Sankhyā : The Indian Journal of Statistics
2006, Volume 68, Part 2, pp. 240-251

Estimation of a Quantile in a Mixture Model of Exponential Distributions with Unknown Location and Scale Parameters

Constantinos Petropoulos
University of Patras, Patras, Greece

Abstract

Estimation of a quantile in a mixture model of exponential distributions is considered. For quadratic loss and specified extreme quantiles, better estimators than the best affine equivariant procedure are established. In particular, improved estimators for a quantile of an exponential-inverse Gaussian distribution and the multivariate Lomax distribution with unknown location and scale parameters are derived.

AMS (2000) *subject classification.* Primary 62C99, 62F10, 62H12.
Keywords and phrases. Decision theory, Stein's estimator of a scale parameter, quantile estimation, mixture of exponential distributions, exponential-inverse Gaussian distribution, multivariate Lomax distribution.

1 Introduction

In this paper, we consider the data set $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$, $n \geq 2$, where, given $\tau > 0$, $X_1, X_2, \ldots, X_n$ are i.i.d. with common exponential distribution $E(\mu, \sigma/\tau)$. Thus, the unconditional joint density of $X_1, X_2, \ldots, X_n$ is

$$f(x_1, x_2, \ldots, x_n; \mu, \sigma) = \int_0^{\infty} \frac{\tau^n}{\sigma^n} \exp\left\{-\frac{\tau}{\sigma}\sum_{i=1}^{n}(x_i - \mu)\right\} I_{[\mu,+\infty)}(x_{(1)})\, dG(\tau), \tag{1.1}$$

where $x_{(1)} = \min\{x_1, x_2, \ldots, x_n\}$, $I_{(a,b)}(\cdot)$ is the indicator function and $\mu \in \mathbb{R}$, $\sigma > 0$ are unknown. The distribution of the mixing parameter τ, $G(\cdot)$, is assumed to be known.

The model (1.1) is called a mixture model of exponential distributions. It was first proposed by Lindley and Singpurwalla (1986) in assessing the reliability of a system of components and was further studied by Nayak (1987). Petropoulos and Kourouklis (2005) investigated estimation of σ in the model (1.1). They showed that the best affine equivariant (b.a.e.) estimator of σ is inadmissible under squared error loss and Stein's loss.

For specific forms of $G(\cdot)$ in (1.1), special distributions are derived. Indeed, two of these are of particular interest in actuarial applications and survival analysis.

First, when $G(\cdot)$ is an inverse Gaussian $IG(\gamma,\lambda)$ distribution with density

$$\frac{\sqrt{\lambda}}{\sqrt{2\pi x^3}}\,\exp\left\{-\frac{\lambda}{2\gamma^2}\,\frac{(x-\gamma)^2}{x}\right\}\,I_{(0,+\infty)}(x),\;\;\lambda,\gamma>0,$$

(1.1) becomes

$$f_2(x_1,x_2,\ldots,x_n;\mu,\sigma)=\frac{\exp\left\{\dfrac{\lambda}{\gamma}-\lambda\sqrt{\dfrac{1}{\gamma^2}+\dfrac{2}{\sigma\lambda}\displaystyle\sum_{i=1}^{n}(x_i-\mu)}\right\}}{\sigma^n\left(\dfrac{1}{\gamma^2}+\dfrac{2}{\sigma\lambda}\displaystyle\sum_{i=1}^{n}(x_i-\mu)\right)^{n/2}}$$

$$\times\sum_{i=1}^{n-1}\frac{(n-1+i)!}{i!(n-1-i)!}\left[2\lambda\sqrt{\frac{1}{\gamma^2}+\frac{2}{\sigma\lambda}\sum_{i=1}^{n}(x_i-\mu)}\right]^{-i}I_{[\mu,+\infty)}(x_{(1)}),\qquad(1.2)$$

which is called exponential-inverse Gaussian (E-IG) distribution with location parameter μ and scale parameter σ. The E-IG distribution has been derived by Bhattacharya and Kumar (1986) in reliability modelling. Also, bivariate and multivariate versions of this distribution have been studied by Whitmore and Lee (1991) and Al-Mutairi (1997).

Another useful case is when $G(\cdot)$ is a Gamma $G(a,1)$ distribution. Then the joint density of $X_1,X_2,\ldots,X_n$ in (1.1) is

$$f_1(x_1,x_2,\ldots,x_n;\mu,\sigma)=\frac{\Gamma(n+a)}{\Gamma(a)\sigma^n}\frac{1}{\left(1+\dfrac{1}{\sigma}\displaystyle\sum_{i=1}^{n}(x_i-\mu)\right)^{n+a}}I_{[\mu,+\infty)}(x_{(1)}),\qquad(1.3)$$

which is called multivariate Lomax distribution with location parameter μ and scale parameter σ, or Mardia's multivariate Pareto II distribution, see

Arnold (1983) and Kotz et al. (2000) for details. Several probabilistic properties of this distribution are reported, and its usefulness in reliability theory is indicated in Arnold (1983) and Nayak (1987).

Based on observations $X_1, X_2, \ldots, X_n$ in the framework (1.1), our aim is the estimation of the linear function $\theta = \mu + \kappa\sigma$, for given $\kappa \geqslant 0$, from a decision theoretic point of view. When $\kappa = -m_\tau(p)$, $0 < p \leqslant 1$, where $m_\tau(\cdot)$ is the moment generating function of the random variable $\tau > 0$, then θ is the $100(1-p)$th quantile of the marginal distribution of X_i. Note that for $p = 1$, the problem reduces to the estimation of the location parameter μ whereas the case $p = m_\tau(-\mathbb{E}\tau^{-1})$ corresponds to the estimation of the common mean of the X_i's.

Previous work on quantile estimation concerns the normal distribution (Zidek, 1971) and the exponential distribution $E(\mu, \sigma)$. In the latter case Rukhin and Strawderman (1982) and Rukhin and Zidek (1985) showed the inadmissibility of the best affine equivariant estimator for $0 \leqslant \kappa < 1/n$ and $\kappa > 1 + 1/n$, whereas Rukhin (1986) was able to establish admissibility for $1/n \leqslant \kappa \leqslant 1 + 1/n$. Analogous inadmissibility results were obtained by Elfessi (1997) based on a doubly censored sample from an exponential distribution, while Petropoulos and Kourouklis (2001) generalized results in Rukhin and Strawderman (1982) to a strictly convex loss. Moreover, they proposed a new minimax estimator for $\mu+\kappa\sigma$ under quadratic loss. Recently, Petropoulos and Kourouklis (2004) considered estimation of $\mu + \kappa\sigma$ in the multivariate Lomax distribution (1.3). Under quadratic loss and for specified "small" and "large" values of κ, they established that the b.a.e. procedure is inadmissible by constructing a better estimator.

In the present work and in Section 2, under quadratic loss, we construct better estimators than the b.a.e. estimator of $\theta = \mu + \kappa\sigma$ in the mixture model (1.1), for extreme values of κ, namely $0 \leqslant \kappa < \frac{1}{n(n+1)}\mathbb{E}\tau^{-2}/\mathbb{E}\tau^{-1}$ and $\kappa > (1 + 1/n)\,\mathbb{E}\tau^{-2}/\mathbb{E}\tau^{-1}$. As a special case, for $\kappa = 0$, it is shown that the b.a.e. estimator of the location parameter μ is inadmissible.

In Sections 3 and 4, we give some examples for specific mixture models, i.e., the E-IG distribution and the multivariate Lomax distribution. In each case, we improve upon the b.a.e. estimator of the quantile $\theta = \mu + \kappa\sigma$, for the above mentioned "small" and "large" values of κ. In the case of the multivariate Lomax distribution, for "small" values of κ, we reproduce the improved estimator in Petropoulos and Kourouklis (2004). Interestingly, however, in the case of "large" κ, our approach is setting off a new improved estimator, which is different from that in Petropoulos and Kourouklis (2004).

Finally, in the Appendix we give the proofs of the two main theorems given in Section 2.

2 Mixture of Exponential Distributions

The (minimal) sufficient statistic in the model (1.1) is (X, S), where $X = X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $S = \sum_{i=1}^{n}(X_i - X_{(1)})$. Conditionally on τ, X and S are independent, with

$$\left.\frac{\tau X}{\sigma}\right| \tau \sim E\left(\frac{\tau\mu}{\sigma}, \frac{1}{n}\right) \quad \text{and} \quad \left.\frac{\tau S}{\sigma}\right| \tau \sim Gamma(n-1, 1). \tag{2.1}$$

The problem of estimating $\theta = \mu + \kappa\sigma$ under the loss $(d - \mu - \kappa\sigma)^2/\sigma^2$ is invariant under the affine group of transformations $(X, S) \to (cX + b, cS)$ and the equivariant estimators are of the form $\delta = X + cS$, where c is a real constant. A particular member of this class of estimators is the minimum variance unbiased estimator corresponding to $c = (\kappa(\mathbb{E}\tau^{-1})^{-1} - 1/n)/(n-1)$. The risk of δ, as a function of c, is minimized at

$$c_0 = -\frac{\mathbb{E}[(X - \mu - \kappa\sigma)S]}{\mathbb{E}S^2}.$$

Upon conditioning on τ and using (2.1) it can be showed that

$$c_0 = \left(\kappa\frac{\mathbb{E}\tau^{-1}}{\mathbb{E}\tau^{-2}} - \frac{1}{n}\right)\frac{1}{n}. \tag{2.2}$$

So, the best affine equivariant estimator of θ is

$$\delta_0 = X + c_0 S. \tag{2.3}$$

Our aim is to provide Stein(1964)-type estimators which improve upon δ_0 in (2.3). To this end, we study the more general class of scale equivariant estimators of the form

$$\delta = \phi(W)S, \tag{2.4}$$

where $W = X/S$ and $\phi(\cdot)$ is a measurable function. Notice that the b.a.e. estimator belongs to the above class, corresponding to

$$\phi_0(W) = W + c_0.$$

For "large" values of κ, Theorem 2.1 establishes the inadmissibility of δ_0 by deriving a Stein(1964)-type estimator of the form (2.4), which is better than δ_0.

THEOREM 2.1. *Assume that the following conditions hold.*

(A1) $\dfrac{\mathbb{E}\tau^{r-1}}{\mathbb{E}\tau^{r-2}} - \dfrac{\mathbb{E}\tau^{-1}}{\mathbb{E}\tau^{-2}} \leqslant c^* r$, *for some* $c^* \geqslant 0$, $\forall\, r \in \{0,1,2,\ldots\}$ *and*

(A2) $\dfrac{\int_0^\infty \tau^n e^{-[(1+nw)s-n\mu]\tau} dG(\tau)}{\int_0^\infty \tau^n e^{-(1+nw)s\tau} dG(\tau)}$ *is decreasing in s.*

Let $\kappa > (1+1/n)\, \mathbb{E}\tau^{-2}/\mathbb{E}\tau^{-1}$ *and*

$$\phi_1(W) = W + \frac{\kappa}{n+1}\frac{\mathbb{E}\tau^{-1}}{\mathbb{E}\tau^{-2}}(1+nW) + \kappa c^* \frac{n(n+2)}{n+1}\frac{\mathbb{E}\tau^{n+1}}{\mathbb{E}\tau^{n}}\frac{\mathbb{E}\tau^{-3}}{\mathbb{E}\tau^{-2}}W.$$

Then the risk of the estimator

$$\delta_1 = \begin{cases} \min\{\phi_1(W), \phi_0(W)\}S & , \quad W > 0, \\ \delta_0 = \phi_0(W)S & , \quad \textit{otherwise}, \end{cases}$$

is strictly smaller than that of δ_0 *in (2.3).*

The next results establishes the inadmissibility of δ_0 in (2.3) for "small" values of κ.

THEOREM 2.2. *Let* $0 \leqslant \kappa < \frac{1}{n(n+1)}\ \mathbb{E}\tau^{-2}/\mathbb{E}\tau^{-1}$. *Then the risk of the estimator*

$$\delta_2 = \begin{cases} \max\left\{\frac{n+2}{n+1}W, \phi_0(W)\right\} S & , \quad W < -1/n, \\ \delta_0 = \phi_0(W)S & , \quad \textit{otherwise}, \end{cases}$$

is nowhere larger than that of δ_0 *in (2.3).*

The usefulness of these two theorems is that we give improved estimators of $\mu+\kappa\sigma$ in a general class of distributions such as the mixture of exponential distribution.

3 Exponential-Inverse Gaussian Model

In this section we consider the framework in (1.1), taking the mixing distribution $G(\cdot)$ to be the inverse Gaussian $IG(\gamma, \lambda)$. In other words we study the problem of estimating $\theta = \mu + \kappa\sigma$ in the exponential-inverse Gaussian model. In this case, $\mathbb{E}\tau^{-1} = 1/\gamma + 1/\lambda$ and $\mathbb{E}\tau^{-2} = 1/\gamma^2 + 3/\lambda^2 + 3/\gamma\lambda$ (see Seshadri, 1993, p.47) so that the b.a.e. estimator of θ in (2.2) is $\delta_0 = X + c_0 S$, where

$$c_0 = \frac{1}{n}\left(\kappa\frac{\lambda\gamma(\lambda+\gamma)}{\lambda^2 + 3\gamma(\lambda+\gamma)} - \frac{1}{n}\right). \tag{3.1}$$

For "large" values of κ, an improved estimator of θ is given in the following result.

THEOREM 3.1. *Let*

$$\kappa > \left(1+\frac{1}{n}\right)\left[\frac{\lambda^2+3\gamma(\lambda+\gamma)}{\lambda\gamma(\lambda+\gamma)}\right],$$

$$\begin{aligned}\phi_1(W) =& W + \frac{\kappa}{n+1}\frac{\lambda\gamma(\lambda+\gamma)}{\lambda^2+3\gamma(\lambda+\gamma)}(1+nW)\\ &+\frac{n(n+2)}{n+1}\kappa\frac{2\gamma^2}{\lambda}\gamma\frac{\sum_{i=0}^{n}\frac{(n+i)!}{i!(n-i)!}\left(\frac{\gamma}{2\lambda}\right)^i}{\sum_{i=0}^{n-1}\frac{(n-1+i)!}{i!(n-1-i)!}\left(\frac{\gamma}{2\lambda}\right)^i}\left[\frac{5}{\lambda}+\frac{\lambda(\lambda+\gamma)}{\lambda^2\gamma+3\gamma^2(\lambda+\gamma)}\right]W,\end{aligned}$$

and $\phi_0(W) = W + c_0$ *with* c_0 *as in (3.1).*

Then the risk of the estimator

$$\delta_1 = \begin{cases} \min\{\phi_1(W), \phi_0(W)\}S & , \quad W > 0, \\ \delta_0 = \phi_0(W)S & , \quad \textit{otherwise}, \end{cases}$$

is strictly smaller than that of δ_0.

PROOF. This theorem is proven provided that (A1) and (A2) in Theorem 2.1 hold for the E-IG model (see Petropoulos, 2006, for details). □

For "small" values of κ, application of Theorem 2.2 yields the following.

THEOREM 3.2. *Let*

$$0 \leqslant \kappa < \frac{1}{n(n+1)}\left[\frac{\lambda^2+3\gamma(\lambda+\gamma)}{\lambda\gamma(\lambda+\gamma)}\right]$$

and $\phi_0(W) = W + c_0$ *with* c_0 *as in (3.1).*

Then the risk of the estimator,

$$\delta_2 = \begin{cases} \max\left\{\frac{n+2}{n+1}W, \phi_0(W)\right\}S & , \quad W < -1/n \\ \delta_0 & , \quad \textit{otherwise}, \end{cases}$$

is nowhere larger than that of δ_0.

It should be noted here that no results are available in the literature for the estimation of $\mu + \kappa\sigma$ for the E-IG model.

4 Multivariate Lomax Model

In this section, we take the mixing distribution in the model (1.1) to be $Gamma(a,1)$. So we study the problem of estimating $\theta = \mu + \kappa\sigma$ in the multivariate Lomax model. In this case, $\mathbb{E}\tau^{-1} = 1/(a-1)$ and $\mathbb{E}\tau^{-2} = 1/[(a-1)(a-2)]$, so that the b.a.e. estimator of θ is $\delta_0 = X + c_0 S$, where

$$c_0 = \left(\kappa(a-2) - \frac{1}{n}\right)\frac{1}{n}. \tag{4.1}$$

Estimation of θ has been studied by Petropoulos and Kourouklis (2004), who proposed better estimators than δ_0 mentioned above. Namely, for $\kappa > (1+1/n)/(a-2)$, they proved that the risk of the estimator

$$\delta_1 = \begin{cases} \min\left\{W + \kappa(1+nW)\left(\frac{a-2}{n+1} + \frac{n(n+a)}{n+1}\right)W, W + c_0\right\} S & , \quad W > 0, \\ \delta_0 & , \quad \text{otherwise,} \end{cases} \tag{4.2}$$

is strictly smaller than that of δ_0, where c_0 as in (4.1), and for $0 \leqslant \kappa < 1/[n(n+1)(a-2)]$, they showed that the risk of the estimator

$$\delta_2 = \begin{cases} \max\left\{\frac{n+2}{n+1}W, \phi_0(W)\right\} S & , \quad W < -1/n, \\ \delta_0 & , \quad \text{otherwise,} \end{cases} \tag{4.3}$$

is nowhere larger than that of δ_0.

In Theorem 4.1 a Stein (1964)-type estimator of $\theta = \mu + \kappa\sigma$ is produced which is better than that of δ_0, for "large" values of κ.

THEOREM 4.1. *Let*

$$\kappa > \left(1 + \frac{1}{n}\right)\frac{1}{a-2},$$

$$\phi_1(W) = W + \frac{\kappa}{n+1}(a-2)(1+nW) + \frac{n\kappa}{n+1}(n+2)\frac{n+a}{a-3}W, \text{ and}$$

$$\phi_0(W) = W + c_0, \text{ where } c_0 = \frac{1}{n}\left(\kappa(a-2) - \frac{1}{n}\right).$$

Then the risk of the estimator

$$\delta_3 = \begin{cases} \min\{\phi_1(W), \phi_0(W)\}S & , \quad W > 0, \\ \delta_0 = \phi_0(W)S & , \quad \text{otherwise,} \end{cases}$$

is strictly smaller than that of δ_0.

PROOF. As in Theorem 3.1, we need to show that the conditions (A1) and (A2) in Theorem 2.1 hold for the mixing distribution $G(\tau) \equiv Gamma(a, 1)$. (See Petropoulos, 2006, for details.) $\square$

For "small" values of κ, we give in Theorem 4.2 a better estimator of θ than δ_0, which coincides with δ_2 in (4.3). The proof is an immediate consequence of Theorem 2.2.

THEOREM 4.2. *Let*

$$0 \leqslant \kappa < \frac{1}{n(n+1)} \frac{1}{a-2}, \quad \textit{and}$$

$$\phi_0(W) = W + c_0, \quad \textit{where } c_0 = \frac{1}{n}\left(\kappa(a-2) - \frac{1}{n}\right).$$

Then the risk of the estimator

$$\delta_4 = \begin{cases} \max\left\{\frac{n+2}{n+1}W, \phi_0(W)\right\} S & , \quad W < -1/n, \\ \delta_0 = \phi_0(W)S & , \quad \textit{otherwise}, \end{cases}$$

is nowhere larger than that of δ_0.

It should be mentioned that Theorem 4.1 provides an alternative estimator of $\theta = \mu + \kappa\sigma$ in the model of Lomax distribution, which improves upon δ_0, and this estimator is different from δ_1 in (4.2).

Appendix

PROOF OF THEOREM 2.1. The risk of δ in (2.4) depends on (μ, σ) only through μ/σ. So without loss of generality, one can take $\sigma = 1$ and write

$$\mathbb{R}(\delta; \mu) = \mathbb{E}_\mu[\mathbb{E}_\mu\{(\phi(W)S - \mu - \kappa)^2 | W = w\}]. \tag{A.1}$$

The conditional expectation $\mathbb{E}_\mu\{(cS - \mu - \kappa)^2 | W = w\}$ is minimized with respect to c at

$$c(\mu; w) = (\mu + \kappa)\frac{\mathbb{E}_\mu(S|W = w)}{\mathbb{E}_\mu(S^2|W = w)} = (\mu + \kappa)\psi(\mu; w). \tag{A.2}$$

The key in Stein's (1964) technique is to find an upper bound for $c(\mu; w)$, as a function of μ for each $w > 0$. Using (2.1), we observe, given $\tau > 0$, that the (conditional) density of $S|W = w$ is proportional to

$$\tau^n s^{n-1} e^{n\tau\mu} e^{-(1+nw)\tau s} \quad , \quad s > \max\{0, \mu/w\}. \tag{A.3}$$

Consider, first, $\mu \leqslant 0$ and fix $w > 0$. Then, from (A.2) and (A.3), we get

$$\psi(\mu; w) = \frac{\mathbb{E}_\mu(S|W=w)}{\mathbb{E}_\mu(S^2|W=w)} = \frac{\int_0^\infty \int_0^\infty s^n \tau^n e^{-[(1+nw)s-n\mu]\tau} ds dG(\tau)}{\int_0^\infty \int_0^\infty s^{n+1} \tau^n e^{-[(1+nw)s-n\mu]\tau} ds dG(\tau)}.$$

In the above expression, we make a change of variable from τs to s and using Lemma 2 in Lehmann (1986, p.85), we conclude that

$$\psi(\mu; w) < \frac{\mathbb{E}\tau^{-1}}{\mathbb{E}\tau^{-2}} \frac{1+nw}{n+1}. \tag{A.4}$$

Recall that $\mu \leqslant 0$. Therefore, from (A.2) and (A.4),

$$c(\mu; w) < \kappa \frac{\mathbb{E}\tau^{-1}}{\mathbb{E}\tau^{-2}} \frac{1+nw}{n+1}. \tag{A.5}$$

Next, let $\mu > 0$ and fix, again, $w > 0$. Then,

$$c(\mu; w) = \mu\psi(\mu; w) + \kappa\psi(\mu; w). \tag{A.6}$$

Our goal now is to bound each term of the right hand side of (A.6) separately. First, because of $s > \mu/w$ and from (A.2), we notice that

$$\mu\psi(\mu; w) < w. \tag{A.7}$$

Also, making a change of variable from τs to s and expanding $e^{n\mu\tau}$ in a Taylor series, we can write,

$$\psi(\mu; w) = \frac{\sum_{k=0}^{+\infty} \frac{(n\mu)^k}{k!} \int_0^\infty \int_{\frac{\tau\mu}{w}}^\infty s^n \tau^{k-1} e^{-(1+nw)s} ds \; dG(\tau)}{\sum_{k=0}^{+\infty} \frac{(n\mu)^k}{k!} \int_0^\infty \int_{\frac{\tau\mu}{w}}^\infty s^{n+1} \tau^{k-2} e^{-(1+nw)s} ds \; dG(\tau)}. \tag{A.8}$$

By Lemma A.1 and condition (A1), (A.8) leads us to the conclusion that

$$\psi(\mu; w) \leqslant \frac{1+nw}{n+1} \frac{\int_0^\infty \tau^{-1} dG(\tau)}{\int_0^\infty \tau^{-2} dG(\tau)} + \frac{1+nw}{n+1} c^* \psi_1(\mu; w) \tag{A.9}$$

where,

$$\psi_1(\mu; w) = n\mu \frac{\sum_{k=0}^{+\infty} \frac{(n\mu)^k}{k!} \int_0^\infty \int_{\frac{\tau\mu}{w}}^\infty s^{n+1} \tau^{k-1} e^{-(1+nw)s} ds \; dG(\tau)}{\sum_{k=0}^{+\infty} \frac{(n\mu)^k}{k!} \int_0^\infty \int_{\frac{\tau\mu}{w}}^\infty s^{n+1} \tau^{k-2} e^{-(1+nw)s} ds \; dG(\tau)}.$$

However, $\sum_{k=0}^{+\infty}(n\mu\tau)^k/k! = e^{n\mu\tau}$ and the above expression becomes,

$$\psi_1(\mu;w) = n\mu\,\frac{\int_0^\infty \tau^{-1}\,e^{n\mu\tau}\int_{\frac{\tau\mu}{w}}^\infty s^{n+1}e^{-(1+nw)s}ds\;dG(\tau)}{\int_0^\infty \tau^{-2}\,e^{n\mu\tau}\int_{\frac{\tau\mu}{w}}^\infty s^{n+1}e^{-(1+nw)s}ds\;dG(\tau)}. \tag{A.10}$$

From conditions (A.6), (A.7), (A.9) and Lemma A.2 below (where condition (A2) has been used), it follows that

$$c(\mu;w) < w + \frac{\kappa}{n+1}\frac{\mathbb{E}\tau^{-1}}{\mathbb{E}\tau^{-2}}(1+nw) + \frac{n(n+2)}{n+1}\kappa c^*\frac{\mathbb{E}\tau^{n+1}}{\mathbb{E}\tau^n}\frac{\mathbb{E}\tau^{-3}}{\mathbb{E}\tau^{-2}}w = \phi_1(w) \tag{A.11}$$

for any μ and $w > 0$.

Furthermore, for $\kappa > (1+1/n)\,\mathbb{E}\tau^{-2}/\mathbb{E}\tau^{-1}$, it is easily verified that $\phi_1(w) < \phi_0(w)$ on a set A of positive w-values having positive probability for all μ. Since $\mathbb{E}_\mu\{(cS-\mu-\kappa)^2|W=w\}$ is strictly increasing in c for $c > c(\mu;w)$, by (A.11), we have $\mathbb{E}_\mu\{(\phi_1(w)S-\mu-\kappa)^2|W=w\} < \mathbb{E}_\mu\{(\phi_0(w)S-\mu-\kappa)^2|W=w\}$, $w \in A$, which in connection with (A.1) implies that $\mathbb{R}(\delta_1;\mu) < \mathbb{R}(\delta_0;\mu)$ for all μ.

PROOF OF THEOREM 2.2. For nonnegative values of μ, δ_2 coincides with δ_0 with probability one. So it suffices to consider $\mu < 0$. The proof of this theorem is analogous to the proof of the Theorem 2.1, that is for $\mu < 0$ and $w < 0$, we will bound from below $c(\mu;w) = (\mu+\kappa)\psi(\mu;w)$ (see (A.2)). Recall that $c(\mu;w)$ is the minimizer of the conditional expectation $\mathbb{E}_\mu\{(cS-\mu-\kappa)^2|W=w\}$ with respect to c. Fixing $\mu < 0$ and using (2.1), given $\tau > 0$, we observe that the (conditional) density of $S|W=w$ is proportional to

$$\tau^n s^{n-1}e^{n\tau\mu}e^{-(1+nw)\tau s}\quad,\quad 0 < s < \mu/w. \tag{A.12}$$

Using (A.2), (A.12) and making a change of variable from s to $\mu s/w$, we get that

$$\psi(\mu;w) = \frac{\mathbb{E}_\mu(S|W=w)}{\mathbb{E}_\mu(S^2|W=w)} = \frac{w}{\mu}\frac{\int_0^\infty \tau^n\int_0^1 s^n e^{-[(1+nw)\frac{\mu}{w}s-n\mu]\tau}ds\;dG(\tau)}{\int_0^\infty \tau^n\int_0^1 s^{n+1}e^{-[(1+nw)\frac{\mu}{w}s-n\mu]\tau}ds\;dG(\tau)}. \tag{A.13}$$

Next, using Lemma 2 in Lehmann (1986, p.85), we establish that

$$\frac{\int_0^1 s^n e^{-[(1+nw)\frac{\mu}{w}s-n\mu]\tau}ds}{\int_0^1 s^{n+1}e^{-[(1+nw)\frac{\mu}{w}s-n\mu]\tau}ds} < \frac{\int_0^1 s^n ds}{\int_0^1 s^{n+1}ds} = \frac{n+2}{n+1},\text{ so that}$$

$$\int_0^1 s^n e^{-[(1+nw)\frac{\mu}{w}s-n\mu]\tau}ds < \frac{n+2}{n+1}\int_0^1 s^{n+1}e^{-[(1+nw)\frac{\mu}{w}s-n\mu]\tau}ds. \tag{A.14}$$

In the numerator of (A.13), we substitute the upper bound of (A.14) and since $\mu < 0$,

$$\mu\psi(\mu; w) > \frac{n+2}{n+1}\, w. \tag{A.15}$$

But $\kappa \geqslant 0$ and hence from (A.2) and (A.15), we obtain that

$$c(\mu; w) = (\mu + \kappa)\psi(\mu; w) \geqslant \mu\psi(\mu; w) > \frac{n+2}{n+1} w.$$

Now, for $w < -1/n$, $\phi_2(w) > \phi_0(w)$ holds for $w > \frac{n+1}{n}\left(\kappa \mathbb{E}\tau^{-1}/\mathbb{E}\tau^{-2} - 1/n\right)$ whereas, provided that $0 \leqslant \kappa < \frac{1}{n(n+1)}\, \mathbb{E}\tau^{-2}/\mathbb{E}\tau^{-1}$, we have

$$c(\mu; w) > \phi_2(w) > \phi_0(w), \tag{A.16}$$

for $w \in B = \left(\frac{n+1}{n}\left(\kappa \mathbb{E}\tau^{-1}/\mathbb{E}\tau^{-2} - 1/n\right), -1/n\right)$. Finally, (A.16) and the fact that $\mathbb{E}_\mu\{(cS - \mu - \kappa)^2 | W = w\}$ is strictly decreasing for $c < c(\mu; w)$, ensure that $\mathbb{E}_\mu\{(\phi_2(w)S - \mu - \kappa)^2 | W = w\} < \mathbb{E}_\mu\{(\phi_0(w)S - \mu - \kappa)^2 | W = w\}$, $w \in B$, which in turn entails $\mathbb{R}(\delta_2; \mu) < \mathbb{R}(\delta_0; \mu)$. This completes the proof of the theorem. □

LEMMA A.1. *For $a = a(\tau) \geqslant 0$ and $t > 0$,*

$$\begin{aligned} &\int_0^\infty \int_a^\infty s^n \tau^{k-1} e^{-ts} ds\, dG(\tau) \\ &\quad \leqslant \frac{\int_0^\infty \tau^{k-1} dG(\tau)}{\int_0^\infty \tau^{k-2} dG(\tau)}\, \frac{t}{n+1} \int_0^\infty \int_a^\infty s^{n+1}\tau^{k-2} e^{-ts} ds\, dG(\tau). \end{aligned}$$

PROOF. It is given in Petropoulos (2006). □

LEMMA A.2. $\psi_1(\mu; w) < nw\, \dfrac{n+2}{1+nw}\, \dfrac{\mathbb{E}\tau^{n+1}}{\mathbb{E}\tau^n}\, \dfrac{\mathbb{E}\tau^{-3}}{\mathbb{E}\tau^{-2}}.$

PROOF. It is given in Petropoulos (2006). □

LEMMA A.3. $g(s) = \sum_{k=0}^{n} A_k a_\mu^k \Big/ \sum_{k=0}^{n} A_k a_0^k$ *is decreasing in s, where*

$$a_\mu = a(s; \mu) = \left[\frac{2}{\lambda}(1 + nw)s + \frac{1}{\gamma^2} - \frac{2}{\lambda} n\mu\right]^{-1/2},\ a_0 = a(s; 0)$$

and $A_k = \dfrac{(n-1+k)!}{k!(n-1-k)!}\, \dfrac{1}{(2\lambda)^k}.$

PROOF. It is given in Petropoulos (2006). □

References

AL-MUTAIRI (1997). Properties of an Inverse Gaussian Mixture of Bivariate Exponential Distribution and its Generalization. *Statist. Probab. Lett.*, **33**, 359-365.

ARNOLD, B.C. (1983). *Pareto Distributions*, International Cooperative Publishing House, Silver Spring, Maryland.

BHATTACHARYA, S.K. and KUMAR, S. (1986). E-IG model in life testing. *Calcutta Statist. Assoc. Bull.*, **35**, 85-90.

ELFESSI, A. (1997). Estimation of a linear function of the parameters of an exponential distribution from doubly censored samples. *Statist. Probab. Lett.*, **36**, 251-259.

KOTZ S., BALAKRISHNAN, N. and JOHNSON, N.L. (2000). *Continuous Multivariate Distributions, Vol.1: Models and Applications*, 2nd ed., Wiley, New York.

LEHMANN, E.L. (1986). *Testing Statistical Hypotheses*, 2nd ed., Wiley, New York.

LINDLEY, D.V. and SINGPURWALLA, N.D. (1986). Multivariate distributions for the life lengths of components of a system sharing a common environment. *J. Appl. Probab.* **23**, 418-431.

NAYAK, T.K. (1987). Multivariate Lomax distribution: properties and usefulness in reliability theory. *J. Appl. Probab.* **24**, 170-177.

PETROPOULOS, C. and KOUROUKLIS, S. (2001). Estimation of an exponential quantile under a general loss and an alternative estimator under quadratic loss. *Ann. Inst. Statist. Math.*, **53**, 746-759.

PETROPOULOS, C. and KOUROUKLIS, S. (2004). Improved estimation of extreme quantiles in the multivariate Lomax (Pareto II) distribution. *Metrika*, **60**, 15-24.

PETROPOULOS, C. and KOUROUKLIS, S. (2005). Estimation of a scale parameter in mixture models with unknown location. *J. Statist. Plann. Inference*, **128**, 191-218.

PETROPOULOS, C. (2006). Estimation of a quantile in a mixture model of exponential distributions with unknown location and scale parameter. Unpublished Manuscript (`http://www.math.upatras.gr/~costas/papers.html`)

RUKHIN, A.L. (1986). Admissibility and minimaxity results in the estimation problem of exponential quantiles. *Ann. Statist.*, **14**, 220-237.

RUKHIN, A.L. and STRAWDERMAN W.E. (1982). Estimating a quantile of an exponential distribution. *J. Amer. Statist. Assoc.*, **77**, 159-162.

RUKHIN, A.L. and ZIDEK, J.V. (1985). Estimation of linear parametric functions for several exponential samples. *Statist. Decisions*, **3**, 225-238.

SESHADRI, V. (1993). *The Inverse Gaussian Distribution : A Case Study in Exponential Families.* Oxford Science Publications.

STEIN, C. (1964). Inadmissibility of the usual estimator for the variance of a normal distribution with unknown mean. *Ann. Inst. Statist. Math.*, **16**, 155-160.

WHITMORE, G.A. and LEE, M.L.T. (1991). A multivariate survival distribution generated by an inverse Gaussian mixture Of exponentials, *Technometrics*, **33**, 39-50.

CONSTANTINOS PETROPOULOS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PATRAS
26 500 RIO, PATRAS, GREECE
E-mail: costas@math.upatras.gr

Paper received January 2005; revised March 2006.