(s) - R_n(0), \quad 2 \leq n \leq N-1, \quad (2.19)$$

$$\begin{aligned} (\mu - s)R_N^*(s) &= \eta P_N^*(s) + a^*(s)R_{N-1}(0) + \mu R_{N+1}^*(s) \\ &\quad + a^*(s)Q_{N-1}(0) - R_N(0), \quad n = N, \end{aligned} \quad (2.20)$$

$$\begin{aligned} (\mu - s)R_n^*(s) &= \eta P_n^*(s) + a^*(s)R_{n-1}(0) + \mu R_{n+1}^*(s) - R_n(0), \\ &\quad N+1 \leq n \leq K-1, \end{aligned} \quad (2.21)$$

$$(\mu - s)R_K^*(s) = \eta P_K^*(s) + a^*(s)[R_{K-1}(0) + R_K(0)] - R_K(0), \quad n = K. \quad (2.22)$$

Our main task is to find the steady-state probabilities  $P_n = P_n^*(0)$  ( $0 \leq n \leq K$ ),  $Q_n = Q_n^*(0)$  ( $0 \leq n \leq N-1$ ), and  $R_n = R_n^*(0)$  ( $1 \leq n \leq K$ ).

### 2.1. $P_n^*(0)$ in terms of $P_0(0)$ .

Inserting  $s = \eta$  in (2.14)-(2.15), we obtain

$$P_n(0) = [a^*(\eta)]^n P_0(0), \quad 1 \leq n \leq K-1, \quad (2.23)$$

$$P_K(0) = \frac{[a^*(\eta)]^K}{1 - a^*(\eta)} P_0(0), \quad n = K. \quad (2.24)$$

Inserting  $s = 0$  in (2.13)-(2.15) and using (2.23)-(2.24), it finally yields

$$P_0 = P_0^*(0) = \frac{\mu R_1^*(0) - P_0(0)}{\eta}, \quad (2.25)$$

$$P_n = P_n^*(0) = \frac{1}{\eta} [1 - a^*(\eta)] [a^*(\eta)]^{n-1} P_0(0), \quad 1 \leq n \leq K-1, \quad (2.26)$$

$$P_K = P_K^*(0) = \frac{1}{\eta} [a^*(\eta)]^{K-1} P_0(0), \quad n = K. \quad (2.27)$$

### 2.2. $Q_n^*(0)$ in terms of $P_0(0)$ and $P_0^*(0)$ .

Substituting  $s = 0$  into (2.16)-(2.17) and using (2.26)-(2.27), we have

$$Q_0(0) = \eta P_0^*(0), \quad (2.28)$$

$$Q_n(0) = [1 - a^*(\eta)^n] P_0(0) + \eta P_0^*(0), \quad n = 1, 2, \dots, N-1. \quad (2.29)$$

Differentiating (2.16)-(2.17) with respect to  $s$  and setting  $s = 0$ , it follows from equations (A.1)-(A.2) of the Appendix that

$$Q_0^*(0) = -\eta P_0^{*(1)}(0) = -\eta a^{*(1)}(0) P_0^*(0) = a_1 \eta P_0, \quad (2.30)$$

$$\begin{aligned}
 Q_n^*(0) &= -\eta P_n^{*(1)}(0) - a^{*(1)}(0)Q_{n-1}(0) = -\eta P_n^{*(1)}(0) + a_1 Q_{n-1}(0) \\
 &= \left( a_1 - \frac{1 - a^*(\eta)}{\eta} \right) [a^*(\eta)]^{n-1} P_0(0) + a_1 Q_{n-1}(0), n=1, 2, \dots, N-1,
 \end{aligned}
 \tag{2.31}$$

where  $a_1 = -a^{*(1)}(0)$  denotes the mean inter-arrival time. It is to be noted that  $Q_n^*(0)$  can be expressed using (2.28)-(2.31) in terms of  $P_0(0)$  and  $P_0^*(0)$ .

We sum (2.13)-(2.22) and finally obtain

$$\begin{aligned}
 &\sum_{n=0}^K P_n^*(s) + \sum_{n=0}^{N-1} Q_n^*(s) + \sum_{n=1}^K R_n^*(s) \\
 &= \frac{1 - a^*(s)}{s} \left[ \sum_{n=0}^K P_n(0) + \sum_{n=0}^{N-1} Q_n(0) + \sum_{n=1}^K R_n(0) \right].
 \end{aligned}
 \tag{2.32}$$

Taking  $\lim_{s \rightarrow 0}$  in (2.32) and use of l'Hôspital's rule gives

$$\begin{aligned}
 &\sum_{n=0}^K P_n^*(0) + \sum_{n=0}^{N-1} Q_n^*(0) + \sum_{n=1}^K R_n^*(0) \\
 &= a_1 \left[ \sum_{n=0}^K P_n(0) + \sum_{n=0}^{N-1} Q_n(0) + \sum_{n=1}^K R_n(0) \right].
 \end{aligned}
 \tag{2.33}$$

Using the normalization condition, namely  $\sum_{n=0}^K P_n^*(0) + \sum_{n=0}^{N-1} Q_n^*(0) + \sum_{n=1}^K R_n^*(0) = 1$ , we have

$$\sum_{n=0}^K P_n(0) + \sum_{n=0}^{N-1} Q_n(0) + \sum_{n=1}^K R_n(0) = \frac{1}{a_1}.
 \tag{2.34}$$

Substituting (2.23)-(2.31) into (2.33) and doing algebraic simplification, it finally yields

$$P_0^*(0) + \sum_{n=1}^K R_n^*(0) = \left[ a_1 + a_1 \frac{[a^*(\eta)]^N}{1 - a^*(\eta)} - \frac{[a^*(\eta)]^{N-1}}{\eta} \right] P_0(0) + a_1 \sum_{n=1}^K R_n(0).
 \tag{2.35}$$

### 2.3. Recursions for $R_n^*(0)$ .

We develop a recursive method to obtain the explicit expressions for the steady-state probabilities  $R_n = R_n^*(0)$  ( $1 \leq n \leq K$ ). The algorithm first obtains  $R_n(0)$ , and then using it, we finally get  $R_n^*(0)$ .

We first derive the expressions of  $R_n(0)$  ( $1 \leq n \leq K - 1$ ) in terms of  $P_0(0)$ ,  $Q_{N-1}(0)$  and  $R_K(0)$ . Setting  $s = \mu$  in (2.19)-(2.22), we finally obtain

$$R_{K-1}(0) = \frac{1 - a^*(\mu)}{a^*(\mu)} R_K(0) - \eta \frac{P_K^*(\mu)}{a^*(\mu)}, \quad (2.36)$$

$$R_{n-1}(0) = \frac{R_n(0) - \mu R_{n+1}^*(\mu)}{a^*(\mu)} - \eta \frac{P_n^*(\mu)}{a^*(\mu)} I_{\{n \geq N\}} - Q_{N-1}(0) I_{\{n=N\}},$$

$$n = K - 1, K - 2, \dots, 2, \quad (2.37)$$

where  $I_{\{\cdot\}}$  is the indicator function and  $P_n^*(\mu)$  are given in (A.3)-(A.4). To obtain  $R_{n+1}^*(\mu)$  ( $2 \leq n \leq K - 1$ ) in (2.37), we differentiate (2.19)-(2.22)  $l$  times with respect to  $s$  and set  $s = \mu$ , and finally get

$$R_K^{*(l-1)}(\mu) = -\frac{1}{l} \left[ a^{*(l)}(\mu) (R_{K-1}(0) + R_K(0)) + \eta P_K^{*(l)}(\mu) \right], \quad l=1, 2, \dots, K-1 \quad (2.38)$$

$$R_n^{*(l-1)}(\mu) = -\frac{1}{l} \left[ a^{*(l)}(\mu) R_{n-1}(0) + \mu R_{n+1}^{*(l)}(\mu) + \eta P_n^{*(l)}(\mu) I_{\{n \geq N\}} \right. \\ \left. + Q_{N-1}(0) I_{\{n=N\}} \right], \quad n = K-1, K-2, \dots, 2; \quad l = 1, 2, \dots, n-1, \quad (2.39)$$

where  $R_n^{*(0)}(\mu) = R_n^*(\mu)$  and  $P_n^{*(l)}(\mu)$  are given in (A.7).

Solving (2.38)-(2.39) recursively, we obtain for  $n = K, K-1, \dots, 2$ ,

$$R_n^*(\mu) = \sum_{j=1}^{K-n} \frac{(-1)^j \mu^{j-1} a^{*(j)}(\mu)}{j!} R_{n+j-2}(0) \\ + \frac{(-1)^{K-n+1} \mu^{K-n} a^{*(K-n+1)}(\mu)}{(K-n+1)!} [R_{K-1}(0) + R_K(0)] \\ + \eta \sum_{j=\max(n, N)}^K \frac{(-1)^{j-n+1} \mu^{j-n} P_j^{*(j-n+1)}(\mu)}{(j-n+1)!} \\ + \frac{(-1)^{N-n+1} \mu^{N-n} a^{*(N-n+1)}(\mu)}{(N-n+1)!} Q_{N-1}(0) I_{\{n \geq N\}}. \quad (2.40)$$

Hence  $R_{K-1}(0), R_{K-2}(0), \dots, R_1(0)$  can be obtained recursively using (2.40) and (A.7) in (2.36)-(2.37) in terms of  $P_0(0), Q_{N-1}(0)$  and  $R_K(0)$ .

To give convenient and explicit expressions for  $R_n(0)$  ( $1 \leq n \leq K-1$ ), we define  $\psi_j$  ( $j = 1, 2, \dots, K-1$ ) as follows:

$$\psi_j = \begin{cases} \frac{1 + \mu a^{*(1)}(\mu)}{a^*(\mu)}, & j = 1, \\ \frac{(-1)^{j+1} \mu^j a^{*(j)}(\mu)}{j! a^*(\mu)}, & j = 2, 3, \dots, K-1, \\ 0, & \text{otherwise.} \end{cases} \quad (2.41)$$

Thus, (2.36)-(2.37) can be re-written as

$$R_{K-1}(0) = \frac{1 - a^*(\mu)}{a^*(\mu)} R_K(0) - \frac{\eta G_K^{(0)}}{a^*(\mu)} P_0(0), \quad (2.42)$$

and

$$\begin{aligned} R_n(0) &= \sum_{j=1}^{K-n-2} \left[ \psi_j R_{n+j}(0) + \psi_{K-n-1-j} I_{\{K=N+j\}} Q_{N-1}(0) \right] \\ &+ \frac{\psi_{K-n-1} - I_{\{n=K-2\}}}{a^*(\mu)} \cdot R_K(0) \\ &- \left( \psi_{K-n-1} G_K^{(0)} + \sum_{j=\max(n+1, N)}^K \frac{(-\mu)^{j-n-1} G_j^{(j-n-1)}}{(j-n-1)!} \right) \frac{\eta}{a^*(\mu)} P_0(0) \\ &- \left( I_{\{n=N-1\}} + \frac{I_{\{n=N-2\}}}{a^*(\mu)} \right) Q_{N-1}(0), \quad n = K-2, K-3, \dots, 1, \end{aligned} \quad (2.43)$$

where  $G_n^{(l)}$  is given in (A.7).

Further, define

$$\Psi(n) = \begin{cases} 1, & n = 0, \\ \sum_{1 \leq k \leq n} \sum_{\substack{\tau_1 + \tau_2 + \dots + \tau_k = n \\ \tau_1, \tau_2, \dots, \tau_k \in \{1, 2, \dots, n\}}} \psi_{\tau_1} \psi_{\tau_2} \dots \psi_{\tau_k}, & n = 1, 2, \dots, k-1 \\ 0, & \text{otherwise.} \end{cases} \quad (2.44)$$

$$\begin{aligned} \Delta_1(n) = & \frac{\eta}{a^*(\mu)} \left[ \sum_{k=1}^{N-n-1} \sum_{j=k}^{N-n-1} \frac{(-1)^k \Psi(j-k) \mu^{k-1}}{(k-1)!} G_{n+j}^{(k-1)} \right. \\ & \left. - \sum_{k=1}^{K-n} \sum_{j=k}^{K-n} \frac{(-1)^k \Psi(j-k) \mu^{k-1}}{(k-1)!} G_{n+j}^{(k-1)} \right], \quad n = 1, 2, \dots, K-2, \end{aligned} \quad (2.45)$$

and

$$\Delta_2(n) = \sum_{k=n}^{K-2} I_{\{k=N-2\}} \cdot \frac{\Psi(k-n)}{a^*(\mu)} + I_{\{n=N-1\}}, \quad n = 1, 2, \dots, K-2. \quad (2.46)$$

REMARK 2.1. The crux of (2.44) is to sum up all possible products of  $k$ -tuples of  $\psi$ 's in which the total of subscript values of  $\psi$  equals  $n$ . We give an easily understood example for  $n = 4$ :

$$\begin{aligned} \Psi(4) &= \psi_4 + \psi_3\psi_1 + \psi_2\psi_2 + \psi_1\psi_3 + \psi_1\psi_1\psi_2 + \psi_1\psi_2\psi_1 + \psi_2\psi_1\psi_1 + \psi_1\psi_1\psi_1\psi_1 \\ &= \psi_4 + 2\psi_3\psi_1 + \psi_2^2 + 3\psi_1^2\psi_2 + \psi_1^4. \end{aligned}$$

Using (2.44)-(2.46) to solve (2.43) recursively, we finally get

$$\begin{aligned} R_n(0) &= [\Psi(K-n-1) - \Psi(K-n-2)] \frac{R_K(0)}{a^*(\mu)} \\ &\quad - \Delta_1(n)P_0(0) - \Delta_2(n)Q_{N-1}(0), \quad n = 1, 2, \dots, K-2. \end{aligned} \quad (2.47)$$

Setting  $s = \mu$  in (2.18) yields

$$\mu R_2^*(\mu) = R_1(0). \quad (2.48)$$

Substituting (2.42) and (2.47)-(2.48) into (2.40) yields

$$\begin{aligned} R_K(0) &= \frac{a^*(\mu) \sum_{j=1}^{K-2} \psi_j \Delta_1(j) + \eta \left[ \sum_{j=N}^K \frac{(-\mu)^{j-1} G_j^{(j-1)}}{(j-1)!} + \psi_{K-1} G_K^{(0)} \right]}{\Psi(K-1) - \Psi(K-2)} \times P_0(0) \\ &\quad + \frac{a^*(\mu) \left[ \sum_{n=1}^{K-2} \psi_n \Delta_2(n) + \psi_{N-1} \right] + I_{\{N=2\}} Q_{N-1}(0)}{\Psi(K-1) - \Psi(K-2)}. \end{aligned} \quad (2.49)$$

Finally, we develop the steady-state probabilities  $R_n^*(0)$  in terms of  $P_0(0)$ . Setting  $s = 0$  in (2.13) and (2.18)-(2.22), and using (2.26) and (2.29), we have

$$\begin{aligned}
 R_1^*(0) &= \frac{\eta P_0^*(0) + P_0(0)}{\mu}, \\
 R_n^*(0) &= \frac{\eta P_0^*(0) + P_0(0) + R_{n-1}(0)}{\mu}, \quad n = 2, 3, \dots, N \\
 R_n^*(0) &= \frac{[a^*(\eta)]^{n-1} P_0(0) + R_{n-1}(0)}{\mu}, \quad n = N + 1, N + 2, \dots, K. \quad (2.50)
 \end{aligned}$$

As  $P_0(0), R_1(0), R_2(0), \dots, R_{K-1}(0)$  are completely known,  $R_1^*(0), R_2^*(0), \dots, R_K^*(0)$  can be determined recursively using (2.50) in terms of  $P_0(0)$ .

Substitution of (2.50) into (2.35) yields

$$\begin{aligned}
 P_0^*(0) &= \left( \left[ a_1 + \frac{a_1 [a^*(\eta)]^N}{1 - a^*(\eta)} - \frac{[a^*(\eta)]^{N-1}}{\eta} - \frac{N}{\mu} + \frac{[a^*(\eta)]^K - [a^*(\eta)]^N}{\mu(1 - a^*(\eta))} \right] \cdot P_0(0) \right. \\
 &\quad \left. + \left( a_1 - \frac{1}{\mu} \right) \sum_{n=1}^{K-1} R_n(0) + a_1 R_K(0) \right) \left( 1 + \frac{N\eta}{\mu} \right)^{-1}. \quad (2.51)
 \end{aligned}$$

Applying (2.47) and (2.49) on (2.51), and doing some arduous algebraic manipulations, it finally gives

$$\begin{aligned}
 P_0 = P_0^*(0) &= \left\{ \frac{[a^*(\eta)]^K - [a^*(\eta)]^N}{\mu(1 - a^*(\eta))} + a_1 \left( 1 + \frac{[a^*(\eta)]^N}{1 - a^*(\eta)} \right) - \frac{[a^*(\eta)]^{N-1}}{\eta} - \frac{N}{\mu} \right. \\
 &\quad \left. - \left( a_1 - \frac{1}{\mu} \right) \times \left( \sum_{n=1}^{K-2} \Delta_1(n) + \frac{\eta G_K^{(0)}}{a^*(\mu)} + H_1(N, K) \cdot (1 - [a^*(\eta)]^{N-1}) \right) \right. \\
 &\quad \left. + H_2(N, K) \cdot \left[ a^*(\mu) \sum_{n=1}^{K-2} \psi_n \Delta_1(n) + \eta \sum_{j=N}^K \frac{(-\mu)^{j-1} G_j^{(j-1)}}{(j-1)!} + \eta \psi_{K-1} G_K^{(0)} \right] \right. \\
 &\quad \left. + H_3(N, K) \cdot (1 - [a^*(\eta)]^{N-1}) \right\} \\
 &\quad \times \frac{P_0(0)}{\left[ 1 + \frac{N\eta}{\mu} + \eta \left( a_1 - \frac{1}{\mu} \right) H_1(N, K) + \eta H_3(N, K) \right]}, \quad (2.52)
 \end{aligned}$$

where

$$\begin{aligned}
 H_1(N, K) &= \sum_{n=1}^{K-2} \Delta_2(n), \\
 H_2(N, K) &= \frac{(a_1 - \frac{1}{\mu}) \frac{\Psi(K-2)}{a^*(\mu)} + \frac{1}{\mu}}{\Psi(K-1) - \Psi(K-2)}, \text{ and} \\
 H_3(N, K) &= H_2(N, K) \cdot \left[ a^*(\mu) \left( \sum_{n=1}^{K-2} \psi_n \Delta_2(n) + \psi_{N-1} \right) + I_{\{N=2\}} \right].
 \end{aligned}$$

So  $R_1^*(0), R_2^*(0), \dots, R_K^*(0)$  are known in terms of  $P_0^*(0)$ , which can be obtained using the normalization condition

$$\sum_{n=0}^K P_n^*(0) + \sum_{n=0}^{N-1} Q_n^*(0) + \sum_{n=1}^K R_n^*(0) = 1. \quad (2.53)$$

We note that  $P_n = P_n^*(0)$  can be determined from (2.26)-(2.27) and (2.52), and  $Q_n = Q_n^*(0)$  can be determined from (2.30)-(2.31) and (2.52).

### 3 Algorithm for Calculating the Steady-state Probabilities

To demonstrate the working schemes of the recursive method, we first describe the algorithm for calculating the steady-state probabilities,  $\{(P_\ell^*(0), Q_m^*(0), R_n^*(0)), 0 \leq \ell \leq K, 0 \leq m \leq N-1, 1 \leq n \leq K\}$ . Next, to illustrate the algorithm, we provide one simple example where the inter-arrival time distribution is exponential. Let  $N$  be the threshold level,  $K$  be the maximum capacity of the system, and let  $a^{*(l)}(s)$  where  $l = 1, 2, \dots, K$  be the  $l$ th derivative of  $a^*(s)$ . Given the values of  $N, K$ , and the LST expression of the inter-arrival time distribution, namely  $a^*(s)$ , the steps of the solution algorithm are stated as follows:

Step 1. For  $n = 1, 2, \dots, K-1$ , compute  $\psi_n$  using (2.41).

Step 2. For  $n = 0, 1, \dots, K-1$ , compute  $\Psi(n)$  using (2.44).

Step 3. For  $l = 0, 1, \dots, K-1$ , and  $n = l+1, l+2, \dots, K$ , compute  $G_n^{(l)}$  using (A.8), (A.9) and (A.10).

Step 4. For  $n = 1, 2, \dots, K-2$ , compute  $\Delta_1(n)$  and  $\Delta_2(n)$  using (2.45) and (2.46), respectively.

Step 5. Set  $P_0 = 1$ .

Step 6. Compute  $P_0(0)$  using (2.52).

Step 7. Compute  $Q_{N-1}(0)$  using (2.29).

Step 8. For  $n = K$ , compute  $R_n(0)$  using (2.49) and the results of Step 6 and Step 7.

Step 9. For  $n = 1, 2, \dots, K-1$ , compute  $R_n(0)$  using (2.47), (2.42) and the results of Step 6 to Step 8.

Step 10. For  $n = 1, 2, \dots, K$ , compute  $R_n^*(0)$  using (2.50) and the results of Steps 6 and 9.

Step 11. For  $n = 1, 2, \dots, K$ , compute  $P_n$  using (2.26)-(2.27) and the results of Step 6.

Step 12. For  $n = 0, 1, 2, \dots, N-1$ , compute  $Q_n = Q_n^*(0)$  using (2.30)-(2.31) and the results of Step 6.

Step 13. Compute  $\text{sum} = 1 + \sum_{n=1}^K P_n^*(0) + \sum_{n=0}^{N-1} Q_n^*(0) + \sum_{n=1}^K R_n^*(0)$ .

Step 14. For  $n = 1, 2, \dots, K$ , compute  $R_n^*(0) := R_n^*(0)/\text{sum}$ .

Step 15. For  $n = 0, 1, \dots, N-1$ , compute  $Q_n^*(0) := Q_n^*(0)/\text{sum}$ .

Step 16. For  $n = 0, 1, \dots, K$ , compute  $P_n := P_n/\text{sum}$ .

We now use the algorithm to illustrate a recursive method, which computes the steady-state probability distributions of the number of customers in the system. We provide analytical results for exponential inter-arrival distribution.

### 3.1. Example for the single vacation queue M/M/1/K with $N$ -policy..

We set the exponential vacation mean  $1/\eta$  and the mean inter-arrival time  $a_1 = 1/\lambda$ , where  $\lambda$  is the inter-arrival rate. Assume that  $N = 3$  and  $K = 5$ . In this case, we have

$$a^*(s) = \frac{\lambda}{\lambda + s}.$$

Step 1: For  $n = 1, 2, 3, 4$ , calculate  $\psi_n$ .

From (2.41), we have

$$\psi_1 = \frac{1 + \rho + \rho^2}{\rho + \rho^2}, \quad \psi_2 = -\frac{1}{(1 + \rho)^2}, \quad \psi_3 = -\frac{1}{(1 + \rho)^3}, \quad \psi_4 = -\frac{1}{(1 + \rho)^4},$$

where  $\rho = \lambda/\mu$ .

Step 2: For  $n = 0, 1, 2, 3, 4$ , calculate  $\Psi(n)$ .

It follows from (2.44) that

$$\begin{aligned} \Psi(0) &= 1, & \Psi(1) &= \frac{1 + \rho + \rho^2}{\rho + \rho^2}, & \Psi(2) &= \frac{1 + \rho^2}{\rho^2}, \\ \Psi(3) &= \frac{1 + \rho + \rho^2 + \rho^3 + \rho^4}{(1 + \rho)\rho^3}, & \Psi(4) &= \frac{1 + \rho^2 + \rho^4}{\rho^4}. \end{aligned}$$

Step 3: For  $l = 0, 1, 2, 3, 4$ , and  $n = l + 1, l + 2, \dots, 5$ , calculate  $G_n^{(l)}$ .

Using (A.8), (A.9) and (A.10), we get

$$\begin{aligned} G_1^{(0)} &= G_2^{(0)} = G_2^{(1)} = 0, \\ G_3^{(0)} &= \frac{\rho\lambda^2}{(1 + \rho)(\eta + \lambda)^3}, & G_4^{(0)} &= \frac{\rho\lambda^3}{(1 + \rho)(\eta + \lambda)^4}, \\ G_5^{(0)} &= \frac{\rho\lambda^4}{\eta(1 + \rho)(\eta + \lambda)^4}, & G_3^{(1)} &= \frac{-\rho^2\lambda}{(1 + \rho)^2(\eta + \lambda)^3}, \\ G_4^{(1)} &= \frac{-\rho^2\lambda^2}{(1 + \rho)^2(\eta + \lambda)^4}, & G_5^{(1)} &= \frac{-\rho^2\lambda^3}{\eta(1 + \rho)^2(\eta + \lambda)^4}, \\ G_3^{(2)} &= \frac{2\rho^3}{(1 + \rho)^3(\eta + \lambda)^3}, & G_4^{(2)} &= \frac{2\rho^3\lambda}{(1 + \rho)^3(\eta + \lambda)^4}, \\ G_5^{(2)} &= \frac{2\rho^3\lambda^2}{\eta(1 + \rho)^3(\eta + \lambda)^4}, & G_4^{(3)} &= \frac{-6\rho^4}{(1 + \rho)^4(\eta + \lambda)^4}, \\ G_5^{(3)} &= \frac{-6\rho^4\lambda}{\eta(1 + \rho)^4(\eta + \lambda)^4}, & G_5^{(4)} &= \frac{24\rho^5}{\eta(1 + \rho)^5(\eta + \lambda)^4}. \end{aligned}$$

Step 4: For  $n = 1, 2, 3$ , calculate  $\Delta_1(n)$  and  $\Delta_2(n)$ .

Using (2.45) and the results of Step 2 and Step 3, we get

$$\begin{aligned} \Delta_1(1) &= \frac{\lambda^2}{(\lambda + \eta)^2} \left[ 1 + \frac{1}{\rho} + \frac{\lambda}{\rho^2(\eta + \lambda)} - \frac{\lambda^2}{\rho^3(\eta + \lambda)^2} \right], \\ \Delta_1(2) &= \frac{\lambda^2}{(\lambda + \eta)^2} \left[ 1 + \frac{\lambda}{\rho(\eta + \lambda)} + \frac{\lambda^2}{\rho^2(\eta + \lambda)^2} \right], \\ \Delta_1(3) &= \frac{\lambda^2}{(\lambda + \eta)^2} \left[ \frac{\lambda}{\eta + \lambda} + \frac{\lambda^2}{\rho(\eta + \lambda)^2} \right]. \end{aligned}$$

Using (2.46) and the results of Step 2, we get

$$\Delta_2(1) = \frac{1 + \rho}{\rho}, \quad \Delta_2(2) = 1, \quad \Delta_2(3) = 0.$$

Step 5: Set  $P_0 = 1$ .

Step 6: From (2.52), we have

$$\begin{aligned} P_0(0) &= \left[ \frac{1}{\lambda} \left( 1 + \frac{[a^*(\eta)]^N}{1 - a^*(\eta)} \right) - \frac{[a^*(\eta)]^3 [1 + a^*(\eta)] + 3}{\mu} - \frac{[a^*(\eta)]^2}{\eta} \right. \\ &\quad \left. - \left( \frac{1}{\lambda} - \frac{1}{\mu} \right) \times \left( \sum_{n=1}^3 \Delta_1(n) + \frac{\eta G_5^{(0)}}{a^*(\mu)} + H_1(3, 5) \cdot (1 - [a^*(\eta)]^2) \right) \right. \\ &\quad \left. + H_2(3, 5) \cdot \left[ a^*(\mu) \sum_{n=1}^3 \psi_n \Delta_1(n) + \eta \left( \sum_{j=3}^5 \frac{(-\mu)^{j-1} G_j^{(j-1)}}{(j-1)!} + \psi_4 G_5^{(0)} \right) \right] \right. \\ &\quad \left. + H_3(3, 5) \cdot (1 - [a^*(\eta)]^2) \right]^{-1} \cdot \left[ 1 + \frac{3\eta}{\mu} + \eta \left( \frac{1}{\lambda} - \frac{1}{\mu} \right) H_1(3, 5) + \eta H_3(3, 5) \right] P_0, \end{aligned}$$

where

$$\begin{aligned} H_1(3, 5) &= \frac{1 + 2\rho}{\rho}, \\ H_2(3, 5) &= \frac{1 + \rho}{\lambda}, \\ \text{and } H_3(3, 5) &= \frac{(1 + 3\rho + 2\rho^2 + 3\rho^3 + \rho^4)}{\lambda\rho(1 + \rho)^2}. \end{aligned}$$

Substituting the results of Step 1 through Step 5 into (2.52), we obtain

$$P_0(0) = \lambda.$$

Step 7: For  $n = 2$ , calculate  $Q_2(0)$  using (2.29)

$$Q_2(0) = \eta + \lambda \left[ 1 - \left( \frac{\lambda}{\lambda + \eta} \right)^2 \right].$$

Step 8: For  $n = 5$ , calculate  $R_5(0)$  using (2.49).

We find from (2.49) that

$$R_5(0) = \frac{a^*(\mu) \sum_{j=1}^3 \psi_j \Delta_1(j) + \eta \left( \sum_{j=3}^5 \frac{(-\mu)^{j-1} G_j^{(j-1)}}{(j-1)!} + \psi_4 G_5^{(0)} \right) P_0(0)}{\Psi(4) - \Psi(3)} \\ + \frac{a^*(\mu) \left( \sum_{j=1}^3 \psi_j \Delta_2(j) + \psi_2 \right)}{\Psi(4) - \Psi(3)} Q_2(0).$$

Thus, using the results of Steps 1-4 and 6-7, we get

$$R_5(0) = \frac{\lambda^3}{(\lambda + \eta)^2} \left[ \frac{\rho^2 \lambda}{\eta + \lambda} + \frac{\rho \lambda^2}{(\eta + \lambda)^2} \right] + [\rho^5 + \rho^4 + \rho^3][\eta + \lambda].$$

Step 9: For  $n = 1, 2, 3, 4$ , calculate  $R_n(0)$ .

Using (2.47) and (2.42), it follows that

$$R_1(0) = (\Psi(3) - \Psi(2)) \frac{R_5(0)}{a^*(\mu)} - \Delta_1(1)P_0(0) - \Delta_2(1)Q_2(0), \\ R_2(0) = (\Psi(2) - \Psi(1)) \frac{R_5(0)}{a^*(\mu)} - \Delta_1(2)P_0(0) - \Delta_2(2)Q_2(0), \\ R_3(0) = (\Psi(1) - \Psi(0)) \frac{R_5(0)}{a^*(\mu)} - \Delta_1(3)P_0(0) - \Delta_2(3)Q_2(0), \\ R_4(0) = \frac{1 - a^*(\mu)}{a^*(\mu)} R_5(0) - \frac{\eta G_5^{(0)}}{a^*(\mu)} P_0(0).$$

Thus, using the results of Steps 2-4 and 6-8, we obtain

$$R_1(0) = \rho[\eta + \lambda], \\ R_2(0) = [\rho^2 + \rho][\eta + \lambda], \\ R_3(0) = [\rho^3 + \rho^2 + \rho][\eta + \lambda], \\ R_4(0) = [\rho^4 + \rho^3 + \rho^2][\eta + \lambda] + \frac{\rho \lambda^4}{(\lambda + \eta)^3}.$$

Step 10: For  $n=1, 2, 3, 4, 5$ , calculate  $R_n^*(0)$  using (2.50).

Using (2.50) and the results of Steps 6 and 9, it follows that

$$\begin{aligned} R_1^*(0) &= \frac{\eta P_0^*(0) + P_0(0)}{\mu} = \frac{\eta + \lambda}{\mu}, \\ R_2^*(0) &= \frac{\mu R_1^*(0) + R_1(0)}{\mu} = [1 + \rho] \frac{\eta + \lambda}{\mu}, \\ R_3^*(0) &= \frac{\mu R_1^*(0) + R_2(0)}{\mu} = [1 + \rho + \rho^2] \frac{\eta + \lambda}{\mu}, \\ R_4^*(0) &= \frac{[a^*(\eta)]^3 P_0(0) + R_3(0)}{\mu} = [\rho + \rho^2 + \rho^3] \frac{\eta + \lambda}{\mu} + \rho \left( \frac{\lambda}{\lambda + \eta} \right)^3, \\ R_5^*(0) &= \frac{[a^*(\eta)]^4 P_0(0) + R_4(0)}{\mu} \\ &= [\rho^2 + \rho^3 + \rho^4] \frac{\eta + \lambda}{\mu} + \rho^2 \left( \frac{\lambda}{\lambda + \eta} \right)^3 + \rho \left( \frac{\lambda}{\lambda + \eta} \right)^4. \end{aligned}$$

Step 11: For  $n = 1, 2, 3, 4, 5$ , calculate  $P_n = P_n^*(0)$  using (2.26)-(2.27).

From (2.26)-(2.27) and the result of Step 6, we obtain

$$P_n = \begin{cases} \frac{\lambda^n}{(\lambda + \eta)^n}, & n = 1, 2, 3, 4, \\ \frac{\lambda^5}{\eta(\lambda + \eta)^4}, & n = 5. \end{cases}$$

Step 12: For  $n = 0, 1, 2$ , compute  $Q_n = Q_n^*(0)$  using (2.30)-(2.31).

We get from (2.30)-(2.31) using (2.28)-(2.29) and the results of Step 6,

$$Q_0^*(0) = \frac{\eta}{\lambda}, \quad Q_1^*(0) = \frac{\eta}{\lambda} \left[ 1 + \frac{\lambda}{\lambda + \eta} \right], \quad Q_2^*(0) = \frac{\eta}{\lambda} \left[ 1 + \frac{\lambda}{\lambda + \eta} + \left( \frac{\lambda}{\lambda + \eta} \right)^2 \right].$$

Step 13: Calculate  $\text{sum} = 1 + \sum_{n=1}^K P_n^*(0) + \sum_{n=0}^{N-1} Q_n^*(0) + \sum_{n=1}^K R_n^*(0)$ .

From the results of Step 10 through Step 12, we obtain

$$\begin{aligned} \text{sum} &= \frac{\eta + \lambda}{\eta} + \frac{\eta}{\lambda} \left[ 3 + \frac{2\lambda}{\lambda + \eta} + \left( \frac{\lambda}{\lambda + \eta} \right)^2 \right] + (\rho + \rho^2) \left( \frac{\lambda}{\lambda + \eta} \right)^3 \\ &\quad + \rho \left( \frac{\lambda}{\lambda + \eta} \right)^4 + (3 + 3\rho + 3\rho^2 + 2\rho^3 + \rho^4) \frac{\eta + \lambda}{\mu}. \end{aligned}$$

Step 14: For  $n = 1, 2, 3, 4, 5$ , using the results of Step 10,  $R_n^*(0)$  can be obtained using  $R_n^*(0) = R_n^*(0)/\text{sum}$ .

Step 15: For  $n = 0, 1, 2$ ,  $Q_n$  can be obtained using  $Q_n = Q_n/\text{sum}$  and the results of Step 12.

Step 16: For  $n = 0, 1, 2, 3, 4, 5$ ,  $P_n$  can be obtained using  $P_n = P_n/\text{sum}$  and the results of Step 11.

It is to be noted that the results listed above are similar to those for the M/M/1/5 queue (under  $N = 3$  policy with a single exponential vacation), which are derived directly through birth and death process.

#### 4 Pre-arrival and Arbitrary Epochs Probability

Let us define

$P_n^- \equiv$  the probabilities of  $n$  customers in the queue immediately prior to an arrival when the server is in vacation ( $0 \leq n \leq K$ );

$Q_n^- \equiv$  the probabilities of  $n$  customers in the queue immediately prior to an arrival when the server is dormant ( $0 \leq n \leq N - 1$ );

$R_n^- \equiv$  the probabilities of  $n$  customers in the queue immediately prior to an arrival when the server is busy ( $1 \leq n \leq K$ ).

Using the same argument as Laxmi and Gupta (1999), we observe that

$$P_n^- = P_n(0) \left[ \sum_{j=0}^K P_j(0) + \sum_{j=0}^{N-1} Q_j(0) + \sum_{j=1}^K R_j(0) \right]^{-1},$$

$$Q_n^- = Q_n(0) \left[ \sum_{j=0}^K P_j(0) + \sum_{j=0}^{N-1} Q_j(0) + \sum_{j=1}^K R_j(0) \right]^{-1},$$

$$R_n^- = R_n(0) \left[ \sum_{j=0}^K P_j(0) + \sum_{j=0}^{N-1} Q_j(0) + \sum_{j=1}^K R_j(0) \right]^{-1}.$$

We obtain from (2.34)

$$P_n^- = a_1 P_n(0), \quad 0 \leq n \leq K,$$

$$Q_n^- = a_1 Q_n(0), \quad 0 \leq n \leq N-1,$$

and

$$R_n^- = a_1 R_n(0), \quad 1 \leq n \leq K.$$

To obtain  $P_n^-$ ,  $Q_n^-$  and  $R_n^-$ , we need to find  $P_n(0)$ ,  $Q_n(0)$  and  $R_n(0)$ . From (2.23)-(2.25), (2.28)-(2.29) and (2.50), the distributions of the numbers of customers in the queue at pre-arrival epochs and at arbitrary epochs have the following relations:

$$\begin{aligned} P_0^- &= \frac{R_1^*(0)}{\rho} - a_1 \eta P_0^*(0), \\ P_n^- &= [a^*(\eta)]^n P_0^-, \quad 1 \leq n \leq K-1, \\ P_K^- &= \frac{[a^*(\eta)]^K}{1 - a^*(\eta)} P_0^-, \quad n = K, \\ Q_0^- &= a_1 \eta P_0^*(0), \\ Q_n^- &= (1 - [a^*(\eta)]^n) P_0^- + a_1 \eta P_0^*(0), \quad n = 1, 2, \dots, N-1, \\ R_n^- &= \frac{R_{n+1}^*(0)}{\rho} - \frac{R_1^*(0)}{\rho}, \quad n = 1, 2, \dots, N-1, \\ R_n^- &= \frac{R_{n+1}^*(0)}{\rho} - [a^*(\eta)]^n P_0^-, \quad n = N, N+1, \dots, K-1, \end{aligned} \quad (4.1)$$

where  $\rho = (\mu a_1)^{-1}$ . Note that  $R_K^-$  can be determined by inserting (4.1) in the normalization condition.

## 5 System Characteristics

*5.1. The mean queue length.* Let  $L_N$  represent the expected number of customers in the system at an arbitrary time. Thus

$$\begin{aligned} L_N &= \frac{N(N-1)}{2} \cdot \frac{R_1^*(0)}{\rho} + \frac{P_0(0)}{\eta} \left( N[a^*(\eta)]^{N-1} + \frac{[a^*(\eta)]^N - [a^*(\eta)]^K}{1 - a^*(\eta)} \right) \\ &\quad + \sum_{n=1}^K n R_n^*(0), \end{aligned}$$

where  $P_0(0)$  is determined by (2.25) or (2.52).

5.2. *Blocking probability.* An important measure of system characteristics for a finite queueing system is the blocking probability. The blocking probability can be easily determined from Section 4:

$$\Pr\{\text{arriving customer is blocked because the system is full}\} = P_K^- + R_K^-.$$

Let  $\lambda'$  represent the effective arrival rate. Since  $1 - P_K^- - R_K^-$  denotes the probability that an arriving customer is accepted, we have

$$\lambda' = \frac{(1 - P_K^- - R_K^-)}{a_1}.$$

5.3. *Waiting time in queue.* Let  $W_q^*(s)$  be the LST of the distribution function of the waiting time in the queue of a customer who is accepted in the system. Due to the memoryless property of the service process, an arrival may occur in one of the following four ways:

*Case 1:* The test customer, who arrives while the server is on vacation and finds  $n$  ( $n < N$ ) customers in the system, must wait while:

- (i)  $(N - n - 1)$  customers arrive before the server comes back from vacation; and
- (ii) the server serves the preceding  $n$  customers after resuming work.

*Case 2:* The test customer arrives while the server is on vacation and finds  $n$  ( $n \geq N$ ) customers in the system. This customer has to wait in the queue till the server returns from the vacation to resume service and then completes the service for  $n$  customers.

*Case 3:* The test customer, who arrives while the server is dormant in the system, finds  $n$  customers in the system, and must wait while:

- (i)  $(N - n - 1)$  customers arrive; and
- (ii) the server serves the preceding  $n$  customers after resuming work.

*Case 4:* The test customer who arrives while the server is busy and finds  $n$  customers in the system. This customer has to wait in the queue till the server completes the service for  $n$  customers.

Combining the Cases 1-4 listed above, we have

$$\begin{aligned}
 W_q^*(s) = & \frac{1}{1 - P_K^- - R_K^-} \left[ \sum_{n=0}^{N-1} P_n^- [a^*(s)]^{N-n-1} \left( \frac{\eta}{\eta + s} \right) \left( \frac{\mu}{\mu + s} \right)^n \right. \\
 & + \sum_{n=N}^{K-1} P_n^- \left( \frac{\eta}{\eta + s} \right) \left( \frac{\mu}{\mu + s} \right)^n \\
 & \left. + \sum_{n=0}^{N-1} Q_n^- [a^*(s)]^{N-n-1} \left( \frac{\mu}{\mu + s} \right)^n + \sum_{n=1}^{K-1} R_n^- \left( \frac{\mu}{\mu + s} \right)^n \right], \quad (5.1)
 \end{aligned}$$

where  $P_n^-$ ,  $Q_n^-$  and  $R_n^-$  are computed by using (4.1).

Note that when  $\eta = \infty$ , the results given above reduce to a special case of the finite capacity G/M/1 queueing system under  $N$ -policy without vacation (see Ke and Wang, 2002). In this case, the expected waiting time in the queue becomes

$$W_q = \frac{a_1}{1 - R_K^-} \left[ \frac{N(N-1)}{2} \cdot \frac{R_1^*(0)}{\rho} + \sum_{n=1}^K n R_n^*(0) \right] - \frac{1}{\mu}.$$

5.4. *Busy period.* Let  $\Pi_m^*(s)$  be the LST of busy period for the G/M/1/K queueing system with no vacation when the system remains idle till  $m$  customers arrive. That is, each busy period is initiated by  $m$  customers. Following Kijima and Makimoto (1992),

$$\pi_m = - \left[ \frac{d\Pi_m^*(s)}{ds} \Big|_{s=0} \right]$$

is obtained by solving simultaneous equations as follows:

$$\begin{aligned}
 \pi_i = & \sum_{j=2}^{i+1} d_{ij}^*(0) \pi_j - \sum_{j=2}^{i+1} d_{ij}^{*(1)}(0) - b_i^{*(1)}(0), \quad i = 1, 2, \dots, K-1, \\
 \pi_K = & \sum_{j=2}^K d_{Kj}^*(0) \pi_j + a^*(\mu) \pi_K - \sum_{j=2}^K d_{Kj}^{*(1)}(0) - a^{*(1)}(\mu) - b_K^{*(1)}(0), \quad (5.2)
 \end{aligned}$$

where

$$d_{ij}^*(s) = \begin{cases} \frac{(-\mu)^{i-j+1} a^{*(i-j+1)}(s+\mu)}{(i-j+1)!}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0, \end{cases} \quad (5.3)$$

and

$$b_i^*(s) = \frac{\mu^{i-1}}{(s+\mu)^i} + (-\mu)^{i-1} \sum_{k=0}^{i-1} \left[ \frac{-1}{s+\mu} \right]^{k+1} \left[ \frac{a^{*(i-k-1)}(s+\mu)}{(i-k-1)!} \right]. \quad (5.4)$$

It is to be noted that  $d_{ij}^{*(1)}(s)$  and  $b_i^{*(1)}(s)$  are the first derivatives of  $d_{ij}^*(s)$  and  $b_i^*(s)$ , respectively.

For the G/M/1/K queue with  $N$ -policy where the server takes a single exponential vacation when the system is empty, there may be more than  $N$  customer in the system when a busy period starts. Let  $Z_B$  be the number of customers in the system while a busy period starts, then we have

$$\begin{aligned} \Pr[Z_B = m] &= \Pr[T_0 + T_1 + \cdots + T_{m-1} \leq V < T_1 + T_2 \cdots + T_m] \\ &= [a^*(\eta)]^{m-1} [1 - a^*(\eta)], \quad N \leq m \leq K, \end{aligned}$$

where  $T_m$  is an inter-arrival time and  $V$  is a vacation length.

From the discussions presented above, we have the distribution and the expected length of the busy period for the G/M/1/K queueing system with  $N$ -policy and a single exponential vacation

$$B_{sv}^*(s) = [1 - a^*(\eta)] \sum_{m=N}^K [a^*(\eta)]^{m-1} \Pi_m^*(s) + \Pi_N^*(s), \quad (5.5)$$

and

$$E[B_{sv}] = - \left[ \frac{dB_{sv}^*(s)}{ds} \Big|_{s=0} \right] = (1 - a^*(\eta)) \sum_{m=N}^K [a^*(\eta)]^{m-1} \pi_m + \pi_N. \quad (5.6)$$

Note that  $E[B_{sv}]$  can be evaluated from (5.2)-(5.4) and (5.6). Furthermore, the second term of RHS in (5.5) is due to the server being dormant in the system until system size reaches  $N$ , because the server may return from vacation and find system size less than  $N$ .

## 6 Numerical Examples

In this section, we study the effects of various parameters on the system characteristics such as the expected number of customers in the system (mean queue length,  $L_N$ ) and the blocking probability ( $P_K^- + R_K^-$ ). For convenience, we first let  $\eta = 0.1$  and  $\mu = 1.0$  in numerical experiments.

Our first set of numerical examples considers various inter-arrival time distributions by varying  $\rho$  from 0.1 to 0.8 and various values of  $N$ . The inter-arrival time distributions are considered to be exponential (denoted by M), 3-stage Erlang (denoted by  $E_3$ ) and deterministic (denoted by D). The effects of the inter-arrival time distributions and different values of  $N$  and  $\rho$  on the  $L_N$  and  $P_K^- + R_K^-$  are shown in Table 1. From Table 1, we find (i) when  $\rho$  is small,  $P_K^- + R_K^-$  approaches zero; (ii) for a smaller  $\rho$ , the effect of D on  $L_N$  is smaller than M or  $E_3$  does; and (iii) for a larger  $\rho$ , the effect of M on  $L_N$  ( $P_K^- + R_K^-$ ) is smaller (greater) than  $E_3$  or D does. In addition, when the inter-arrival distribution is given, one can see that (i)  $L_N$  and  $P_K^- + R_K^-$  are increasing in  $\rho$ ; (ii)  $L_N$  is decreasing in  $N$ ; and (iii)  $P_K^- + R_K^-$  is non-increasing in  $N$ . One also observes when all parameters are given, the impact of the inter-arrival time distributions on the  $L_N$  is not significant for a larger  $\rho$ .

A second set of numerical examples deals with the cases regarding the effect of different values of  $K$  and  $\rho$  on the  $L_N$  and  $P_K^- + R_K^-$ . The inter-arrival time distribution and the threshold value are assumed to be  $E_3$  and 3, respectively. The effects of different values of  $K$  and  $\rho$  on  $L_N$  and  $P_K^- + R_K^-$  are displayed in Table 2. One observes from Table 2 that (i)  $L_N$  and  $P_K^- + R_K^-$  increase as  $\rho$  increases; (ii)  $L_N$  decreases as  $K$  decreases; and (iii)  $P_K^- + R_K^-$  increases as  $K$  decreases.

Finally, we set  $\lambda = 0.5$ ,  $\mu = 2.0$ , and perform numerical study of system characteristics for  $E_3/M/1$  system by considering different values of  $\eta$  and  $N$ , which are depicted in Table 3. We observe from Table 3 that (i)  $L_N$  increases as  $\eta$  decreases or  $N$  increases; and (ii)  $P_K^- + R_K^-$  is close to zero for a larger  $\eta$ . One also sees that  $\eta$  significantly affects  $L_N$  for a smaller  $N$ .

Our numerical investigations indicate that (i) the capacity  $K$  has a more significant effect on the blocking probability than the threshold  $N$  (see Tables 1 and 2), and (ii) the influences of  $N$  and  $\rho$  on the mean queue length are more significant than inter-arrival time distributions.

TABLE 1. THE MEAN QUEUE LENGTH AND BLOCKING PROBABILITY FOR DIFFERENT VALUES OF  $\rho$  AND  $N$  IN THE SINGLE VACATION G/M/1/K SYSTEM WHERE  $G \equiv M, E_3,$  AND  $D,$  RESPECTIVELY. ( $\eta = 0.1, \mu = 1.0, K = 15$ )

	$N = 1$		$N = 5$		$N = 10$	
	$L_N$	$P_K^- + R_K^-$	$L_N$	$P_K^- + R_K^-$	$L_N$	$P_K^- + R_K^-$
$\rho = 0.1$						
$M$	0.740	0.000	2.182	0.000	4.784	0.000
$E_3$	0.777	0.000	2.167	0.000	4.784	0.000
$D$	0.024	0.000	1.545	0.000	2.890	0.000
$\rho = 0.2$						
$M$	1.853	0.002	2.598	0.001	5.098	0.001
$E_3$	1.904	0.001	2.552	0.000	5.100	0.000
$D$	0.135	0.000	1.727	0.000	4.134	0.000
$\rho = 0.3$						
$M$	2.921	0.011	3.272	0.008	5.481	0.006
$E_3$	2.975	0.008	3.233	0.005	5.472	0.004
$D$	3.013	0.006	3.200	0.004	5.479	0.003
$\rho = 0.4$						
$M$	3.876	0.029	4.052	0.024	5.936	0.019
$E_3$	3.936	0.023	4.042	0.019	5.920	0.015
$D$	3.976	0.020	4.043	0.016	5.900	0.012
$\rho = 0.5$						
$M$	4.701	0.052	4.814	0.046	6.424	0.040
$E_3$	4.769	0.044	4.832	0.039	6.413	0.033
$D$	4.806	0.040	4.854	0.036	6.393	0.029
$\rho = 0.6$						
$M$	5.401	0.076	5.502	0.071	6.901	0.065
$E_3$	5.465	0.067	5.537	0.062	6.902	0.057
$D$	5.506	0.063	5.566	0.059	6.892	0.052
$\rho = 0.7$						
$M$	5.992	0.099	6.100	0.095	7.341	0.091
$E_3$	6.051	0.090	6.143	0.086	7.353	0.082
$D$	6.089	0.085	6.170	0.082	7.353	0.077
$\rho = 0.8$						
$M$	6.493	0.120	6.614	0.117	7.732	0.116
$E_3$	6.542	0.110	6.655	0.108	7.751	0.106
$D$	6.573	0.105	6.677	0.103	7.758	0.101

TABLE 2. THE MEAN QUEUE LENGTH AND BLOCKING PROBABILITY FOR DIFFERENT VALUES OF  $K$  AND  $\rho$  IN THE SINGLE VACATION  $E_3/M/1/K$  SYSTEM. ( $\eta = 0.1, \mu = 1.0, N = 3$ ).

	$K = 5$		$K = 10$		$k = 15$	
	$L_N$	$P_K^- + R_K^-$	$L_N$	$P_K^- + R_K^-$	$L_N$	$P_K^- + R_K^-$
$\rho = 0.1$	1.698	0.010	1.711	0.000	1.711	0.000
$\rho = 0.2$	2.219	0.089	2.413	0.009	2.436	0.001
$\rho = 0.3$	2.666	0.189	3.200	0.038	3.333	0.008
$\rho = 0.4$	2.982	0.273	3.873	0.078	4.192	0.023
$\rho = 0.5$	3.204	0.339	4.409	0.119	4.947	0.044
$\rho = 0.6$	3.368	0.391	4.834	0.157	5.589	0.067
$\rho = 0.7$	3.492	0.432	5.174	0.189	6.132	0.090
$\rho = 0.8$	3.591	0.467	5.450	0.216	6.589	0.110
$\rho = 0.8$	3.670	0.497	5.676	0.237	6.977	0.127
$\rho = 0.8$	3.737	0.525	5.864	0.253	7.305	0.141

TABLE 3. THE MEAN QUEUE LENGTH AND BLOCKING PROBABILITY FOR DIFFERENT VALUES OF  $\eta$  AND  $N$  IN THE SINGLE VACATION  $E_3/M/1/K$  SYSTEM ( $\lambda = 0.5, \mu = 2.0, K = 15$ ).

	$\eta = 0.1$		$\eta = 0.5$		$\eta = 1.0$	
	$L_N$	$P_K^- + R_K^-$	$L_N$	$P_K^- + R_K^-$	$L_N$	$P_K^- + R_K^-$
$N = 1$	4.360	0.042	1.079	0.000	0.583	0.000
$N = 2$	4.467	0.042	1.375	0.000	1.211	0.000
$N = 3$	4.522	0.040	1.830	0.000	1.683	0.000
$N = 4$	4.589	0.040	2.330	0.000	2.283	0.000
$N = 5$	4.749	0.038	2.867	0.000	2.864	0.000
$N = 6$	5.026	0.037	3.491	0.000	3.477	0.000
$N = 7$	5.183	0.037	4.126	0.000	4.124	0.000
$N = 8$	5.776	0.032	5.320	0.000	5.104	0.000
$N = 9$	5.989	0.031	5.360	0.000	5.308	0.000
$N = 10$	6.325	0.031	5.929	0.000	5.920	0.000
$N = 11$	6.875	0.030	6.541	0.000	6.533	0.000
$N = 12$	7.370	0.030	7.155	0.000	7.150	0.000
$N = 13$	7.892	0.028	7.772	0.000	7.768	0.000
$N = 14$	8.436	0.027	8.390	0.000	8.388	0.000

**Appendix: Derivations of  $P_n^{*(1)}(0)$  and  $P_n^{*(l)}(\mu)$  in terms of  $P_0(0)$**

Differentiating (2.12) and (2.14) with respect to  $s$  and inserting  $s = 0$ , it follows from (2.23) that we have

$$P_0^{*(1)}(0) = -a^{*(1)}(0)P_0 = a_1P_0, \tag{A.1}$$

$$P_n^{*(1)}(0) = \left( \frac{[a^*(\eta)]^{n-1}(1-a^*(\eta))}{\eta^2} - \frac{a_1[a^*(\eta)]^{n-1}}{\eta} \right) P_0(0), \tag{A.2}$$

$n = 1, 2, \dots, K-1.$

For  $n = N, N + 1, \dots, K$ , we use  $s = \mu$  in (2.14)-(2.15). It yields

$$P_n^*(\mu) = \frac{1}{\eta - \mu} [a^*(\mu) - a^*(\eta)] [a^*(\eta)]^{n-1} P_0(0), \quad N \leq n \leq K - 1, \quad (A.3)$$

$$P_K^*(\mu) = \frac{1}{\eta - \mu} \left[ \frac{a^*(\mu) - a^*(\eta)}{1 - a^*(\eta)} \right] [a^*(\eta)]^{K-1} P_0(0), \quad n = K. \quad (A.4)$$

We differentiate (2.14)-(2.15)  $l$  ( $l = 0, 1, 2, \dots, K - 1$ ) times with respect to  $s$  and then insert  $s = \mu$ . Thus, we get

$$P_n^{*(l)}(\mu) = l! \left[ \sum_{j=0}^{l-1} \frac{a^{*(j+1)}(\mu)}{(j+1)!(\eta - \mu)^{l-j}} P_{n-1}(0) + \frac{P_n^*(\mu)}{(\eta - \mu)^l} \right],$$

$$n = l, l + 1, \dots, K - 1, \quad (A.5)$$

$$P_K^{*(l)}(\mu) = l! \left[ \sum_{j=0}^{l-1} \frac{a^{*(j+1)}(\mu)}{(j+1)!(\eta - \mu)^{l-j}} [P_{K-1}(0) + P_K(0)] + \frac{P_K^*(\mu)}{(\eta - \mu)^l} \right]. \quad (A.6)$$

Substituting (A.3)-(A.4) and (2.23)-(2.24) into (A.5)-(A.6), it finally yields

$$P_n^{*(l)}(\mu) = G_n^{(l)} P_0(0), \quad (A.7)$$

where

$$G_n^{(l)} = 0 \quad \text{for } n = l + 1, l + 2, \dots, N - 1, \quad (A.8)$$

$$G_n^{(l)} = l! [a^*(\eta)]^{n-1} \left\{ \sum_{j=0}^{l-1} \left[ \frac{a^{*(j+1)}(\mu)}{(j+1)!(\eta - \mu)^{l-j}} \right] + \frac{a^*(\mu) - a^*(\eta)}{(\eta - \mu)^{l+1}} \right\}$$

$$\text{for } n = N, N + 1, \dots, K - 1, \quad (A.9)$$

$$\text{and } G_K^{(l)} = \frac{l! [a^*(\eta)]^{K-1}}{1 - a^*(\eta)} \left\{ \sum_{j=0}^{l-1} \left[ \frac{a^{*(j+1)}(\mu)}{(j+1)!(\eta - \mu)^{l-j}} \right] + \frac{a^*(\mu) - a^*(\eta)}{(\eta - \mu)^{l+1}} \right\}$$

$$(A.10)$$

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