On the Distribution of the Product and Ratio of Independent Generalized Gamma-Ratio Random Variables

Carlos A. Coelho and João T. Mexia
The New University of Lisbon, Portugal

Abstract

Using a decomposition of the characteristic function of the logarithm of the product of independent generalized gamma-ratio random variables (r.v.’s), we obtain explicit expressions for both the probability density and cumulative distribution functions of the product of independent r.v.’s with generalized F or generalized gamma-ratio distributions in the form of particular mixtures of generalized Pareto and inverted Pareto distributions. The expressions obtained do not involve any unsolved integrals and are convenient for computer implementation. By considering power parameters which are not required to be positive, we were able to obtain, as particular cases, not only the distributions for the product of folded T and folded Cauchy r.v.’s but also for the ratio of two independent products of generalized gamma-ratio r.v.’s. Theoretical applications of the results as well as simulations are presented.


Keywords and phrases. Particular mixtures, Pareto and inverted Pareto distributions, GIG distribution, sum of exponentials, difference of exponentials, folded T, folded Cauchy, beta prime, beta second kind.

1 Introduction

The problem of obtaining an explicit expression, without involving any unsolved integrals, for both the probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of the product of independent generalized gamma-ratio random variables (r.v.’s) or generalized F r.v.’s, as they are also called, is a challenging one, especially since the characteristic function is not readily available for such r.v.’s. In this paper, we determine the distribution of the product of independent generalized gamma-ratio (GGR)
random variables under the form of particular mixtures of Pareto and inverted Pareto distributions. Expressions for the p.d.f. of such a product were obtained by Shah and Rathie (1974) in terms of Fox’s $H$ function. Also Pham-Gia and Turkkan (2002) obtained expressions for the p.d.f. of the product of only two independent generalized $F$ r.v.’s in terms of the Lauricella hypergeometric $D$-function of two variables. However, even nowadays, when good softwares for symbolic and numeric computation are available, Fox’s $H$ function and the Lauricella function are not readily computable, being usually computed in terms of the integrals that define them. Although Pham-Gia and Turkkan (2002) developed an efficient computer code to compute the Lauricella hypergeometric function, their approach is not extensible to the product of more than two r.v.’s and these authors point out that when we consider the product of more than two GGR r.v.’s, it seems that “frequently, however, no closed form solution for these operations can be obtained, and one has to resort to approximate approaches, including simulation”. The same authors, when referring to the results in Shah and Rathie (1974) say that “these results, although very convenient notationwise, are difficult to be programmed on a computer and hence are difficult to be used in applications”. Yet the same authors say that “they are, however, essential when the number of variables in the product, or quotient, is larger than 2”. In this paper, our aim is to obtain explicit simpler expressions for the p.d.f. and the c.d.f. of the product of independent generalized gamma-ratio r.v.’s which may be readily implemented computationally and that, given its structure, may also give us ready access to asymptotic and near-exact distributions. Given the approach followed, not only the distributions for the non-central case are readily at hand but in addition the distribution for the product of any non-null power of GGR r.v.’s is also available.

As particular immediate cases, we have the product of independent generalized second kind beta or beta prime, folded $T$ and folded Cauchy r.v.’s and yet, of course, $F$ r.v.’s.

Given the fact that the characteristic functions for the GGR random variables are not readily available and given the fact that we are dealing with a product of random variables, it is more convenient to carry out our work through the decomposition of the characteristic function of the logarithm of the product of the GGR random variables.

Another novelty is that, although usually only positive power parameters are considered for the GGR distributions, nothing actually forces those parameters to be positive. In fact, to consider a negative power parameter
in the GGR distribution is clearly equivalent to considering the reciprocal of that given random variable with the symmetrical positive power parameter. Given the way the problem is approached, even negative power parameters may be easily considered in the GGR distributions and also the distribution of the ratio of two GGR random variables or of the ratio of two products of GGR random variables are particular cases of the results obtained in the paper.

Products of several independent GGR random variables arise in several areas of application in statistics as it is shown in Section 4 and are also related to a test statistic used in the multivariate linear functional model (Provost, 1986).

2 Some Preliminary Results

2.1. The generalized gamma-ratio (GGR) distribution. In order to establish some of the notation, nomenclature and a result used, we will start by defining what we mean by a generalized gamma-ratio (GGR) distribution. Let

\[ X_1 \sim \Gamma(r_1, \lambda_1) \quad \text{and} \quad X_2 \sim \Gamma(r_2, \lambda_2) \]

be two independent r.v.’s with gamma distributions with shape parameters \( r_1 \) and \( r_2 \) and rate parameters \( \lambda_1 \) and \( \lambda_2 \), that is, for example, \( X_1 \) has p.d.f.

\[
f_{X_1}(x) = \frac{\lambda_1^{r_1}}{\Gamma(r_1)} \exp(-\lambda_1 x) x^{r_1-1}, \quad r_1, \lambda_1 > 0; \ x > 0.
\]

Let then

\[ Y_1 = X_1^{1/\beta}, \quad Y_2 = X_2^{1/\beta}, \quad \beta \in \mathbb{R}\{0\} \]

and

\[ Z = Y_1/Y_2. \]

We will say that \( Y_1 \) and \( Y_2 \) have generalized gamma distributions and that \( Z \) has a GGR distribution. Using standard methods, we have the p.d.f.’s of \( Y_i \) (\( i = 1, 2 \)) and \( Z \) given by

\[
f_{Y_i}(y_i) = \frac{|\beta| \lambda_i^{r_i}}{\Gamma(r_i)} \exp\left(-\lambda_i y_i^{\beta}\right) y_i^{\beta r_i - 1}, \quad y_i > 0, \quad (i = 1, 2)
\]

and

\[
f_{Z}(z) = \frac{|\beta| k^{r_1}}{B(r_1, r_2)} \left(1 + k z^{\beta}\right)^{-r_1-r_2} z^{\beta r_1 - 1}, \quad z > 0,
\]
where \( k = \lambda_1/\lambda_2 \), and \( B(\cdot, \cdot) \) is the beta function.

We will denote the fact that \( Z \) has the GGR distribution with parameters \( k, r_1, r_2 \) and \( \beta \) by

\[
Z \sim GGR(k, r_1, r_2, \beta).
\]

The \( h \)th moment of \( Z \) is easily derived as

\[
E(Z^h) = k^{-h/\beta} \frac{\Gamma(r_1 + h/\beta)}{\Gamma(r_1)} \frac{\Gamma(r_2 - h/\beta)}{\Gamma(r_2)}, \quad (-r_1 < h/\beta < r_2). \tag{2.1}
\]

If \( \beta = 1 \), \( k = m/n \), \( r_1 = m/2 \) and \( r_2 = n/2 \), with \( m, n \in \mathbb{N} \), then \( Z \) has an \( F \) distribution with \( m \) and \( n \) degrees of freedom. This is the reason why the distribution of \( Z \) is also called a generalized \( F \) distribution (Shah and Rathie, 1974).

2.2. The generalized integer gamma (GIG) distribution. In this subsection and in the two following ones, we will establish some distributions that will be used in the next section. Let

\[
X_j \sim \Gamma(r_j, \lambda_j), \quad j = 1, \ldots, p,
\]

be \( p \) independent r.v.’s with gamma distributions with shape parameters \( r_j \in \mathbb{N} \) and rate parameters \( \lambda_j > 0 \) \( (j = 1, \ldots, p) \). We will say that then the r.v.

\[
Y = \sum_{j=1}^{p} X_j
\]

has a GIG distribution of depth \( p \), with shape parameters \( r_j \) and rate parameters \( \lambda_j \), \( (j = 1, \ldots, p) \), and we will denote this fact by

\[
Y \sim GIG(r_j, \lambda_j; p).
\]

The p.d.f. and the c.d.f. of \( Y \) are respectively given by (Coelho, 1998)

\[
f_Y(y) = K \sum_{j=1}^{p} P_j(y) \exp(-\lambda_j y) \tag{2.2}
\]

and

\[
F_Y(y) = 1 - K \sum_{j=1}^{p} P_j^*(y) \exp(-\lambda_j y), \tag{2.3}
\]
where

\[ K = \prod_{j=1}^{p} \lambda_j^{r_j}, \quad P_j(y) = \sum_{k=1}^{r_j} c_{j,k} y^{k-1} \]  

(2.4)

and

\[ P_j^*(y) = \sum_{k=1}^{r_j} c_{j,k} (k-1)! \sum_{i=0}^{k-1} \frac{y^i}{i! \lambda_j^{k-i}}, \]

with

\[ c_{j,r_j} = \frac{1}{(r_j-1)!} \prod_{i=1, i \neq j}^{p} (\lambda_i - \lambda_j)^{-r_i}, \quad j = 1, \ldots, p, \]  

(2.5)

and

\[ c_{j,r_j-k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(r_j - k + i - 1)!}{(r_j - k - 1)!} R(i, j, p) c_{j,r_j-(k-i)}, \quad (k = 1, \ldots, r_j - 1) \]

(2.6)

where

\[ R(i, j, p) = \sum_{k=1}^{p} r_k (\lambda_j - \lambda_k)^{-i} \quad (i = 1, \ldots, r_j - 1). \]  

(2.7)

2.3. The distribution of the sum of random variables with exponential distribution and the distribution of the difference of two of these random variables. Let

\[ X_i \sim \text{Exp}(\lambda_i) \sim \Gamma(1, \lambda_i), \quad i = 1, \ldots, p, \]

be \( p \) independent exponential r.v.’s with rate parameters \( \lambda_i \) \((i = 1, \ldots, p)\), and let

\[ Y = \sum_{i=1}^{p} X_i. \]

The distribution of the r.v. \( Y \) is a particular case of the GIG distribution (Coelho, 1998) of depth \( p \), with all the shape parameters equal to 1, whose p.d.f. may be written as

\[ f_Y(y) = K \sum_{j=1}^{p} c_{j,1} \exp(-\lambda_j y), \]  

(2.8)
where
\[ K = \prod_{j=1}^{p} \lambda_j \quad \text{and} \quad c_{j,1} = \prod_{k=1, k \neq j}^{p} \frac{1}{\lambda_k - \lambda_j} \quad (j = 1, \ldots, p). \tag{2.9} \]

We will denote the fact that \( Y \) has this distribution by
\[ Y \sim SE(\lambda_j; p) \sim GIG(1, \lambda_j; p). \tag{2.10} \]

**Theorem 2.1.** Let
\[ Y_1 \sim SE(\lambda_j; p) \quad \text{and} \quad Y_2 \sim SE(\nu_j; p) \tag{2.11} \]
be two independent r.v.’s and let
\[ Z = Y_1 - Y_2. \]

The p.d.f. and c.d.f. of \( Z \) are given by
\[
 f_Z(z) = \begin{cases} 
 K_1 K_2 \sum_{j=1}^{p} H_{2j} d_{j,1} \exp(\nu_j z), & z \leq 0 \\
 K_1 K_2 \sum_{j=1}^{p} H_{1j} c_{j,1} \exp(-\lambda_j z), & z \geq 0 
\end{cases} \tag{2.12}
\]

and
\[
 F_Z(z) = \begin{cases} 
 K_1 K_2 \sum_{j=1}^{p} \frac{H_{2j}}{\nu_j} \frac{d_{j,1}}{\nu_j} \exp(\nu_j z), & z \leq 0 \\
 K_1 K_2 \sum_{j=1}^{p} \left( \frac{H_{2j}}{\nu_j} \frac{d_{j,1}}{\nu_j} + H_{1j} c_{j,1} \frac{1}{\lambda_j} (1 - \exp(-\lambda_j z)) \right), & z \geq 0 
\end{cases} \tag{2.13}
\]

respectively, where
\[ K_1 = \prod_{j=1}^{p} \lambda_j, \quad K_2 = \prod_{j=1}^{p} \nu_j \tag{2.14} \]

and, for \( j = 1, \ldots, p, \)
\[ c_{j,1} = \prod_{k=1, k \neq j}^{p} \frac{1}{\lambda_k - \lambda_j}, \quad d_{j,1} = \prod_{k=1, k \neq j}^{p} \frac{1}{\nu_k - \nu_j}, \tag{2.15} \]
\[ H_{1j} = \sum_{h=1}^{p} \frac{d_{h,1}}{\lambda_j + \nu_h} \quad \text{and} \quad H_{2j} = \sum_{h=1}^{p} \frac{c_{h,1}}{\lambda_h + \nu_j}. \tag{2.16} \]
Proof. The p.d.f.’s of $Y_1$ and $Y_2$ may then be written as

$$f_{Y_1}(y_1) = K_1 \sum_{j=1}^{P} c_{j,1} \exp(-\lambda_j y_1) \quad \text{and} \quad f_{Y_2}(y_2) = K_2 \sum_{j=1}^{P} d_{j,1} \exp(-\nu_j y_2)$$

respectively, with $K_1$ and $K_2$ given by (2.14), $c_{j,1}$ and $d_{j,1}$ given by (2.15) and $H_{1j}$ and $H_{2j}$ given by (2.16), so that the p.d.f. of $Z$ will be given by

$$f_Z(z) = \int_{\max(z,0)}^{+\infty} K_1 K_2 \left( \sum_{j=1}^{P} c_{j,1} \exp(-\lambda_j y_1) \right) \left( \sum_{j=1}^{P} d_{j,1} \exp\{-\nu_j(y_1 - z)\} \right) dy_1$$

$$= K_1 K_2 \sum_{j=1}^{P} \sum_{k=1}^{P} \exp(\nu_k z) c_{j,1} d_{k,1} \int_{\max(z,0)}^{+\infty} \exp\{- (\lambda_j + \nu_k) y_1 \} dy_1,$$

where

$$\int_{\max(z,0)}^{\infty} \exp\{- (\lambda_j + \nu_k) y_1 \} dy_1 = \begin{cases} \frac{1}{\lambda_j + \nu_k}, & z \leq 0, \\ \frac{\exp\{- (\lambda_j + \nu_k) z\}}{\lambda_j + \nu_k}, & z \geq 0 \end{cases}$$

so that, after some small rearrangements, we have $f_Z(z)$ given by (2.12). The c.d.f. of $Z$ is then easily derived from (2.12) as given by (2.13). \[ \square \]

**Corollary 2.1.** If we take, for some $k > 0$,

$$W = k^{-1} \exp(Z),$$

then, the p.d.f. and the c.d.f. of $W$ will be given by

$$f_W(w) = \begin{cases} K_1 K_2 \sum_{j=1}^{P} H_{1j} c_{j,1} (kw)^{-\lambda_j} \frac{1}{w}, & w \geq k^{-1}, \\ K_1 K_2 \sum_{j=1}^{P} H_{2j} d_{j,1} (kw)^{\nu_j} \frac{1}{w}, & 0 < w \leq k^{-1}, \end{cases}$$

and

$$F_W(w) = \begin{cases} K_1 K_2 \sum_{j=1}^{P} \left( H_{2j} \frac{d_{j,1}}{\nu_j} + H_{1j} \frac{c_{j,1}}{\lambda_j} \left( 1 - (kw)^{-\lambda_j} \right) \right), & w \geq k^{-1}, \\ K_1 K_2 \sum_{j=1}^{P} H_{2j} \frac{d_{j,1}}{\nu_j} (kw)^{\nu_j}, & 0 < w \leq k^{-1}, \end{cases}$$

respectively.
The distribution of $Z$ is also the distribution of the sum of $p$ independent r.v.’s with the distribution of the difference of two independent r.v.’s with exponential distribution, either with similar or different parameters, i.e., the distribution of the sum of $p$ independent r.v.’s with Laplace or generalized Laplace distributions.

We should note that although, namely in (2.14), it may seem that it would not be reasonable to take $p \to \infty$, as a matter of fact in both (2.12) and (2.18) taking $p \to \infty$ will yield genuine p.d.f.’s (see Appendix).

Taking into account that if the r.v. $X$ has an exponential distribution with rate parameter $\lambda$ and p.d.f.

$$f_X(x) = \lambda \exp(-\lambda x), \quad x > 0; \lambda > 0,$$

for $k > 0$, the r.v. $Y = k \cdot \exp(X)$ has a Pareto distribution with rate parameter $\lambda$, lower bound parameter $k$, and p.d.f.

$$f_Y(y) = \lambda \left(\frac{y}{k}\right)^{-\lambda} \frac{1}{y}, \quad y \geq k; \lambda > 0.$$

We will say that the r.v. $X_1 = -X$ with p.d.f.

$$f_{X_1}(x) = \lambda \exp(\lambda x), \quad x < 0; \lambda > 0$$

has a symmetric exponential distribution with rate parameter $\lambda$ and that the r.v. $Y_1 = 1/Y = (1/k) \exp(X_1) = (1/k) \exp(-X)$, with p.d.f.

$$f_{Y_1}(y) = \lambda \left(\frac{y}{k^{-1}}\right)^{-\lambda} \frac{1}{y}, \quad 0 < y \leq \frac{1}{k}; \lambda > 0,$$

has an inverted Pareto distribution with rate parameter $\lambda$ and lower bound parameter $k^{-1}$.

We may then note that the distribution of $Z$, for $z \geq 0$, may be seen as a particular mixture of exponential distributions with rate parameters $\lambda_j$ ($j = 1, \ldots, p$), with weights

$$p_j = K_1K_2H_1j c_j \lambda_j, \quad j = 1, \ldots, p,$$

such that

$$\sum_{j=1}^{p} p_j = P[Z \geq 0];$$
and for \( z \leq 0 \), as a particular mixture of symmetric exponential distributions with rate parameters \( \nu_j \), with weights
\[
q_j = K_1 K_2 H_{2j} \frac{d_j}{\nu_j}, \quad j = 1, \ldots, p,
\]
such that
\[
\sum_{j=1}^{p} q_j = P[Z \leq 0].
\]
On the other hand, the distribution of \( W \) may, for \( w \geq k^{-1} \), be seen as a particular mixture of Pareto distributions with rate parameters \( \lambda_j \) (\( j = 1, \ldots, p \)) and lower bound parameters \( k^{-1} \), with weights \( p_j \) (\( j = 1, \ldots, p \)), such that
\[
\sum_{j=1}^{p} p_j = P[W \geq k^{-1}],
\]
and for \( w \leq k^{-1} \), it may be seen as a mixture of inverted Pareto distributions with rate parameters \( \nu_j \) (\( j = 1, \ldots, p \)) and lower bound parameters \( k \), with weights \( q_j \) (\( j = 1, \ldots, p \)), such that
\[
\sum_{j=1}^{p} q_j = P[W \leq k^{-1}].
\]

2.4. The distribution of the difference of two GIG distributions.

**Theorem 2.2.** Let, for \( j = 1, \ldots, p_1 \) and \( l = 1, \ldots, p_2 \),
\[
Y_1 \sim GIG(r_{1j}, \lambda_j; p_1), \quad \text{and} \quad Y_2 \sim GIG(r_{2l}, \nu_l; p_2),
\]
be two independent r.v.’s, and let
\[
Z = Y_1 - Y_2.
\]
Then, by taking \( K_1 \) and \( c_{jk} \) defined in a manner similar to the definitions of \( K \) and \( c_{jk} \) in (2.4) and (2.5)-(2.7) respectively, and \( K_2 \) and \( d_{lh} \) defined in a corresponding manner, replacing \( p_1 \) by \( p_2 \), \( r_{1j} \) by \( r_{2l} \) and \( \lambda_j \) by \( \nu_l \), the p.d.f. of \( Z \) is given by
\[
f_Z(z) = \begin{cases} 
K_1 K_2 \sum_{j=1}^{p_1} P_{1j}^{**}(z) \exp(-\lambda_j z), & z \geq 0, \\
K_1 K_2 \sum_{j=1}^{p_1} P_{2j}^{**}(z) \exp(\nu_j z), & z \leq 0,
\end{cases}
\]
(2.20)
where

\[
P_{1j}^{**}(z) = \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{r_{2j}} \sum_{h=1}^{r_{2lj}} d_{lh} \sum_{i=0}^{k-1} \binom{k-1}{i} z^{k-1-i} \frac{(h+i-1)!}{(\lambda_j + \nu_j)^{h+i}}
\]

(2.21)

and

\[
P_{2j}^{**}(z) = \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{r_{2j}} \sum_{h=1}^{r_{2lj}} d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} (-z)^{h-1-i} \frac{(k+i-1)!}{(\lambda_j + \nu_j)^{k+i}}
\]

(2.22)

and the c.d.f. by

\[
F_Z(z) = \begin{cases} 
K_1 K_2 \sum_{j=1}^{p_1} P_{1j}^{***}(z) \exp(-\lambda_j z), & z \geq 0, \\
K_1 K_2 \sum_{j=1}^{p_1} P_{2j}^{***}(z) \exp(\nu_j z), & z \leq 0, 
\end{cases}
\]

(2.23)

with

\[
P_{1j}^{***}(z) = \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{r_{2j}} \sum_{h=1}^{r_{2lj}} d_{lh} \sum_{i=0}^{k-1} \binom{k-1}{i} (h+i-1)! \frac{(k-1-i)!}{t!} \frac{z^t}{\lambda_j^{k-i-t}}
\]

(2.24)

and

\[
P_{2j}^{***}(z) = \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{r_{2j}} \sum_{h=1}^{r_{2lj}} d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} (k+i-1)! \frac{(h-1-i)!}{t!} \frac{(-z)^t}{\nu_j^{h-i-t}}
\]

(2.25)

**Proof.** Considering (2.2) and taking \(K_1\) and \(c_{jk}\) defined in a manner similar to the definitions of \(K\) and \(c_{jk}\) in (2.4) and (2.5)-(2.7) respectively, and \(K_2\) and \(d_{lh}\) defined in a corresponding manner, the p.d.f. of \(Z\) is given
Product and ratio of gamma-ratio random variables

\[ f_Z(z) = K_1 K_2 \int_{\max(z,0)}^{+\infty} \left\{ \sum_{j=1}^{p_1} \left( \sum_{k=1}^{r_{1j}} c_{jk} y_1^{k-1} \right) \exp(-\lambda_j y_1) \right\} \]

\[ \times \left\{ \sum_{l=1}^{p_2} \left( \sum_{h=1}^{r_{2l}} d_{lh} (y_1 - z)^{h-1} \right) \exp\{-(\nu_l y_1)\} \right\} dy_1 \]

\[ = K_1 K_2 \int_{\max(z,0)}^{+\infty} \sum_{j=1}^{p_1} \sum_{l=1}^{p_2} \left( \sum_{k=1}^{r_{1j}} c_{jk} y_1^{k-1} \right) \left( \sum_{h=1}^{r_{2l}} d_{lh} (y_1 - z)^{h-1} \right) \]

\[ \times \exp\{-(\lambda_j + \nu_l) y_1\} \exp(\nu_l z) dy_1 \]

\[ = K_1 K_2 \sum_{j=1}^{p_1} \sum_{l=1}^{p_2} \exp(\nu_l z) \left( \sum_{k=1}^{r_{1j}} c_{jk} d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} (-z)^{h-1-i} \right) \]

\[ \int_{\max(z,0)}^{+\infty} \exp\{-(\lambda_j + \nu_l) y_1\} y_1^{k+i-1} dy_1, \quad (2.26) \]

where, for \( m > 0 \) and \( k \in \mathbb{N}_0, \)

\[ \int_{\max(z,0)}^{+\infty} \exp(-my) y^k dy = \begin{cases} \exp(-mz) \sum_{i=0}^{k} \frac{k!}{i!} \frac{z^i}{m^{k+i+1}}, & z \geq 0, \\ \frac{k!}{m^{k+1}}, & z \leq 0, \end{cases} \quad (2.27) \]

(which is, for \( z > 0 \), a particular version of an incomplete gamma function) yielding (2.20) with \( P_{1j}^{**}(z) \) given by

\[ P_{1j}^{**}(z) = \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} (-z)^{h-1-i} \sum_{t=0}^{k+i-1} \frac{(k+i-1)!}{t!} \frac{z^t}{(\lambda_j + \nu_l)^{k+i-t}}. \quad (2.28) \]

It is however interesting and useful to observe that, given that the distribution of \( Y_1 - Y_2 \) and \( Y_2 - Y_1 \) are symmetric, and that we may in (2.26)
integrate with respect to \( y_2 \) instead of \( y_1 \), and we may obtain the p.d.f. of \( Z \) given by an expression similar to the one in (2.20) with \( P_{1j}^{**}(z) \) given by (2.21) and

\[
P_{2j}^{**}(z) = \sum_{k=1}^{r_{1j}} c_{jk} \sum_{\ell=1}^{p_2} \sum_{h=1}^{r_{2\ell}} d_{\ell h} \sum_{i=0}^{k-1} \binom{k-1}{i} z^{k-1-i} \sum_{t=0}^{b+i-1} \frac{(h+i-1)!}{t!} \frac{z^t}{(\lambda_j + \nu_t)^{h+i-t}}.
\]

(2.29)

We may further note that

\[
\sum_{i=0}^{h-1} \binom{h-1}{i} (-z)^{h-1-i} \sum_{t=0}^{k+i-1} \frac{(k+i-1)!}{t!} \frac{z^t}{(\lambda_j + \nu_t)^{k+i-t}}
= \sum_{i=0}^{k-1} \binom{k-1}{i} z^{k-1-i} \frac{(h+i-1)!}{(\lambda_j + \nu_t)^{h+i-t}}.
\]

This way, in order to obtain a simpler expression for the p.d.f. of \( Z \), we may consider the p.d.f. in (2.20) with \( P_{1j}^{**}(z) \) given by (2.21) and \( P_{2j}^{**}(z) \) given by (2.22).

Then, from (2.20) through (2.22), it is easy to derive the c.d.f. of \( Z \) as given by (2.23) through (2.25).

\[
\text{Corollary 2.2. If we consider, for some } k > 0, \text{ the r.v. } W = k^{-1}\exp(Z),
\]

we have

\[
f_W(w) = \begin{cases} 
K_1 K_2 \sum_{j=1}^{p_1} P_{1j}^{**}(\log(kw)) (kw)^{-\lambda_j} \frac{1}{w}, & w \geq k^{-1}, \\
K_1 K_2 \sum_{j=1}^{p_1} P_{2j}^{**}(\log(kw)) (kw)^{\nu_j} \frac{1}{w}, & 0 < w \leq k^{-1}, 
\end{cases}
\]

(2.30)

and

\[
F_W(w) = \begin{cases} 
K_1 K_2 \sum_{j=1}^{p_1} P_{1j}^{***}(\log(kw)) (kw)^{-\lambda_j}, & w \geq k^{-1}, \\
K_1 K_2 \sum_{j=1}^{p_1} P_{2j}^{***}(\log(kw)) (kw)^{\nu_j}, & 0 < w \leq k^{-1}. 
\end{cases}
\]

(2.31)
Product and ratio of gamma-ratio random variables

As we did with (2.12) and (2.18) and also in (2.20) through (2.31), we may take both $p_1 \to \infty$ and $p_2 \to \infty$, which still yields genuine distributions (see Appendix).

3 The Distribution of the Product of $m$ Independent Random Variables with GGR Distributions

3.1. The case with all distinct shape parameters for the numerator and the denominator. In this section, we will obtain explicit and concise expressions for the p.d.f. and the c.d.f. of the r.v.

$$W = \prod_{j=1}^{m} X_j,$$

where the $X_j$’s are independent r.v.’s with all distinct shape parameters. The p.d.f. and c.d.f. of $W$ are stated in the following Theorem.

**Theorem 3.1.** Let

$$X_j \sim GGR(k_j, r_{1j}, r_{2j}, \beta_j), \quad j = 1, \ldots, m,$$

be $m$ independent r.v.’s, where the power parameters $\beta_j$’s ($j = 1, \ldots, m$) are not necessarily all positive. Let

$$\beta_j^* = |\beta_j|, \quad s_{1j} = \begin{cases} r_{1j} & \text{if } \beta_j > 0, \\ r_{2j} & \text{if } \beta_j < 0, \end{cases} \quad \text{and} \quad s_{2j} = \begin{cases} r_{2j} & \text{if } \beta_j > 0, \\ r_{1j} & \text{if } \beta_j < 0, \end{cases}$$

(3.1)

Then, if $\beta_j^* s_{1j} \neq \beta_k^* s_{1k}$ and $\beta_j^* s_{2j} \neq \beta_k^* s_{2k}, \forall j \neq k, j, k \in \{1, \ldots, m\}$, the p.d.f. and the c.d.f. of the r.v.

$$W = \prod_{j=1}^{m} X_j,$$

(3.2)

are given by

$$f_W(w) = \begin{cases} \lim_{n \to \infty} K_1 K_2 \sum_{j=1}^{m} \sum_{h=0}^{n} H_{2jh} d_{hj} (w/K^*)^{s_{1j}h} \frac{1}{w}, & 0 < w \leq K^*, \\ \lim_{n \to \infty} K_1 K_2 \sum_{j=1}^{m} \sum_{h=0}^{n} H_{1jh} c_{hj} (w/K^*)^{-s_{2j}h} \frac{1}{w}, & w \geq K^*, \end{cases}$$

(3.3)
and

\[
F_W(w) = \begin{cases}
\lim_{n \to \infty} K_1 K_2 \sum_{j=1}^{m} \sum_{h=0}^{n} H_{2jh} \frac{d_{hj}}{s_{1jh}} \left( \frac{w}{K^*} \right)^{s_{1jh}}, & 0 < w \leq K^*, \\
\lim_{n \to \infty} K_1 K_2 \sum_{j=1}^{m} \sum_{h=0}^{n} \left\{ H_{2jh} \frac{d_{hj}}{s_{1jh}} + H_{1jh} \frac{c_{hj}}{s_{2jh}} \left( 1 - \left( \frac{w}{K^*} \right)^{-s_{2jh}} \right) \right\}, & w \geq K^*
\end{cases}
\]

respectively, where

\[
K^* = \prod_{j=1}^{m} k_j^{-1/\beta_j},
\]

\[
K_1 = \prod_{j=1}^{m} \prod_{h=0}^{n} s_{1jh}^*, \quad K_2 = \prod_{j=1}^{m} \prod_{h=0}^{n} s_{2jh}^*,
\]

and, for \( j = 1, \ldots, m \) and \( n = 0, 1, \ldots \),

\[
\begin{align*}
c_{hj} &= \prod_{\eta=1}^{m} \prod_{\nu=0}^{n} \frac{1}{s_{2\eta\nu}^* - s_{2jh}^*}, \\
d_{hj} &= \prod_{\eta=1}^{m} \prod_{\nu=0}^{n} \frac{1}{s_{1\eta\nu}^* - s_{1jh}^*},
\end{align*}
\]

and

\[
H_{1jh} = \sum_{k=1}^{m} \sum_{l=0}^{n} s_{2jh}^* s_{1kl}^*, \quad H_{2jh} = \sum_{k=1}^{m} \sum_{l=0}^{n} s_{1jh}^* s_{2kl}^*,
\]

with

\[
s_{1jh}^* = \beta_j^*(s_{1jh} + h) \quad \text{and} \quad s_{2jh}^* = \beta_j^*(s_{2jh} + h) \quad (j = 1, \ldots, m).
\]

**Proof.** From (2.1), we have

\[
E \left( X_j^h \right) = k_j^{-h/\beta_j} \frac{\Gamma(r_{1j} + h/\beta_j) \Gamma(r_{2j} - h/\beta_j)}{\Gamma(r_{1j}) \Gamma(r_{2j})},
\]

so that if we take

\[
Y_j = k_j^{1/\beta_j} X_j,
\]

then we have

\[
E \left( Y_j^h \right) = \frac{\Gamma(r_{1j} + h/\beta_j) \Gamma(r_{2j} - h/\beta_j)}{\Gamma(r_{1j}) \Gamma(r_{2j})}.
\]
Thus, if we define
\[ W' = \prod_{j=1}^{m} Y_j, \]
then we have
\[ W = \prod_{j=1}^{m} X_j = \left( \prod_{j=1}^{m} k_j^{-1/\beta_j} \right) \prod_{j=1}^{m} Y_j = K^* W', \]
where we may take
\[ Z = \log W' = \sum_{j=1}^{m} \log Y_j, \]
so that the characteristics function \( \Phi_Z(t) \) is given, under the form of a Mellin transform, for \( i = \sqrt{-1} \), by
\[
\Phi_Z(t) = \prod_{j=1}^{m} \Phi_{Y_j}(t) = \prod_{j=1}^{m} E \left( Y_j^{it} \right) = \prod_{j=1}^{m} \frac{\Gamma(r_{1j} + it/\beta_j) \Gamma(r_{2j} - it/\beta_j)}{\Gamma(r_{1j}) \Gamma(r_{2j})}.
\]
Using in (3.10), the relation
\[
\Gamma(z) = \frac{\exp(-\gamma z)}{z} \prod_{i=1}^{\infty} \left( \frac{i}{i+z} \exp(z/i) \right),
\]
which is valid for any \( z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \) (Ahlfors, 1979, § 2.4, pp. 198; Lang, 1999, Chap. XV, § 2, pp. 413), where \( \gamma \) is the Euler gamma constant, we may, after some simplifications, write
\[
\Phi_Z(t) = \prod_{j=1}^{m} \prod_{k=0}^{\infty} \beta_j (r_{1j}+k) (\beta_j (r_{1j}+k) + it)^{-1} \beta_j (r_{2j}+k) (\beta_j (r_{2j}+k) - it)^{-1}.
\]
(3.11)

Since the \( \beta_j \)'s \((j = 1, \ldots, m)\) are not necessarily all positive, we take \( \beta^*_j \), \( s_{1j} \) and \( s_{2j} \) as given by (3.1), so that we may write
\[
\Phi_Z(t) = \prod_{j=1}^{m} \prod_{k=0}^{\infty} \beta^*_j (s_{1j}+k) (\beta^*_j (s_{1j}+k) + it)^{-1} \beta^*_j (s_{2j}+k) (\beta^*_j (s_{2j}+k) - it)^{-1},
\]
(3.12)
where now $\beta^*_j(s_{1j} + k) > 0$ and $\beta^*_j(s_{2j} + k) > 0$ for all $j = 1, \ldots, m$ and $k = 0, 1, \ldots$. Expression (3.12) shows that, if $\beta^*_j s_{1j} \neq \beta^*_k s_{1k}$ and $\beta^*_j s_{2j} \neq \beta^*_k s_{2k}$, $\forall j, k \in \{1, \ldots, m\}$, the distribution of $Z$ is the same as the distribution of a sum of infinitely many independent r.v.'s distributed as the difference of two independent exponential distributions, with parameters $\beta^*_j(s_{2j} + k)$ and $\beta^*_j(s_{1j} + k)$ ($j = 1, \ldots, m; k = 0, 1, \ldots$). Alternatively, $Z$ is distributed as the difference of two independent r.v.'s, each one having the distribution of the sum of infinitely many independent exponential r.v.'s.

Thus, since
\[ W = K^* \exp(Z), \] (3.13)
with $K^*$ given by (3.5), taking $s^*_{1jh} = \beta^*_j(s_{1j} + k)$ and $s^*_{2jh} = \beta^*_j(s_{2j} + k)$ and using (2.18) and (2.19) in Corollary 2.1 as a basis, we may write the p.d.f. and the c.d.f. of $W$ as in (3.3) through (3.9).

To simplify the notation, we may replace the pair of indices $(k, j)$ by $h = km + j$, setting
\[ s^*_ih = \beta^*_j(s_{ij} + k), \quad \text{for} \quad i = 1, 2; \; j = 1, \ldots, m; \; k = 0, \ldots, n. \] (3.14)
We may now define $K_1, K_2, c_j, d_j, H_{1j}$ and $H_{2j}$ ($j = 1, \ldots, m(n+1)$), (with $n \to \infty$) in a way similar to the one used in Theorem 3.1, that is,

\[ K_1 = \prod_{j=1}^{m(n+1)} s^*_1j, \quad K_2 = \prod_{j=1}^{m(n+1)} s^*_2j, \]

and, for $j = 1, \ldots,$

\[ c_j = \prod_{k=1 \atop k \neq j}^{m(n+1)} \frac{1}{s^*_1j - s^*_1k}, \quad d_j = \prod_{k=1 \atop k \neq j}^{m(n+1)} \frac{1}{s^*_2j - s^*_2k}, \]

and

\[ H_{1j} = \sum_{h=1}^{m(n+1)} \frac{d_h}{s^*_1j + s^*_2h}, \quad H_{2j} = \sum_{h=1}^{m(n+1)} \frac{c_h}{s^*_1h + s^*_2j}, \]
so that we may, for example, write the c.d.f. of the r.v. $W$ as

\[
F_W(w) = \begin{cases} 
\lim_{n \to \infty} K_1 K_2 \sum_{j=1}^{m(n+1)} H_{2j} \frac{d_j}{s_{1j}} \left( \frac{w}{K^{*}} \right)^{s_{1j}}, & 0 < w \leq K^{*}, \\
\lim_{n \to \infty} K_1 K_2 \sum_{j=1}^{m(n+1)} \left[ H_{2j} \frac{d_j}{s_{1j}} + H_{1j} \frac{c_j}{s_{2j}} \left( 1 - \left( \frac{w}{K^{*}} \right)^{s_{2j}} \right) \right], & w \geq K^{*}.
\end{cases}
\]

(3.15)

3.2. The general case. In the general case, we will allow for the possibility that some of the parameters $\tilde{s}_{1j} = \beta_j^{*} s_{1j}$, $j \in \{i, \ldots, m\}$, will be equal, and that also some of the parameters $\tilde{s}_{2j} = \beta_j^{*} s_{2j}$, $j \in \{i, \ldots, m\}$, will be equal. More precisely, and without any loss of generality, let us suppose that $\tau_1$ of the $m$ parameters $\tilde{s}_{1j}$ are equal to $\tilde{s}_{11}$, $\tau_2$ are equal to $\tilde{s}_{12}$, and so on, and that $\tau_{p_1}$ are equal to $\tilde{s}_{1p_1}$, with $p_1 < m$ and

\[
\sum_{j=1}^{p_1} \tau_j = m
\]

and that, similarly, $\eta_1$ of the $m$ parameters $\tilde{s}_{2j}$ are equal to $\tilde{s}_{21}$, $\eta_2$ are equal to $\tilde{s}_{22}$, and so on, and that $\eta_{p_2}$ are equal to $\tilde{s}_{2p_2}$, with $p_2 < m$ and

\[
\sum_{j=1}^{p_2} \eta_j = m
\]

Then the characteristic function in (3.10) may be written as

\[
\Phi_Z(t) = \prod_{j=1}^{p_1} \prod_{k=0}^{\infty} \left( \tilde{s}_{1j} + \beta_j^{*} k \right)^{\tau_j} \left( \tilde{s}_{1j} + \beta_j^{*} k - it \right)^{-\tau_j} \prod_{j=1}^{p_2} \prod_{k=0}^{\infty} \left( \tilde{s}_{2j} + \beta_j^{*} k \right)^{\eta_j} \left( \tilde{s}_{2j} + \beta_j^{*} k - it \right)^{-\eta_j}
\]

\[
= \lim_{n \to \infty} \prod_{j=1}^{p_1} \prod_{k=0}^{n} \left( \tilde{s}_{1j} + \beta_j^{*} k \right)^{\tau_j} \left( \tilde{s}_{1j} + \beta_j^{*} k - it \right)^{-\tau_j} \prod_{j=1}^{p_2} \prod_{k=0}^{n} \left( \tilde{s}_{2j} + \beta_j^{*} k \right)^{\eta_j} \left( \tilde{s}_{2j} + \beta_j^{*} k - it \right)^{-\eta_j},
\]

that is, the characteristic function of the difference of two independent r.v.’s with GIG distributions, where the first one, that is, the one with positive
sign, with depth $p_1 \times (n+1)$ (with $n \to \infty$), with rate parameters and associated shape parameters

$$\tilde{s}_{11} + \beta^*_1 k, \ldots, \tilde{s}_{11} + \beta^*_1 k, \ldots, \tilde{s}_{1p_1} + \beta^*_p k$$

$k = 0, \ldots, n$

$$\tau_1, \ldots, \tau_1$$

$n+1$

and the second one, that is, the one with negative sign, with depth $p_2 \times (n+1)$ (with $n \to \infty$), with shape parameters and associated rate parameters

$$\tilde{s}_{21} + \beta^*_1 k, \ldots, \tilde{s}_{21} + \beta^*_1 k, \ldots, \tilde{s}_{2p_2} + \beta^*_p k$$

$k = 0, \ldots, n$

$$\eta_1, \ldots, \eta_1$$

$n+1$

Let us consider the vectors

$$\tilde{\tau}^* = \left[ \tau_1, \ldots, \tau_{p_1}, \tau_1, \ldots, \tau_{p_1} \right]'$$

$n+1$ times

and

$$\tilde{\eta}^* = \left[ \eta_1, \ldots, \eta_{p_2}, \eta_1, \ldots, \eta_{p_2} \right]'$$

$n+1$ times

where, for $j = 1, \ldots, p_1(n+1)$ and $l = 1, \ldots, p_2(n+1)$, with $j = kp_1 + h$ and $l = kp_2 + i$, for $k = 0, \ldots, n$, $h = 1, \ldots, p_1$ and $i = 1, \ldots, p_2$,

$$\tau^*_j = \tau_h, \quad \text{and} \quad \eta^*_l = \eta_i,$$

and (similarly to the vectors $\tilde{s}_1^*$ and $\tilde{s}_2^*$ considered in the previous subsection) the vectors

$$\tilde{s}_1^* = \left[ \tilde{s}_{11}, \ldots, \tilde{s}_{1p_1}, \tilde{s}_{11} + \beta^*_1, \ldots, \tilde{s}_{1p_1} + \beta^*_p, \ldots, \tilde{s}_{11} + \beta^*_1 n, \ldots, \tilde{s}_{1p_1} + \beta^*_p n \right]'$$

and

$$\tilde{s}_2^* = \left[ \tilde{s}_{21}, \ldots, \tilde{s}_{2p_2}, \tilde{s}_{21} + \beta^*_1, \ldots, \tilde{s}_{2p_2} + \beta^*_p, \ldots, \tilde{s}_{21} + \beta^*_1 n, \ldots, \tilde{s}_{2p_2} + \beta^*_p n \right]'$$

where, once again, for $j = 1, \ldots, p_1(n+1)$ and $l = 1, \ldots, p_2(n+1)$, with $j$, $l$, $k$, $h$ and $i$ defined as above,

$$\tilde{s}_{1j}^* = \tilde{s}_{1j} + \beta^*_h k \quad \text{and} \quad \tilde{s}_{2l}^* = \tilde{s}_{2l} + \beta^*_i k.$$
The p.d.f. and the c.d.f. of $W = K^* \exp(Z)$, for $K^*$ defined as in (3.13), may then be derived respectively from (2.30) and (2.31) in Corollary 2.2, taking into account the shape and the rate parameters mentioned above, as

$$f_W(w) = \begin{cases} \lim_{n \to \infty} K_1 K_2 \sum_{j=1}^{p_2(n+1)} P_{2j}^{**} \left( \log \left( \frac{w}{K^*} \right) \right) \left( \frac{w}{K^*} \right)^{s_{2j}} \frac{1}{w}, & 0 < w \leq K^* \\ \lim_{n \to \infty} K_1 K_2 \sum_{j=1}^{p_1(n+1)} P_{1j}^{**} \left( \log \left( \frac{w}{K^*} \right) \right) \left( \frac{w}{K^*} \right)^{-s_{1j}} \frac{1}{w}, & w \geq K^* \end{cases}$$

(3.16)

and

$$F_W(w) = \begin{cases} \lim_{n \to \infty} K_1 K_2 \sum_{j=1}^{p_2(n+1)} P_{2j}^{***} \left( \log \left( \frac{w}{K^*} \right) \right) \left( \frac{w}{K^*} \right)^{s_{2j}} , & 0 < w \leq K^* \\ \lim_{n \to \infty} K_1 K_2 \sum_{j=1}^{p_1(n+1)} P_{1j}^{***} \left( \log \left( \frac{w}{K^*} \right) \right) \left( \frac{w}{K^*} \right)^{-s_{1j}}, & w \geq K^* \end{cases}$$

(3.17)

where now $K_1, K_2, P_{1j}^{**}()$ and $P_{2j}^{**}()$ are defined as in Theorem 2.2 and Corollary 2.2, with $p_1$ replaced by $p_1(n+1)$, $p_2$ replaced by $p_2(n+1)$, $\lambda_j$ replaced by $\tilde{s}_{1j}$, $\nu_l$ replaced by $\tilde{s}_{2j}$, $r_{1j}$ replaced by $\tau_j$, and $r_{2l}$ replaced by $\eta_l^*$.

4 Applications

4.1. Testing the (mutual) independence of a set of variables. Let us suppose that the random vector $X$ ($p \times 1$) has a $p$-multivariate Normal distribution

$$X \sim N_p(\mu, \Sigma),$$

and we want to test the hypothesis that the $p$ random variables in $X$ are mutually independent. This hypothesis may be written as

$$\Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_p^2).$$

(4.1)

If we have a sample of size $N$ ($> p$), the power $2/N$ of the likelihood ratio test statistic is (Anderson, 1984, Chap. 9, sec. 9.2; Coelho, 1992, 1998, 2004)

$$\Lambda = \frac{|V|}{\prod_{j=1}^p V_{jj}},$$

(4.2)
where $V$ is either the sample variance-covariance matrix or the maximum likelihood estimator of $\Sigma$, and $V_{jj}$ is the $j$-th diagonal element of $V$. With $V^*$ the corresponding matrix for the last $p-j$ variables (a lower right submatrix of $V$), we have (see Anderson, 1984, Chap. 9, Thm. 9.3.2; Coelho, 1992, Chap. 4)

$$\Lambda = \prod_{j=1}^{p-1} \Lambda_{j(j+1,\ldots,p)}$$  \hspace{1cm} (4.3)

with

$$\Lambda_{j(j+1,\ldots,p)} = \frac{|V^*_j|}{V_{jj} |V^*_{j+1}|}$$

as the statistic for testing the independence between the $j$-th variable and those with indices $j + 1, \ldots, p$. Moreover, when $H_0$ holds, $\Lambda_{j(j+1,\ldots,p)}$ will have a beta distribution with parameters $(n^* - p + j)/2$ and $(p - j)/2$, where $n^* = N - 1$ (Anderson, 1984, Chap. 9, sec. 9.3).

When $H_0$ holds,

$$\frac{\Lambda_{j(j+1,\ldots,p)}}{1 - \Lambda_{j(j+1,\ldots,p)}} \sim GGR\left(1, \frac{n^* - p + j}{2}, \frac{p - j}{2}, \frac{1}{1} \right),$$

and

$$\frac{1 - \Lambda_{j(j+1,\ldots,p)}}{\Lambda_{j(j+1,\ldots,p)}} \sim GGR\left(1, \frac{p - j}{2}, \frac{n^* - p + j}{2}, \frac{1}{1} \right).$$

We may thus replace $\Lambda$ as test statistic by

$$G^* = \prod_{j=1}^{p-1} \frac{\Lambda_{j(j+1,\ldots,p)}}{1 - \Lambda_{j(j+1,\ldots,p)}} \quad \text{or} \quad G^{**} = \prod_{j=1}^{p-1} \frac{1 - \Lambda_{j(j+1,\ldots,p)}}{\Lambda_{j(j+1,\ldots,p)}},$$

both of which, under $H_0$ in (4.1), have the distribution of the product of independent gamma-ratio random variables with parameters $k_j = \beta_j = 1$ and $r_{1j} = (n^* - p + j)/2$, $r_{2j} = (p - j)/2$, and $r_{1j} = (p - j)/2$, $r_{2j} = (n^* - p + j)/2$, $(j = 1, \ldots, p - 1)$, respectively.

4.2. Random effects models with balanced cross-nesting. Let us assume that there are $L$ groups with $u_1, \ldots, u_L$ factors. The first factors in the groups will have $a_{\ell}(1)$ levels and, if $u_\ell > 1$, the $h$-th factor in the $\ell$-th group will have $a_{\ell}(h)$ levels nested inside each level of the preceding factor ($h = 2, \ldots, u_\ell$). There will be $c_\ell(h) = \prod_{k=1}^{h} a_{\ell}(k)$ level combinations of the first $h$ factors in the $\ell$-th group, each nesting $b_{\ell}(h) = c_\ell(u_\ell)/c_\ell(h)$ level
combinations of the remaining factors \((h = 1, \ldots, u; \ell = 1, \ldots, L)\). We also take \(c_i(1) = c_i(0) = 1, \ell = 1, \ldots, L\).

Since when a factor is nested in another they do not interact, the variance components will correspond to sets of factors belonging to different groups. These sets of factors are indicated by the vectors \(h\) with components \(h_{\ell} = 0, \ldots, u\), \(\ell = 1, \ldots, L\). If \(h_{\ell} = 0\), no factor is taken from the \(\ell\)-th group, otherwise \(h_{\ell}\) will be the index of the factor taken in that group. Let \(\Gamma\) represent the set of vectors \(h\) and \(\sigma^2(h)\) the variance component for the set of factors corresponding to \(h\).

If \(r\) observations are taken for each treatment, the sum \(S(h)\) of squares of the effects or interactions indicated by \(h\) will be (see Fonseca et al., 2003) the product of

\[
\gamma(h) = \sigma^2 + \sum_{k: k \leq h} b(k) \sigma^2(k), \quad h \in \Gamma,
\]

where \(b(k) = r \prod_{\ell=1}^k b_{\ell}(k_{\ell})\), and a central chi-square with \(g(h) = \prod_{\ell=1}^L \{c_{\ell}(k_{\ell}) - c_{\ell}(k_{\ell}-1)\}\) degrees of freedom.

When only one component, say \(h_{\ell}\), of \(h\), is less than the corresponding bound \(u_{\ell}\), let \(h^+\) be the vector obtained increasing that component by 1. Then

\[
\gamma(h) = \gamma(h^+) + b(h) \sigma^2(h),
\]

and we may use

\[
F(h) = \frac{g(h^+) S(h)}{g(h) S(h^+)}
\]

to test

\[
H_0(h) : \sigma^2(h) = 0.
\]

We point out that \(F(h)\) will be the product of \(1 + \sigma^2(h)/\gamma(h^+)\) and a variable having a central \(F\) distribution with \(g(h)\) and \(g(h^+)\) degrees of freedom.

The \(S(h), h \in \Gamma\), are independent (Fonseca et al., 2003). Thus if the pairs \((h_j, h^+_j), j = 1, \ldots, d\), are formed by distinct vectors, we may use

\[
\mathcal{F} = \prod_{j=1}^d F(h_j),
\]

to test

\[
H_0 : \sigma^2(h_1) = \cdots = \sigma^2(h_d) = 0.
\]
This test statistic, when $H_0$ holds, will be the product of $d$ independent variables with central $F$ distributions.

4.3. Combining independent $F$ tests in meta-analysis. An emergent problem in meta-analysis is how to combine independent $F$ tests (Khuri et al., 1998). One possible way to do so is by considering an overall test statistic, for the whole meta-analysis being considered, which will be the product of the elementary $F$ statistics for each single analysis. By using this procedure, we may also obtain the distribution of the overall statistic under the alternative hypothesis, which will be the distribution of the product of non-central $F$ distributions. This distribution may be derived from the distributions obtained in this manuscript and will be the subject of a future work.

4.4. A test for equality of two generalized variances. Let us suppose that $X_1 \sim N_p(\mu_1, \Sigma_1)$ and $X_2 \sim N_p(\mu_2, \Sigma_2)$, and that we want to test the hypothesis

$$H_0 : |\Sigma_1| = |\Sigma_2|,$$

(4.4)

based on two independent samples of sizes $n_1$ and $n_2$ of $X_1$ and $X_2$ respectively.

Taking the work of Cao (2006, Chap. 4, sec. 4.1, 4.2) as a basis, we may use the statistic

$$T = \frac{|S_1|}{|S_2|},$$

(4.5)

where $S_1$ and $S_2$ are either the sample variance-covariance matrices or the maximum likelihood estimators of $\Sigma_1$ and $\Sigma_2$, based on the two independent random samples of sizes $n_1$ and $n_2$ respectively of $X_1$ and $X_2$. Alternatively, $S_1$ and $S_2$ may yet be taken as the matrices of squares and cross-products of the deviations from the sample means. A test of size $\alpha$ for $H_0$ in (4.4) may then be constructed by rejecting $H_0$ in (4.4) if, for the statistic $T$ in (4.5),

$$T_{\text{calc}} > T_{1-\alpha/2} \quad \text{or} \quad T_{\text{calc}} < T_{\alpha/2},$$

where $T_{1-\alpha/2}$ and $T_{\alpha/2}$ represent respectively the $1-\alpha/2$ and the $\alpha/2$ quantiles of the statistic $T$ in (4.5).

Taking $S_1$ and $S_2$ as the maximum likelihood estimators of $\Sigma_1$ and $\Sigma_2$, we will have

$$S_1 \sim W_p\left(n_1 - 1, \frac{1}{n_1} \Sigma_1\right) \quad \text{and} \quad S_2 \sim W_p\left(n_2 - 1, \frac{1}{n_2} \Sigma_2\right),$$
with

\[ |S_1| \sim \left| \frac{1}{n_1} \Sigma_1 \right| \prod_{j=1}^{p} W_j \quad \text{and} \quad |S_2| \sim \left| \frac{1}{n_2} \Sigma_2 \right| \prod_{j=1}^{p} Y_j , \]

where

\[ W_j \sim \chi^2_{n_1-j} \quad \text{and} \quad Y_j \sim \chi^2_{n_2-j} \quad (j = 1, \ldots, p) \]

are all independent. Thus, under \( H_0 \) in (4.4), the distribution of the statistic \( T \) in (4.5) is the same as the distribution of

\[ \left( \frac{1}{n_1} \right)^p \prod_{j=1}^{p} W_j \]
\[ \left( \frac{1}{n_2} \right)^p \prod_{j=1}^{p} Y_j \]
\[ = \prod_{j=1}^{p} Z_j , \]

where

\[ W_j^* \sim \Gamma \left( \frac{n_1-j}{2}, \frac{n_1}{2} \right) \quad \text{and} \quad Y_j^* \sim \Gamma \left( \frac{n_2-j}{2}, \frac{n_2}{2} \right) \quad (j = 1, \ldots, p) , \]

i.e., under \( H_0 \) in (4.4), the exact distribution of the statistic \( T \) in (4.5) is the product of \( p \) r.v.’s

\[ Z_j \sim GGR \left( \frac{n_1}{n_2}, \frac{n_1-j}{2}, \frac{n_2-j}{2}, 1 \right) \quad (j = 1, \ldots, p) . \]

5 A Few Simulations and Numerical Studies for the Distributions Obtained

We will consider in this section a few numerical studies, together with the results from simulations in order to show the correct behaviour of the distributions obtained and the correctness of the corresponding expressions. To show the agreement between the simulated data and the distributions obtained, simultaneous plots of the histograms of relative frequencies (simple and cumulative) and the exact p.d.f.’s and c.d.f.’s are shown.

The simulations were carried out by generating five vectors, each with \( 10^6 \) randomly generated values of the distribution under study. These vectors were then ordered and their average used to compute the simulated quantiles and plot the relative frequency histograms.

We should note that although for any value of \( n \in \mathbb{N} \), the expressions for the p.d.f.’s and the c.d.f.’s in (3.3), (3.4), (3.16) and (3.17) yield proper p.d.f.’s and c.d.f.’s, we need to consider sufficiently high values of \( n \in \mathbb{N} \) in
order that the p.d.f.’s and the c.d.f.’s in (3.3) and (3.4) or (3.16) and (3.17)
converge to the exact distribution of the desired product of GGR r.v.’s.

All simulations and computations were done with the software Mathematica, from Wolfram Research, version 5.2.

5.1. The test of independence in a set of $p$ variables. The case presented in this subsection corresponds to the example of application discussed in subsection 4.1.

For $p = 5$ and $N = 10$, according to what was exposed in subsection 4.1, if we want to test the independence of the $p = 5$ variables, we may, in this case, use the test statistic

$$G^* = G_1G_2G_3G_4,$$

(5.1)

where

$$G_1 \sim GGR(1, 5/2, 2, 1), \quad G_2 \sim GGR(1, 6/2, 3/2, 1),$$

$$G_3 \sim GGR(1, 7/2, 1, 1), \quad G_4 \sim GGR(1, 8/2, 1/2, 1),$$

(5.2)

so that, using the result that states that if

$$X_1 \sim \chi^2_m \quad \text{and} \quad X_2 \sim \chi^2_{m-1}$$

are two independent r.v.’s, then

$$X_1X_2 \sim \frac{(\chi^2_{2m-2})^2}{4},$$

we may also write

$$G^* = F_1^*F_2^*,$$

(5.3)

where

$$F_1^* \sim GGR(1, 10/2, 2/2, 1/2) \quad \text{and} \quad F_2^* \sim GGR(1, 14/2, 6/2, 1/2).$$

(5.4)

In order to obtain the exact distribution of the statistic $G^*$, we may then either use (5.1) and (5.2) or (5.3) and (5.4).

In Figure 5.1, we may observe the overlayed plots of the histograms of relative frequencies generated from the simulated data, obtained from (5.2), and the exact p.d.f. and c.d.f. for the statistic $G^*$ in (5.1), for $n = 100$, where we may observe the very long tail that the distribution has in this case.
Figure 5.1. Overlayed plots of the histograms of relative frequencies generated from simulated data and the exact p.d.f. and c.d.f. for \( n = 100 \) for the statistic in (5.1): (a) histogram of relative frequencies and exact p.d.f.; (b) histogram of cumulative relative frequencies and exact c.d.f.

In Table 5.1, we have the simulated 0.90, 0.95 and 0.99 quantiles for the statistic \( G^* \) in (5.1), together with the quantiles computed from different truncations of the exact distributions corresponding to (5.1)-(5.2) and (5.3)-(5.4).

<table>
<thead>
<tr>
<th>( q )</th>
<th>simulat.</th>
<th>From the exact distribution in (5.1), (5.2) with ( n = )</th>
<th>From the exact distribution in (5.3), (5.4) with ( n = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.90</td>
<td>23748</td>
<td>19887</td>
<td>21729</td>
</tr>
<tr>
<td>0.95</td>
<td>107386</td>
<td>89567</td>
<td>98030</td>
</tr>
<tr>
<td>0.99</td>
<td>2935411</td>
<td>2469174</td>
<td>2706682</td>
</tr>
</tbody>
</table>

We may note that although, as expected, both exact distributions have quantiles that converge to the simulated ones as \( n \to \infty \), the performance of the truncations with \( n = 50 \) and \( n = 100 \) is rather poor, with the truncations of the exact distribution based on (5.3) and (5.4) exhibiting a much worse behaviour. The exact distribution of \( G^* \) based on (5.3) and (5.4) is indeed simpler than the one based on (5.1) and (5.2) since it only involves two r.v.'s instead of four, and as such it is also much easier and quicker to compute. But, on the other hand, it seems to need twice the number of terms to attain a similar accuracy as the one exhibited by the distribution based on (5.1) and (5.2). Although this worse performance of the exact distribution based on the product of two GGR r.v.'s may come out as a surprise, in the end
it seems that the result points towards somewhat similar number of terms being necessary while one uses either one of the two exact distributions.

Similar conclusions may be drawn from Table 5.2, where we may analyse the values of the relative errors for the quantiles obtained from the truncations of the exact distributions used in Table 5.1. Now that we do not have the exact quantiles, the relative errors are computed relative to the simulated quantiles as \((\text{approximate} - \text{simulated}) / \text{simulated}\). Once again, it is interesting to note the almost linear relation between the values of the relative error and the value of \(n\) used for both exact distributions.

| Quant. | Value of \(n\) in the exact distribution in (5.1) and (5.2) | Value of \(n\) in the exact distribution in (5.3) and (5.4) |
|--------|------------------------------------------------------------|
|        | 50 | 100 | 200 | 50 | 100 | 200 |
| 0.90   | -0.16257 | -0.08500 | -0.04187 | -0.29396 | -0.16402 | -0.08541 |
| 0.95   | -0.16594 | -0.08713 | -0.04325 | -0.29908 | -0.16741 | -0.08755 |
| 0.99   | -0.15883 | -0.07792 | -0.03281 | -0.29518 | -0.16034 | -0.07834 |

The alternative approach is based on using the statistic

\[
G^{**} = G^*_1 G^*_2 G^*_3 G^*_4 ,
\]

where

\[
\begin{align*}
G^*_1 & \sim \text{GGR}(1, 2, 5/2, 1) , & G^*_2 & \sim \text{GGR}(1, 3/2, 6/2, 1) , \\
G^*_3 & \sim \text{GGR}(1, 1, 7/2, 1) , & G^*_4 & \sim \text{GGR}(1, 1/2, 8/2, 1) 
\end{align*}
\]

or,

\[
G^{**} = F^{**}_1 F^{**}_2 ,
\]

where

\[
\begin{align*}
F^{**}_1 & \sim \text{GGR}(1, 2/2, 10/2, 1/2) & & \text{and} & & F^{**}_2 & \sim \text{GGR}(1, 6/2, 14/2, 1/2) . 
\end{align*}
\]

In Figure 5.2, we have the overlayed plots of the histograms of relative frequencies generated from the simulated data, obtained from (5.6), and the exact p.d.f. and c.d.f. for the statistic \(G^{**}\) in (5.5), for \(n = 100\).
In Table 5.3, we have the simulated 0.90, 0.95 and 0.99 quantiles and corresponding quantiles from different truncations of the exact distributions derived from (5.6) and (5.8), and in Table 5.4, the relative errors of these quantiles computed taking as reference the values of the simulated quantiles.

**Table 5.3. Quantiles \( q \) (simulated and from different truncations of the exact distribution) for the statistic \( G^{**} \) in (5.5) and (5.7)**

<table>
<thead>
<tr>
<th>( q )</th>
<th>simulat. 50</th>
<th>100</th>
<th>200</th>
<th>From the exact distribution in (5.5), (5.6) with ( n = )</th>
<th>From the exact distribution in (5.7), (5.8) with ( n = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.90</td>
<td>0.09090</td>
<td>0.10169</td>
<td>0.09619</td>
<td>0.09345</td>
<td>0.11315</td>
</tr>
<tr>
<td>0.95</td>
<td>0.21654</td>
<td>0.23908</td>
<td>0.22307</td>
<td>0.21714</td>
<td>0.26257</td>
</tr>
<tr>
<td>0.99</td>
<td>1.03876</td>
<td>1.08365</td>
<td>1.08246</td>
<td>1.06167</td>
<td>1.20176</td>
</tr>
</tbody>
</table>

**Table 5.4. Relative errors for quantiles obtained from different truncations of the exact distributions of the statistic \( G^{**} \) in (5.5) and (5.7)**

<table>
<thead>
<tr>
<th>Quant.</th>
<th>Value of ( n ) in the exact distribution in (5.5), (5.6)</th>
<th>Value of ( n ) in the exact distribution in (5.7), (5.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>0.90</td>
<td>0.11876</td>
<td>0.05827</td>
</tr>
<tr>
<td>0.95</td>
<td>0.10409</td>
<td>0.03015</td>
</tr>
<tr>
<td>0.99</td>
<td>0.04322</td>
<td>0.04207</td>
</tr>
</tbody>
</table>

We may observe in Table 5.4 that the relative errors are, for a given value of \( n \), generally smaller when using the statistic \( G^{**} \), with a noticeable small value of the relative error for the 0.95 quantiles for \( n = 200 \).
Simultaneous tests for variance components in random effects models with balanced cross-nesting. Let us assume we have three groups of factors, the first and the third of which have a single factor and the second with five factors sequentially nested. Let us further suppose that all factors have two levels and random effects. When we want to test the simultaneous nullity of the variance components associated with the factor in the first group and with its interaction with the second and fourth factors in the second group, we have to use the statistic

\[ F^* = F_1 F_2 F_3, \]

where

\[ F_1 = \frac{1}{1} \frac{S\left(\begin{array}{c} 1 \\ 0 \end{array}\right)}{S\left(\begin{array}{c} 1 \\ 1 \end{array}\right)} \sim F_{1,1} \sim GGR\left(1, \frac{1}{2}, \frac{1}{2}, 1\right) \]

\[ F_2 = \frac{4}{2} \frac{S\left(\begin{array}{c} 2 \\ 1 \end{array}\right)}{S\left(\begin{array}{c} 3 \\ 1 \end{array}\right)} \sim F_{2,4} \sim GGR\left(2, \frac{4}{3}, \frac{4}{3}, 1\right) \]

\[ F_3 = \frac{16}{8} \frac{S\left(\begin{array}{c} 4 \\ 1 \end{array}\right)}{S\left(\begin{array}{c} 5 \\ 1 \end{array}\right)} \sim F_{8,16} \sim GGR\left(\frac{8}{16}, \frac{8}{5}, \frac{16}{2}, 1\right). \]

Figure 5.3 shows the good graphical agreement between the relative frequency histograms generated from the simulated data and the exact p.d.f.
Table 5.5. Quantiles (simulated and from different truncations of the exact distribution) for the statistic $F^*$ in (5.9)

<table>
<thead>
<tr>
<th>Quant.</th>
<th>Simulat.</th>
<th>From the exact distribution with $n =$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>0.90</td>
<td>48.112</td>
<td>51.448</td>
</tr>
<tr>
<td>0.95</td>
<td>203.590</td>
<td>216.070</td>
</tr>
<tr>
<td>0.99</td>
<td>5169.479</td>
<td>5505.706</td>
</tr>
</tbody>
</table>

In Table 5.6, we report the values of the relative errors for the quantiles obtained from the truncations of the exact distribution used in Table 5.5. Once again, we may note the almost linear decay of the values of the relative errors for increasing values of $n$. Only for the quantile 0.95, this rate seems to be slightly higher, what may indeed be due to the fact that the value of the simulated quantile, relative to which the relative errors are computed, may be a bit higher than what it should be.

Table 5.6. Relative error for quantiles obtained from different truncations of the exact distribution of the statistic $F^*$ in (5.9)

<table>
<thead>
<tr>
<th>Quant.</th>
<th>Value of $n$ in the exact distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50</td>
</tr>
<tr>
<td>0.90</td>
<td>0.06934</td>
</tr>
<tr>
<td>0.95</td>
<td>0.06130</td>
</tr>
<tr>
<td>0.99</td>
<td>0.06504</td>
</tr>
</tbody>
</table>

### 6 Relations with Other Distributions

We should emphasize that the results obtained may be easily and directly generalized to the case where we are interested in the distribution of the r.v.

$$Z = \prod_{j=1}^{n} \gamma_j Y_j^{\alpha_j},$$

where $\gamma_j \in \mathbb{R}^+$, $\alpha_j \in \mathbb{R}\{0\}$ and $Y_j$ are independent r.v.'s with GGR distributions, since it is straightforward to show that if

$$Y_j \sim GGR(k_j, r_{1j}, r_{2j}, \beta_j),$$

(6.1)
then

\[ \gamma_j Y_j^{\alpha_j} \sim GGR \left( \frac{k_j}{\gamma_j^{-\beta_j\alpha_j}}, r_{1j}, r_{2j}, \beta_j \right). \]

The distributions obtained in section 3 may also be seen as the distributions of the product of independent beta prime r.v.’s. Recall that if the r.v. \( X \) has a standard beta distribution, the distribution of either \( (1 - X)/X \) or \( X/(1 - X) \) is then usually called a standard beta prime or beta second kind distribution. However, we should note that the distributions of either \( (1 - X)/X \) or \( X/(1 - X) \) are then only particular GGR distributions. Actually, it is easy to show that if \( Y_j \sim GGR(k_j, r_{1j}, r_{2j}, \beta_j) \) with \( k_j = 1 \) and \( \beta_j = 1 \), then the r.v.’s \( 1/(1 + Y_j) \) and \( Y_j/(1 + Y_j) \) have standard beta distributions with parameters \( r_{2j} \) and \( r_{1j} \) or \( r_{1j} \) and \( r_{2j} \), respectively, while for general \( k_j > 0 \) and general \( \beta_j \in \mathbb{R} \setminus \{0\} \) the r.v. \( X_j = Y_j/(1 + Y_j) \) has what we call a generalized beta distribution with p.d.f.

\[
f_{X_j}(x) = \frac{|\beta_j| k_j^{\frac{r_{1j}}{r_{2j}}} (1 + k_j \frac{x}{1-x})^{-r_{1j} - r_{2j}}}{B(r_{1j}, r_{2j})} (1 - x)^{-\beta_j r_{1j} - 1} x^{\beta_j r_{1j} - 1},
\]

which clearly reduces to the standard beta p.d.f. for \( k_j = 1 \) and \( \beta_j = 1 \), while the r.v. \( 1 - X_j = 1/(1 + Y_j) \) has of course a similar p.d.f. with \( r_{1j} \) and \( r_{2j} \) swapped. For this reason, the distribution of \( Y_j \) is also called, for \( k_j = 1 \) and \( \beta_j = 1 \), a beta prime distribution and for general \( k_j \) and \( \beta_j \) a generalized beta prime distribution. Thus, for the distribution in (6.1), we may also say that \( Y_j \) has a generalized beta prime distribution and thus, the distributions obtained in section 3 are also the distributions of the product of independent generalized beta prime r.v.’s.

Clearly, if in (6.1) we have \( r_{1j} = m_j/2, r_{2j} = n_j/2 \), with \( m_j, n_j \in \mathbb{N} \), \( k = m_j/n_j \) and \( \beta_j = 1 \), the r.v.’s \( Y_j \) will have \( F \) distributions with \( m_j \) and \( n_j \) degrees of freedom, and thus the results in section 3 will then give the distribution of the product of independent \( F \) distributed r.v.’s.

Also, if in (6.1) we have \( r_{1j} = 1/2, r_{2j} = n_j/2 \), \( k_j = 1/n_j \) and \( \beta_j = 2 \), with \( n_j \in \mathbb{N} \), we have \( Y_j \) with the so-called folded \( T \) distribution, which is the distribution of a r.v. \( Y_j = |T_j| \), where \( T_j \) has a student’s \( t \) distribution with \( n_j \) degrees of freedom. If instead we take \( r_{2j} = 1/2 \) and \( k_j = 1 \), we will have \( Y_j \) with the so-called folded Cauchy distribution, that is the distribution
of the absolute value of a r.v. with a standard Cauchy distribution, or a student’s \( t \) distribution with only 1 degree of freedom. In both cases, once again, the results in section 3 may be readily applied.

Since, contrary to what is commonly done, we considered the power parameters as real and not necessarily positive (only non-zero), by allowing them to be negative, the results obtained may be directly extended to the distribution of the ratio of two independent GGR random variables or the distribution of the ratio of two independent products of independent GGR random variables. In order to obtain the distribution of the ratio of two independent GGR random variables, one simply has to consider \( m = 2 \) in (3.2), taking then for the random variable in the denominator the negative of its power parameter. The distribution for the ratio of two independent products of independent GGR random variables may then be obtained by considering the distribution of the product of the whole set of random variables, taking the power parameters for the random variables in the denominator with a minus sign.

As a by-product, in subsections 2.3 and 2.4, we also obtain closed form representations, not involving any infinite series or unsolved integrals, for the distribution of the difference of two independent sums of a finite number of exponential random variables with all different rate parameters or for the distribution of the difference of two independent sums of gamma random variables with all different rate parameters and integer shape parameters, under the form of particular mixtures of either exponential or gamma distributions, according to the case. Also, if we consider the exponential of a gamma random variable with integer shape parameter as a generalized Pareto distribution, then the distribution of the random variable \( W \) in subsection 2.3 is the distribution of the ratio of two independent products of Pareto distributions, while the distribution of the same random variable in subsection 2.4 is the distribution of the ratio of two independent products of generalized Pareto distributions, expressed as a particular mixture of Pareto and inverted Pareto distributions.

7 Conclusions and Final Remarks

Given the form of the exact distributions obtained for the product of GGR r.v.’s and the problems related to the need for the evaluation of a large number of terms in the series in order to obtain the desired precision, the development of near-exact distributions seems to be a much desirable
goal. Such near-exact distributions are indeed not too hard to obtain, being not treated in this paper due to length limitations. The simple truncation of the series obtained gives indeed unsatisfactory results, namely in terms of the c.d.f. and the quantiles, owing to the fact that in this case the weights do not add up to 1, preventing the c.d.f. from reaching this value of 1. Much better results may be obtained if we consider near-exact distributions, based on the concept of keeping the major part of the exact characteristic function unchanged and approaching the remaining part by an asymptotic result (Coelho, 2003, 2004). Given the form obtained for the exact distributions, such an approach would lead us to consider, for example, the truncation of the infinite series obtained, coupled with one or two more terms, which would at the same time enable the weights to add up to 1 and the first few moments to match the exact ones.

Yet, the extension of the results presented in this paper to non-central distributions is almost straightforward, as it only requires consideration of the mixtures with Poisson weights that would account for the non-centrality. Once again, this topic is not addressed in this paper as the paper is already quite long.

Appendix

The facts that in (2.2)-(2.3), and thus also in (2.4)-(2.7) and (2.8)-(2.19), we may take \( p \to \infty \), and also in (2.26)-(2.31) we may take \( p_1 \to \infty \) and \( p_2 \to \infty \), with no need to worry about the convergence of the series involved, as well as about the convergence of the infinite products in (3.11) and (3.12), are justified as follows. If

\[
X_i \sim \Gamma(r_j, \lambda_j), \quad i = 1, 2, \ldots,
\]

then the characteristic function of

\[
Z = \sum_{j=1}^{\infty} X_j
\]

is

\[
\prod_{j=1}^{\infty} \lambda_j^{r_j} (\lambda_j - it)^{-r_j} = \lim_{p \to \infty} \prod_{j=1}^{p} \lambda_j^{r_j} (\lambda_j - it)^{-r_j},
\]

with

\[
\left| \prod_{j=1}^{\infty} \lambda_j^{r_j} (\lambda_j - it)^{-r_j} \right| = \left| \lim_{p \to \infty} \prod_{j=1}^{p} \lambda_j^{r_j} (\lambda_j - it)^{-r_j} \right| \leq 1,
\]
not only because \( \prod_{j=1}^{\infty} \lambda_j^{r_j} (\lambda_j - it)^{-r_j} \) is a characteristic function but also because
\[
\left| \prod_{j=1}^{\infty} \lambda_j^{r_j} (\lambda_j - it)^{-r_j} \right| \leq \prod_{j=1}^{p_1} \lambda_j^{r_j} (\lambda_j - it)^{-r_j} \leq 1 \quad (A.2)
\]
for any \( p_1 < \infty \), and because for any \( j \), we clearly have
\[
\left| \lambda_j^{r_j} (\lambda_j - it)^{-r_j} \right| \leq 1.
\]

But then, let
\[
f_{Z^*}(z; \mathcal{R}, \Delta, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itz} \prod_{j=1}^{p} \lambda_j^{r_j} (\lambda_j - it)^{-r_j} \, dt
\]
represent the p.d.f. of the random variable
\[
Z^* = \sum_{j=1}^{p} X_j,
\]
and let
\[
f_{Z}(z; \infty) = \lim_{p \to \infty} f_{Z^*}(z; \mathcal{R}, \Delta; p)
\]
represent the p.d.f. of the random variable \( Z \) in (A.1). The fact that \( f_{Z}(z; \infty) \) is a legitimate p.d.f. follows directly from (A.2) and the dominated convergence theorem, since thus
\[
f_{Z}(z; \infty) = \lim_{p \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itz} \prod_{j=1}^{p} \lambda_j^{r_j} (\lambda_j - it)^{-r_j} \, dt
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itz} \prod_{j=1}^{\infty} \lambda_j^{r_j} (\lambda_j - it)^{-r_j} \, dt
\]
(where the integral is uniformly convergent given the result in (A.2)).

If all the shape parameters \( r_j \) (\( j = 1, 2, \ldots \)) are integer, the p.d.f. \( f_{Z}(z; \infty) \) is the p.d.f. of a GIG distribution with infinite depth, which shows that in expressions (2.14) through (2.20) we may take \( p \to \infty \).

Similar arguments could then be used for the distribution of the difference of the two sums of infinitely many exponential random variables considered after expression (2.23), or the distribution of the difference of the two sums of
infinitely many gamma random variables all with integer shape parameters, considered at the end of Section 2, both of these distributions being the distribution of the difference of two GIG distributions with infinite depth.

Acknowledgements. The authors wish to thank the remarks and suggestions made by the referees and editors which improved the presentation of the paper. The reviewers suggested the need for Sections 4 and 5, which help in understanding the scope, usefulness and functioning of the distributions presented in the paper. The authors also want to thank Professor Thomas Mathew of the University of Maryland, who, as the adviser, brought to our attention and made available to us the material in Cao (2006).

References


Carlos A. Coelho and João T. Mexia
Department of Mathematics
Faculty of Science and Technology
The New University of Lisbon
2829-516 caparica, Portugal
E-mail: cmac@fct.unl.pt
jtm@fct.unl.pt