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Invariant Probabilities of Markov-Feller Operators
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On a locally compact separable metric space X, let $\mathcal{M}(X)$ be the space of all finite signed Borel measures equipped with the total variation norm, and $C_b(X)$ be the space of all bounded real continuous functions equipped with the usual sup norm. A positive contraction $T : \mathcal{M}(X) \to \mathcal{M}(X)$ is called a Markov operator if $\|T\mu\| = \|\mu\|$ for all non-negative μ. A pair (S, T), where S is a linear operator on $C_b(X)$ and T a Markov operator on $\mathcal{M}(X)$, is called a *Markov-Feller pair* if $\langle Sf, \mu\rangle = \langle f, T\mu\rangle$ for all $f \in C_b(X)$ and $\mu \in \mathcal{M}(X)$. T is called a Markov-Feller operator. A classical result of M. Rosenblatt states that any Markov-Feller pair (S, T) on a locally compact separable metric space X is generated by a transition probability, i.e., there is a transition probability P on $X \times \mathcal{B}(X)$ such that $Sf(x) = \int f(y)P(x, dy)$ and $T\mu(A) = \int P(x, A)\, d\mu(x)$. Markov-Feller pairs appear in the study of Feller processes and are used extensively in many areas such as dynamical systems, random iterated function systems and the study of convolution of measures. For a Markov-Feller operator, an invariant probability is often an important object of interest. The main aim of the present book is to offer "formulas" for supports of various types of invariant probabilities. These include formulas for ergodic probabilities and for the unique invariant probability of a uniquely ergodic Markov-Feller operator.

As one of the key tools used to derive such formulas, the author first obtains what he calls the *weak KBBY decomposition.* The basic idea is to use a given Markov-Feller operator T to decompose the whole space X into its *dissipative* and *non-dissipative* parts. This is a nice extension of well-known ergodic decomposition results of Krylov, Bogolioubov, Beboutoff and Yoshida to a locally compact separable metric space. Concepts of topological limits and Banach limits of sequences of subsets are brought in to get this extension. For a non-initiated reader, it will perhaps be helpful to go through earlier works of Beboutoff and Yoshida for the compact case, to get a feel for the author's present work for the locally compact case.

The other important tool is the extension of the notion of orbits used in dynamical systems to Markov-Feller operators. These orbits are used to study the supports of ergodic measures and what are called *elementary invariant measures.* For example, here is a specific result. For any point x in the non-dissipative part of X and any Banach limit L, let $\epsilon_x^{(L)}(f) = L\left(\{\langle f, T^n\delta_x\rangle\}_{n\geq 0}\right)$ for $f \in C_b(X)$. There is a natural way in which one may think of $\epsilon_x^{(L)}$ as an element of $\mathcal{M}(X)$, and then one can show that it is a T-invariant measure. It is called an *elementary* invariant measure if it is not identically zero. Now, the *orbit* of x under the action of T is the subset of X defined by $\mathcal{O}(x) = \bigcup_{n\geq 0}$ supp $(T^n\delta_x)$, and its closure $\overline{\mathcal{O}(x)}$ is called the *orbit-closure* of x. The author proves that if $\epsilon_x^{(L)}$ is an elementary T-invariant measure, then its support is contained in the orbit-closure of x, and in case x belongs to the support, then the support is exactly equal to the orbit-closure of x. For supports of ergodic measures also, similar descriptions in terms of orbit-closures are obtained. All these results seem very natural, especially if one thinks in terms of an underlying Markov chain (a viewpoint which is somehow missing in the book), but are fairly non-trivial to prove. Another equally natural result says that in case X has no dissipative part (in particular if X is compact), the support of any T-invariant probability equals the whole of X if and only if the Markov-Feller operator T is *minimal* (i.e., every orbit is dense in X). Finally, for uniquely ergodic Markov-Feller operators, a precise description is given for the support of the unique invariant probability, which in the case of compact spaces turns out to be just the intersection of all the orbit closures.

Overall, the book offers a completely thorough treatment of the specific problem of describing supports of invariant probabilities for Markov-Feller operators in a fairly abstract setting. One of the problems of a book dealing with such a highly abstract and focussed problem is that its appeal will be somewhat limited. The reviewer can think of at least two areas where the book could have been improved and hence could have generated wider interest. First, more examples would have certainly thrown light into some of the intricacies of the theoretical developments in the book. The author essentially gives only one example and uses it throughout the book. The other issue is more a matter of interest. Being a probabilist, the reviewer would have liked to see the connections with the familiar results in Markov chain theory highlighted. The book has plenty of scope at various places to do that. In case a revised edition is contemplated in the future, it will be nice to see some of these issues addressed.

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