Maximal Rank - Minimum Aberration Regular Two-Level Split-Plot Fractional Factorial Designs

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Abstract

Regular two-level fractional factorial designs are often used in industrial experiments as screening experiments. When some factors have levels which are hard or expensive to change, restrictions are often placed on the order in which runs can be performed, resulting in a split-plot factorial design. In these cases, the hard or expensive to change factors are applied to whole plots, whereas the easier or less expensive to change factors are applied to the subplots within the split plot designs. For such experimental situations the minimum aberration criterion has been used by a number of authors to find optimal regular fractional factorial split plot designs. In this paper, we suggest an alternative criterion called the maximal rank-minimum aberration criterion for selecting optimal fractional factorial split plot designs and study how this alternative criterion performs in terms of the optimal designs it selects, and how it compares to the minimum aberration criterion.


Keywords and phrases. Minimum aberration, subplot, whole plot, word length pattern, estimable effect.

1 Introduction

Many industrial experiments are performed using a fractional factorial (FF) design. These designs are often used in the initial stages of an experimental study to determine those experimental factors that have a significant effect on the response being studied. The most common screening experiments are those in which $m$ factors occur at two levels.

When an FF experiment is run, it is required that experimental runs be completely randomized. Frequently, it is impractical to perform the experimental runs in a completely random order. Reasons for this might be
because some factors are hard or extremely expensive to change. This results in a split plot structure, and the resulting experiments are fractional factorial split-plot (FFSP) designs. The problem to be considered in this paper is the construction of FFSP designs using a modified version of the widely used minimum aberration (MA) criterion.

In section 2, preliminary notation and definitions are given. In section 3, a modified version of the MA criterion, called the maximal rank-minimum aberration (MR-MA) criterion is suggested for use in the construction of FFSP designs. However, it often happens with both the MA and the MR-MA criterion that there exist several nonisomorphic MA optimal or MR-MA optimal FFSP designs for a given set of design parameters. Hence, a secondary criterion is also suggested in Section 3 for distinguishing between nonisomorphic MA optimal or MR-MA optimal designs. This secondary criterion is a modification of the one given in Mukerjee and Fang (2002). Examples are given in Section 4 of usage of the various criteria defined in Section 3. Finally, in Section 5, a table of optimal FFSP designs is given for 16 run designs having 10 or fewer experimental factors.

2 Basic Notation and Definitions

We shall henceforth use \( d \) to represent an arbitrary factorial design having \( m \) factors occurring at two levels, a high level denoted by +1 and a low level denoted by -1, respectively. We shall interchangeably represent \( d \) by an \( n \times m \) matrix

\[
X_d = (x_{d1}, \cdots, x_{dm}) = (x_{dij})
\]

whose \( i^{th} \) row corresponds to run \( i \) of the design, and whose \( j^{th} \) column corresponds to experimental factor \( j \), and where

\[
x_{dij} = \begin{cases} 
1 & \text{if factor } j \text{ is at a high level in run } i, \\
-1 & \text{if factor } j \text{ is at a low level in run } i.
\end{cases}
\]

In this paper, we shall only be considering what are typically referred to as regular \( 2^{m-k} \) FF designs. A \( 2^{m-k} \) regular FF design has \( m \) factors and \( 2^{m-k} \) runs. In a given \( 2^{m-k} \) design \( d \), there are \( m-k \) factors called basic factors which we denote by \( B_1, \cdots, B_{m-k} \) and are such that design \( d \) is a complete factorial in the basic factors. The other \( k \) factors are called added factors and are denoted by \( A_1, \cdots, A_k \) and are obtained by associating each added factor with an interaction in the basic factors. In each run, the level of an added
factor is determined from those of the basic factors through the associated basic factor interaction. By combining each added factor label with those of its associated basic factor interaction, we obtain what are called the defining words or generating words of the design. A word is simply a set of letters corresponding to factor labels. The defining words are used to generate the defining relations group of the design. The defining relations group of $d$ is an algebraic group obtained by taking all possible products of the $k$ defining words according to the rule that if a letter appears an odd number of times in the product, it is kept, whereas if it occurs an even number of times, it is deleted. The identity element of the group is denoted by $I$, and the number of letters in any given word of the group is called its word length. There are $2^k$ words in the defining relations group of $d$, and this group is used to determine the aliasing relationships among the main effects and interactions of the design. In particular, the $2^m$ main effects and interactions of $d$ are divided into $2^{m-k} - 1$ mutually exclusive alias sets, where the alias set of a given interaction is obtained by multiplying all elements of the defining relations group by that interaction.

For given values of $m$ and $k$, there are typically a large number of $2^{m-k}$ regular FF designs that can be constructed using different defining relations. To aid in the construction of good designs, the criteria of resolution and minimum aberration were introduced. The resolution of a given design is given by the length of the shortest word in its defining relations group (Box and Hunter, 1951). However, there are often a number of $2^{m-k}$ designs having the same resolution. Fries and Hunter (1986) proposed a refinement of the resolution criterion, which they called MA. For a $2^{m-k}$ design $d$, let $w_i(d)$ be the number of words of length $i$ in the defining relations group. Then $W(d) = (w_3(d), \cdots, w_m(d))$ is called the word length pattern (WLP) vector of the design.

**Definition 2.1.** Let $d_1$ and $d_2$ be two $2^{m-k}$ regular FF designs, and let $r$ be the smallest value of $i$ such that $w_i(d_1) \neq w_i(d_2)$. Then, $d_1$ is said to have less aberration than $d_2$ if $w_r(d_1) < w_r(d_2)$. If no such $r$ exists, we say that $d_1$ and $d_2$ have equal aberration. If no design has less aberration than $d_1$, then $d_1$ is called an MA design.

The construction of regular FF MA designs has been considered by a number of authors, e.g., see Chen, Sun and Wu (1993) for a catalogue containing many MA designs for various values of $m$ and $k$ as well as many references.
When the primary interest of an experimenter is to estimate main effects, the MA criterion has desirable properties. For example, Cheng and Tang (2005) show that, when one fits a model with only main effects included, a MA regular FF design minimizes the magnitude of the bias present in the resulting least squares estimates for main effects when the true model might contain other terms such as two-factor or higher order interactions. However, if one is interested in considering a larger model which not only contains main effects but two-factor interactions also as is done often in practice, then the MA criterion has at least one possible deficiency. In particular, designs which are found to be MA optimal for given values of \( m \) and \( k \) often do not allow for the estimation of as many main effects and two-factor interactions as do other potentially available designs. To find designs which allow for the estimation of as many main effects and two-factor interactions as possible, Jacroux (2004) introduced an alternative to the MA criterion, which is called the MR-MA criterion. This is a two-step criterion which, when applied to \( 2^{m-k} \) designs, satisfies the conditions given in the following definition.

**Definition 2.2.** To find an MR-MA optimal regular FF \( 2^{m-k} \) design, proceed as follows.

1. Find all \( 2^{m-k} \) regular FF designs which allow for the estimation of as many main effects and two-factor interactions as possible. Denote this class of designs by \( E(m,k) \).
2. Among all designs in \( E(m,k) \), find those that are MA designs.

Any design which satisfies these conditions is called an MR-MA optimal design.

Jacroux (2004) gives a catalogue of MR-MA optimal designs for various values of \( m \) and \( k \).

**Remark.** Chen and Cheng (2004) introduced the idea of estimation index of a design. For a given \( 2^{m-k} \) design \( d \), let \( p_i(d) \) be the length of the shortest word in the \( i^{th} \) alias set of \( d \), \( i = 1, \ldots, 2^{m-k} - 1 \). Then the estimation index of \( d \) is \( p(d) = \max\{p_i(d), i = 1, \ldots, 2^{m-k} - 1\} \). A design which is determined to be optimal under the MR-MA criterion can also be viewed as being a design which has MA among those designs having minimum estimation index.

As mentioned above, it often happens in industrial and other experiments that it is expensive or difficult to change the levels of some factors. When
this happens, restrictions on the randomization of experimental runs occur and regular FFSP designs are often used.

A two level regular FFSP design is a split plot design in which $m_1$ of the factors are applied to the whole plots (WP’s) and are arranged as a regular FF design with fractional element $k_1$. These factors are referred to as WP factors. In addition, $m_2$ factors are applied to split plots (SP’s) within WP’s and are arranged as a FF design with fractional element $k_2$. The factors applied to SP’s are called SP factors. In each of the $2^{m_1-k_1}$ WP’s, the $m_1$ WP factors are set to their respective levels, and then a subset of treatment combinations involving the SP factors are applied to the SP’s within each WP. We will refer to such a design as a regular $2^{(m_1+m_2)-(k_1+k_2)}$ FFSP design. The combined design matrix corresponds to a regular $2^{m-k}$ FF design, where $m = m_1 + m_2$ and $k = k_1 + k_2$. We note that in a regular $2^{(m_1+m_2)-(k_1+k_2)}$ FFSP design, the WP design must be a regular $2^{m_1-k_1}$ FF but that the SP design need not be a $2^{m_2-k_2}$ FF. However, when selecting the $k_2$ SP generators, certain care must be taken. In particular, although the fractional generators at the SP level may contain WP factors, the WP fractional generators must be free of SP factors, and the SP generators must contain at least two SP factors. Otherwise, the split plot nature of the experiment is destroyed.

To help facilitate discussion later, we introduce some specialized notation for FFSP designs. For a given $2^{(m_1+m_2)-(k_1+k_2)}$ FFSP design $d$, there are $m_1$ WP factors ($m_1 - k_1$ of which are basic factors) and $m_2$ SP factors ($m_2 - k_2$ of which are basic factors). We shall henceforth denote the WP basic factors by $B_1, \cdots, B_{m_1-k_1}$, the SP basic factors by $b_1, \cdots, b_{m_2-k_2}$, the WP added factors by $A_1, \cdots, A_{k_1}$ and the SP added factors by $a_1, \cdots, a_{k_2}$.

We shall only consider $2^{(m_1+m_2)-(k_1+k_2)}$ designs in which no main effects are aliased with one another. When analyzing the data from a given $2^{(m_1+m_2)-(k_1+k_2)}$ design $d$, we shall assume, as is often done in practice, that three-factor and higher order interactions are negligible. Within this set-up, the model for analysis for a given observation $y$ is

$$y = X_{d(y)} \beta_1 + \epsilon + M_{d(y)} \beta_2 + e,$$

where $X_{d(y)}$ and $M_{d(y)}$ are the rows of the WP and SP design matrices corresponding to $y$, $\beta_1$ and $\beta_2$ are the vectors of WP and SP parameters, and $\epsilon$ and $e$ are the WP and SP error terms. It is assumed that $\epsilon$ and $e$ are mutually independent random variables with $E(\epsilon) = E(e) = 0$, $\text{Var}(\epsilon) = \sigma^2_{\epsilon}$.
and \( \text{Var}(e) = \sigma^2_e \). Typically, partially because WP’s are larger, it is assumed that \( \sigma^2_e > \sigma^2_{\epsilon} \).

### 3 Optimality Criteria

In this section, we introduce the optimality criteria to be considered in this paper. In Section 2, we introduced the MA and the MR-MA optimality criterion as applied to regular \( 2^{m-k} \) FF designs. These same criteria can also be applied to regular \( 2^{(m_1 + m_2) - (k_1 + k_2)} \) FFSP designs. In particular, let \( w_i(d) \) denote the number of words of length \( i \) in the defining relations group of an FFSP design \( d \) and let \( W(d) = (w_3(d), \ldots, w_{m_1 + m_2}(d)) \). We call \( W(d) \) the primary WLP vector. The idea of resolution is the same here as for regular \( 2^{m-k} \) designs.

**Definition 3.1.** Suppose that \( d_1 \) and \( d_2 \) are \( 2^{(m_1 + m_2) - (k_1 + k_2)} \) FFSP designs. Let \( r \) be the smallest value of \( i \) such that \( w_i(d_1) \neq w_i(d_2) \). Then, \( d_1 \) is said to have less aberration than \( d_2 \) if \( w_r(d_1) < w_r(d_2) \). If no such \( i \) exists, then \( d_1 \) and \( d_2 \) have equal aberration. We say that \( d_1 \) is an MA FFSP design if no other design has less aberration.

Using the definition of MA given above, Bingham and Sitter (1999) used computer search methods to find MA FFSP designs for various values of \( m_1, m_2, k_1 \) and \( k_2 \). They also gave a catalogue of MA FFSP designs.

**Definition 3.2.** To find an MR-MA optimal \( 2^{(m_1 + m_2) - (k_1 + k_2)} \) design, proceed as follows.

1. Find all \( 2^{(m_1 + m_2) - (k_1 + k_2)} \) regular FFSP designs which allow for the estimation of as many main effects and two-factor interactions as possible under model (2.1). Denote this class of designs by \( E(m_1, m_2, k_1, k_2) \).

2. Among all designs in \( E(m_1, m_2, k_1, k_2) \), find those that are MA designs. Any design which satisfies these conditions is called an MR-MA optimal FFSP design.

With regard to the MR-MA criterion, we note that in performing step 1, an experimenter is essentially finding all \( 2^{(m_1 + m_2) - (k_1 + k_2)} \) designs which yield the maximal number of between WP and within WP contrasts that can be used to estimate all main effects and two-factor interactions. This stage can also be viewed as selecting those \( 2^{(m_1 + m_2) - (k_1 + k_2)} \) designs which yield the maximal amount of information concerning main effects and two-factor
interactions in terms of estimability under model (2.1). Step 2 of the MR-MA criterion provides as much separation as possible between main effects and two-factor interactions.

Comparisons between the MA and the MR-MA optimal designs which were mentioned in Section 2 for $2^{m-k}$ FF designs also hold for $2^{(m_1+m_2)-(k_1+k_2)}$ designs, i.e., MA optimal designs allow for the better estimation of main effects, whereas MR-MA designs allow for the estimation of more main effects and two-factor interactions. However, two additional statements also generally seem to hold, when comparing MA and MR-MA optimal designs.

1. Designs which are MR-MA optimal, partly because they allow for the estimation of more effects, generally also allow for the estimation of more main effects and two-factor interactions, which are not aliased with other two-factor interactions than do MA-optimal designs.

2. MR-MA optimal designs generally have an advantage over MA-optimal designs in the use of follow-up design strategies in terms of using techniques such as the foldover. These latter advantages will be reported elsewhere.

The main disadvantage of using the MR-MA criterion is that in some cases, where the MR-MA and MA optimal designs differ, the MR-MA criterion selects a design of resolution 3, whereas the MA criterion selects a design of resolution 4. In these cases, the MR-MA optimal design does not provide for the estimation of as many main effects which are not aliased with two-factor interactions as does the MA-optimal design.

**Remark.** As mentioned previously, and MR-MA optimal design can also be viewed in a regular design setting as having MA among designs having minimal estimation index as defined in Chen and Cheng (2004). This same relationship holds between the MR-MA criterion and the estimation index as applied to FFSP designs. Chen and Cheng (2004) provide a good discussion concerning properties of the estimation index as applied to regular $2^{m-k}$ designs, and this same discussion carries over, where applicable, to FFSP designs. The reader is referred to Chen and Cheng (2004) for specific details.

However, as pointed out in Bingham and Sitter (2001) and Mukerjee and Fang (2002), one of the problems occurring with both the MA and the MR-MA optimality criteria in trying to find an optimal $2^{(m_1+m_2)-(k_1+k_2)}$ design is that there may exist several nonisomorphic optimal designs having the same
characteristics. These designs have identical estimation properties and word length patterns. The problem for FFSP designs is even more pronounced than for FF designs since in the former the roles of a WP and an SP factor are not interchangeable. Because of this difficulty, Bingham and Sitter (2001) suggested a secondary criterion for distinguishing between nonisomorphic MA designs, and Mukerjee and Fang (2002) gave a formal definition for a minimum aberration secondary (MAS) criterion. The MAS-criterion is based on the well known fact that in an FFSP design, all interactions involving the WP factors are tested against the WP level error while all interactions involving at least one SP factor are tested against the SP level error. Since the WP error level is typically larger than the SP level error, a good FFSP design should try to avoid assignment of interactions involving SP factors, especially those representing lower order factorial effects, to WP alias sets. For a given regular FFSP design $d_1$, let $\beta_i(d)$ be the number of distinct $i$ factor interactions involving only SP factors that appear in a WP alias set of $d$. As mentioned previously, no SP factor can appear in a WP alias set, hence $\beta_1(d) = 0$. We shall call $WS(d) = (\beta_2(d), \ldots, \beta_m(d))$ the secondary WLP vector. Bingham and Sitter (2001) essentially suggested minimizing $\beta_i(d)$ to discriminate among nonisomorphic MA FFSP designs. Mukerjee and Fang (2002) extended this criterion to the following.

**Definition 3.3.** Let $d_1$ and $d_2$ be two nonisomorphic MA or MR-MA optimal $2^{(m_1+m_2)-(k_1+k_2)}$ regular FFSP designs. We say that $d_1$ has less secondary aberration than $d_2$ if $\beta_i(d_1) = \beta_i(d_2), i = 1, \ldots, t - 1$ and $\beta_t(d_1) < \beta_t(d_2)$. An MA optimal or an MR-MA optimal design is a minimum aberration secondary (MAS) or a maximal rank-minimum aberration secondary (MR-MAS) design if no MA or MR-MA optimal has less secondary aberration.

However, even this secondary criterion is not always successful in distinguishing between MA or MR-MA optimal designs as the following example illustrates.

**Example 3.4.** Consider the class of designs having $m_1 = 3, m_2 = 5, k_1 = 0$ and $k_3 = 3$. Also, consider the following four designs.

<table>
<thead>
<tr>
<th>Design</th>
<th>Generators</th>
<th>Primary WLP</th>
<th>Secondary WLP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>$B_1B_2b_1a_1, B_1B_3b_1b_2a_2$</td>
<td>(0,1,2)</td>
<td>(1,2,0)</td>
</tr>
<tr>
<td>$d_2$</td>
<td>$B_1B_2B_3b_1a_1, B_1b_1b_2a_2$</td>
<td>(0,1,2)</td>
<td>(1,2,0)</td>
</tr>
<tr>
<td>$d_3$</td>
<td>$B_1B_2b_1a_1, B_1B_2B_3b_2a_2$</td>
<td>(0,1,2)</td>
<td>(2,0,1)</td>
</tr>
<tr>
<td>$d_4$</td>
<td>$B_1B_2B_3b_1a_1, B_1B_2b_2a_2$</td>
<td>(0,1,2)</td>
<td>(2,0,0)</td>
</tr>
</tbody>
</table>
As can be seen, \(d_1, d_2, d_3\) and \(d_4\) all have the same primary WLP. However, according to the secondary word length patterns and Definition 3.3, \(d_1\) and \(d_2\) are both MAS better than \(d_3\) and \(d_4\), but \(d_1\) and \(d_2\) cannot be distinguished from one another. A similar situation occurs for other values of \(m_1, m_2, k_1\) and \(k_2\).

Because of situations such as illustrated in Example 3.4 and others, we consider a slight modification of the MAS criterion. In particular, if an \(i\) factor SP interaction is aliased with a WP interaction and tested against WP error, one would prefer that the WP interaction has as many letters in it as possible since under the hierarchial principle, such an interaction is less likely to be significant than a lower order interaction. This is particularly true for SP interactions involving few SP factors. Thus, if a lower order SP interaction is to be aliased with a WP interaction, one would like the order of the WP interaction to be as large as possible. In line with this reasoning, for a given design \(d\), let \(\beta_{m_1}(d)\) denote the number of distinct \(m\) factor SP interactions that have in their corresponding alias sets a lowest order WP interaction containing \(i\) WP factor letters. We shall call \(W_{m_1}(d) = (\beta_{m_1}(d), \beta_{m_2}(d), \cdots, \beta_{m_{m_1}}(d))\) the secondary WLP vector of order \(m\) for \(m = 2, \cdots, m_2\). Using this notation, we give the following modification of Definition 3.3.

**Definition 3.5.** Let \(d_1\) and \(d_2\) be two nonisomorphic MA or MR-MA optimal \(2^{(m_1 + m_2) - (k_1 + k_2)}\) regular FFSP designs. If \(\beta_2(d_1) < \beta_2(d_2)\), we say that \(d_1\) has less secondary aberration than \(d_2\). If \(\beta_2(d_1) = \beta_2(d_2)\), we say that \(d_1\) has less secondary aberration than \(d_2\) if \(\beta_{2i}(d_1) = \beta_{2i}(d_2)\) for \(i = 1, \cdots, s-1\), and \(\beta_{2s}(d_1) < \beta_{2s}(d_2)\). If \(\beta_{2i}(d_1) = \beta_{2i}(d_2)\) for \(i = 1, \cdots, m_1\), apply the same process to \(\beta_j(d_1)\) and \(\beta_{jk}(d_1)\), \(i = 1, 2, j = 3, \cdots, m_2, k = 1, \cdots, m_1\). We say that \(d_1\) is an MA secondary (MAS) or an MR-MA secondary (MR-MAS) optimal design if \(d_1\) is an MA or an MR-MA optimal design, and no other MA or MR-MA optimal design has less secondary aberration than \(d_1\).

**Example 3.4 (cont'd).** For designs \(d_1\) and \(d_2\) in the example, it is seen that \(\beta_2(d_1) = \beta_2(d_2)\) and \(\beta_{21}(d_1) = 0, \beta_{22}(d_1) = 1\) and \(\beta_{23}(d_1) = 0\) whereas \(\beta_{21}(d_2) = 0, \beta_{22}(d_2) = 0\) and \(\beta_{23}(d_2) = 1\). Thus, \(d_2\) is MAS better than \(d_1\), and \(d_2\) would be optimal under the criterion given in Definition 3.5.

**Remark.** Example 3.4 is typical of what happens when two FFSP designs \(d_1\) and \(d_2\) are equivalent under the MA or the MR-MA criterion. In particular, the author has not yet found an example where one must go
beyond the smallest SP interactions which are aliased with WP interactions to distinguish between $d_1$ and $d_2$.

4 Examples

In this section, we give several examples demonstrating usage of the MAS and the MR-MAS criteria and compare and contrast the types of optimal FFSP designs these criteria select.

Example 4.1. Consider the class of FFSP designs having $m_1 = 5$, $m_2 = 4$, $k_1 = 1$ and $k_2 = 3$. The MAS optimal FFSP design $d_1$ given in Bingham and Sitter (2001) has generators $B_1 B_2 B_3 A_1$, $B_1 B_2 b_1 a_1$, $B_1 B_3 b_1 a_2$ and $B_2 B_3 b_1 a_3$ and primary WLP vector $W(d_1) = (0, 6, 8, 0, 0)$. Design $d_1$ also has secondary WLP values $\beta_{21}(d_1) = 0$, $\beta_{22}(d_1) = 2$, $\beta_{23}(d_1) = 4$ and allows for the estimation of 30 main effects and two-factor interactions under model (2.1). The MR-MAS design $d_2$ has generators $B_1 B_2 B_3 B_4 A_1$, $B_2 B_3 b_1 a_2$ and $B_3 b_1 a_3$ and primary WLP vector $W(d_2) = (0, 7, 7, 0)$. The design $d_2$ also has secondary WLP values $\beta_{21}(d_2) = 0$, $\beta_{22}(d_2) = 6$, $\beta_{23}(d_2) = 0$ and allows for the estimation of 31 main effects and two-factor interactions under model (2.1).

Remark. In Example 4.1, we see that the MAS optimal and the MR-MAS optimal designs have the same resolution 4, and the MR-MAS optimal design allows for the estimation of one more main effect and two-factor interaction than does the MAS optimal design.

Example 4.2. Consider the class of FFSP designs having $m_1 = 2$, $m_2 = 4$, $k_1 = 0$ and $k_2 = 2$. The MAS optimal FFSP design $d_1$ given in Bingham and Sitter (2001) has generators $B_1 B_2 b_1 a_1$ and $B_1 b_1 b_2 a_2$ and primary WLP vector $W(d_1) = (0, 3, 0)$. The design $d_1$ also has secondary WLP values $\beta_{21}(d_1) = 0$, $\beta_{22}(d_1) = 1$, $\beta_{23}(d_1) = 0$ and allows for the estimation of 13 main effects and two-factor interactions under model (2.1). The MR-MAS optimal design $d_2$ has generators $B_1 B_2 b_2 a_2$ and $B_1 b_2 b_1 a_2$ and primary WLP vector $W(d_1) = (1, 1, 1)$. The design $d_2$ also has secondary WLP values $\beta_{21}(d_2) = 0$, $\beta_{22}(d_2) = 1$, $\beta_{23}(d_2) = 0$ and allows for the estimation of 15 main effects and two-factor interactions under model (2.1).

Remark. In Example 4.2, the MAS optimal design has resolution 4 and allows for the estimation of 13 effects under model (2.1). The MR-MAS
optimal design \( d_2 \) on the other hand has resolution 3 and allows for the estimation of 15 effects under model (2.1). Thus estimation capability is increased in this example for the MR-MAS design at the cost of resolution.

**Example 4.3.** Consider the class of FFSP designs having \( m_1 = 8, m_2 = 2, k_1 = 4 \) and \( k_2 = 1 \). The MAS optimal FFSP design \( d_1 \) given in Bingham and Sitter (2001) has generators \( B_1B_2B_3A_1, B_1B_2B_4A_2, B_1B_3B_4A_3, B_2B_3B_4A_1 \) and \( B_1B_2b_1a_1 \) and primary WLP vector \( W(d_1) = (0, 18, 0) \). The design \( d_1 \) also has secondary WLP values \( B_{21}(d_1) = 0, B_{22}(d_1) = 1, B_{23}(d_1) = 0 \) and allows for the estimation of 25 main effects under model (2.1). The MR-MAS optimal design \( d_2 \) has generators \( B_1B_3B_4A_1, B_1B_2B_3A_2, B_1B_2B_4A_3, B_2B_3B_4A_4 \) and \( B_2b_1a_1 \) and primary WLP vector \( W(d_2) = (1, 14, 7) \). The design \( d_2 \) has secondary WLP values \( B_{21}(d_2) = 1, B_{22}(d_2) = 0, B_{23}(d_2) = 0 \) and allows for the estimation of 31 main effects and two-factor interactions under model (2.1).

**Remark.** In Example 4.3, the MAS optimal design is of resolution 4 but only allows for the estimation of 25 main effects and two-factor interactions, whereas the MR-MAS optimal design is of resolution 3 (it has one three-letter word in its defining relations group) but allows for the estimation of 31 main effects and two-factor interactions. In this example, resolution for the MR-MAS optimal design \( d_2 \) is sacrificed in order to allow \( d_2 \) to estimate five more effects of interest.

5 The Table

Table 1 contains a list of FFSP designs having 16 runs, which are MAS and MR-MAS optimal for given values of \( m_1, m_2, k_1 \) and \( k_2 \). A similar table is available from the author for designs having 32 runs and 10 or fewer factors. Columns one through four of the table give numerical values for \( m_1, m_2, k_1 \) and \( k_2 \) to identify the parameters of each design given. This notation corresponds to that used in Bingham and Sitter (2001). Column five gives the generating words for the treatment defining relations group of each FFSP design given. Column six gives the first three entries of the primary WLP vector of the corresponding design, i.e., for the design \( d \), the entries of column six correspond to \( w_3(d), w_4(d) \) and \( w_5(d) \). In column seven, the first three entries of the secondary WLP vectors of order 2 for each design are given, i.e., the entries in column seven are \( \beta_{21}(d), \beta_{22}(d) \) and \( \beta_{23}(d) \). The reason for giving only these three entries is that in most cases, they were all
that were needed to distinguish between the nonisomorphic MA optimal and
MR-MA optimal FFSP designs associated with the given values of \( m_1, m_2, k_1 \)
and \( k_2 \). Columns eight and nine identify whether the given design is MAS
optimal, MR-MAS optimal or both. The last column gives the number
of main effects and two factor interactions which are estimable under model
(2.1) for each design and is given for comparison purposes. The MAS optimal
designs correspond to one of the nonisomorphic MA FFSP designs given in
Bingham and Sitter (2001) as determined by the MAS criterion when
more than one nonisomorphic design was given. The MR-MAS designs were
obtained using a computer search algorithm, which essentially identified the
MR-MA optimal designs given in Jacroux (2005) and then found the MR-
MAS optimal design using the secondary WLP pattern associated with the
nonisomorphic MR-MA designs obtained when more than one existed.

\[
\begin{array}{ccccccccccc}
& & & & & & & & & & \\
\text{Generators} & \text{W(d)} & \text{WS(d)} & \text{MAS} & \text{MR-MAS} & \text{Estim-} \\
\hline
1 & 5 & 0 & 2 & B_1b_1b_2a_1, B_1b_1b_3a_2 & 0 & 3 & 0 & 0 & \text{yes} & \text{no} & 13 \\
1 & 5 & 0 & 2 & b_1b_2a_1, B_1b_2b_3a_2 & 1 & 1 & 1 & 0 & \text{no} & \text{yes} & 15 \\
2 & 4 & 0 & 2 & B_1B_2b_1a_1, B_1b_2b_3a_2 & 0 & 3 & 0 & 0 & \text{yes} & \text{no} & 13 \\
2 & 4 & 0 & 2 & b_2b_1a_1, B_1B_2b_3a_2 & 1 & 1 & 1 & 0 & \text{no} & \text{yes} & 15 \\
3 & 3 & 0 & 2 & B_1B_2b_1a_1, B_1B_3b_1a_2 & 0 & 3 & 0 & 0 & \text{yes} & \text{no} & 13 \\
3 & 3 & 0 & 2 & b_3b_1a_1, B_1B_3b_1a_2 & 1 & 1 & 1 & 0 & \text{no} & \text{yes} & 15 \\
3 & 3 & 1 & 1 & B_1B_2A_1, B_1b_1b_3a_1 & 1 & 1 & 1 & 0 & \text{yes} & \text{yes} & 15 \\
4 & 2 & 1 & 1 & B_1B_3b_1a_1 & 0 & 3 & 0 & 0 & \text{yes} & \text{no} & 13 \\
4 & 2 & 1 & 1 & B_1B_2A_1, B_1B_3b_1a_1 & 1 & 1 & 1 & 0 & \text{no} & \text{yes} & 15 \\
1 & 6 & 0 & 3 & B_1b_1b_2a_1, B_1b_1b_3a_2 & 0 & 7 & 0 & 0 & \text{yes} & \text{no} & 14 \\
1 & 6 & 0 & 3 & b_1b_2a_1, B_1b_2b_3a_2 & 2 & 3 & 2 & 0 & \text{no} & \text{yes} & 15 \\
2 & 5 & 0 & 3 & B_1B_2b_1a_1, B_1B_2b_3a_2 & 0 & 7 & 0 & 0 & \text{yes} & \text{no} & 14 \\
2 & 5 & 0 & 3 & b_2b_3a_1, B_1B_2b_3a_2 & 2 & 3 & 2 & 0 & \text{no} & \text{yes} & 15 \\
3 & 4 & 0 & 3 & B_1B_2b_1a_1, B_1B_3b_1a_2 & 0 & 6 & 0 & 0 & \text{yes} & \text{no} & 14 \\
3 & 4 & 0 & 3 & b_2b_3a_1, B_1B_3b_1a_2 & 2 & 3 & 2 & 3 & \text{no} & \text{yes} & 15 \\
3 & 4 & 1 & 2 & B_1B_2A_1, B_1b_1a_1 & 2 & 3 & 2 & 1 & \text{yes} & \text{yes} & 15 \\
4 & 3 & 1 & 2 & B_1B_2A_1, B_1B_3b_1a_1, B_2b_1a_2 & 0 & 7 & 0 & 3 & \text{yes} & \text{no} & 14 \\
4 & 3 & 1 & 2 & B_1B_2A_1, B_1B_3b_1a_1, B_2b_1a_2 & 2 & 3 & 2 & 0 & \text{no} & \text{yes} & 15 \\
5 & 2 & 2 & 1 & B_1B_2A_1, B_1B_3a_2 & 2 & 3 & 2 & 0 & \text{yes} & \text{yes} & 15 \\
\end{array}
\]
Table 1. (contd.) Sixteen run MAS and MR-MAS FFSP Designs

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<th>$m_2$</th>
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<th>WS(d)</th>
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Table 1. (contd.) Sixteen run MAS and MR-MAS FFSP Designs

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References


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