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Expansions for Quantiles and Moments of Extremes for Distributions of Exponential Power Type

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Abstract

Let M_{nr} be the rth largest of a random sample of size n from a distribution F of *exponential power type* on R. That is, $1-F(z) = O(x^d \exp(-x))$ as $x = (z/\sigma)^\alpha \to \infty$. For example, the exponential, gamma, chi-square, Laplace and normal distributions are of this type. We obtain an asymptotic expansion in powers of $u_1 = -\log(1-u)$ and $u_2 = \log u_1$, for the quantile $F^{-1}(u)$ near $u = 1$. From this, we obtain a double expansion in inverse powers of $(\log\ n, n)$ for the moments of $M_{nr}/n^{1/\alpha}$, with the coefficient a polynomial in $\log\log n$. We also discuss a possible application to an optimal stopping problem.

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1 Introduction and Summary

For $1 \le r \le n$, let M_{nr} be the rth largest of a random sample of size n from a distribution F on R. Let f denote the density of F when it exists. Suppose F belongs to the domain of attraction of an extreme value distribution, i.e. there exists $a_n > 0$ and b_n such that $(M_{n1} - b_n)/a_n$ approaches a non-degenerate limit belonging to one of the three classes of extreme value distributions.

The study of the asymptotics of the moments of M_{nr} has been of considerable interest. McCord (1964) gave a first approximation to the moments of M_{n1} for three classes. This showed that a moment of M_{n1} can behave like a positive power of n or $n_1 = \log\ n$. (Here, log is to the base e.) In Appendix A, we show that if $1-F(x) \approx p_i(x)^{-1}$ as $x \to \infty$, where $p_i(x) = p_{i-1}(\exp(x))$ and $p_0(x) = x$, then M_{nr} may behave like a positive power of $n_i = \log n_{i-1}$, where $n_0 = n$. Here, $a(x) \approx b(x)$ means $a(x)/b(x) \to 1$ as $x \to \infty$. Pickands (1968) explored the conditions under which various moments of $(M_{n1}-b_n)/a_n$ converge to the corresponding moments of the extreme value distribution. It was proved that this is indeed true for all F in the domain of attraction of an extreme value distribution provided that the moments are finite for sufficiently large n. Nair (1981) investigated the limiting behavior of the distribution and the moments of M_{n1} for large n when F is standard normal.

The results provided rates of convergence of the distribution and the moments of M_{n1}. Downey (1990) derived explicit bounds for EM_{n1} in terms of the moments associated with F. The bounds were given up to the order $o(n^{1/\rho})$, where $\int \mid x \mid^\rho dF(x)$ is defined, so EM_{n1} grows slowly with the sample size. For other work, we refer the readers to Ramachandran (1984), Hill and Spruill (1994) and Hüsler, Piterbarg and Seleznjev (2003).

Most recently, Withers and Nadarajah (2007) have given expansions for the multivariate moments of $(M_{n1}, \ldots, M_{nr})$ in powers of $(n^{-1}, n^{-\beta/\alpha})$ when F can be expanded as

$$1 - F(x) = x^{-\alpha} \sum_{i=0}^{\infty} c_i \, x^{-i\beta},$$

where $\alpha, \beta > 0$. (For example, Student's t and Fisher's F distributions can be expanded in this way.) In this case, Withers and Nadarajah (2007) showed that the quantiles of F could be expanded in the form

$$F^{-1}(u) = \sum_{i=0}^{\infty} (1-u)^{(i\beta-1)/\alpha} C_i,$$

so that its singularity at $u = 1$ is of power type.

Asymptotics of the moments of M_{nr} have received applications in many areas. An example involving solutions of stochastic traveling salesman problems is discussed in Leipala (1978). Other areas include adaptive designs in linear models, channel-aware scheduling, minimum variance estimation, order restricted maximum likelihood estimation, signal processing, statistical physics, stochastic delay systems, telecommunications and urn problems. We refer the readers to Resnick (2000) and references therein.

In this note, we show how to obtain expansions for moments of M_{nr} for distributions which can be expanded in the form

$$1 - F(z) = k x^d \, \exp(-x) S(x) \tag{1.1}$$

for $S(x) = \sum_{i=0}^{\infty} a_i \, x^{-ip}$ and $x = (z/\sigma)^\alpha$, where $p > 0$, $a_0 = 1$ and $\sigma\alpha > 0$. We call such distributions *exponential power type*. For the case $p = 1$, which includes the exponential, gamma, chi-square, Laplace and normal distributions, we show in Section 2 that its quantile may be expanded in the form

$$F^{-1}(u) = \sum_{i=0}^{\infty} u_1^{1-i} \sum_{j=0}^{i} u_2^j \, \Lambda_{ij}, \tag{1.2}$$

where Λ_{ij} are defined later by (2.7), $u_1 = \log(1-u)$ and $u_2 = \log \, u_1$, so that its singularity at $u = 1$ is of log type. In particular, as $u \to 1$,

$$F^{-1}(u) = \sigma u_1^{1/\alpha} \, \left\{ 1 + \alpha^{-1} \, u_1^{-1} \left(u_2 d + k_1 \right) + O\left(u_1^{-2} \, u_2^2 \right) \right\},$$

where $k_1 = \log k$. In Section 3, we apply this to obtain our main result: when (1.1) holds with $p = 1$, we have an expansion for the moments of M_{nr} in powers of (n^{-1}, n_1^{-1}):

$$E\left(M_{nr} n_1^{-1/\alpha} \sigma^{-1}\right)^{\varphi} = \sum_{a=0}^{\infty} n^{-a} \sum_{b=0}^{\infty} n_1^{-b} \sum_{c=0}^{b} n_2^c \, g_{abc}^r(\varphi/\alpha), \tag{1.3}$$

where $g_{abc}^r(\cdot)$ are defined later by Theorem 3.1, $n_1 = \log n$ and $n_2 = \log\log n$. So, M_{nr} behaves like $\sigma n_1^{1/\alpha}$. In particular, the right hand side of (1.3) is equal to

$$1 + n_1^{-1}\left\{n_2 \theta d + \theta k_1 - \psi(r)\right\} + O\left(n_1^{-2}\, n_2^2\right),$$

where $\theta = \varphi/\alpha$ and $\psi(x) = (d/dx)\log\Gamma(x)$. Finally, in Section 4, we discuss a possible application of these results.

We express the general coefficient in terms of the Bell polynomials B_{ri}, $\widehat{B}_{ri}$ defined for $i = 0,\ 1, \ldots$ by

$$\left(\sum_{j=1}^{\infty} t^j\ x_j\right)^i = \sum_{r=i}^{\infty} t^r \widehat{B}_{ri}(\mathbf{x}) \text{ for } \mathbf{x} = (x_1,\ x_2, \ldots)$$

and

$$\left(\sum_{j=1}^{\infty} t^j y_j/j!\right)^i /i! = \sum_{r=i}^{\infty} t^r B_{ri}(\mathbf{y}) \text{ for } \mathbf{y} = (y_1,\ y_2, \ldots)$$

and tabled on pages 307-309 of Comtet (1974). Their applications needed here are given in Withers and Nadarajah (2007). They are most easily obtained using recurrence formulas. Set $(r)_j = r(r+1)\cdots(r+j-1) = \Gamma(r+j)/\Gamma(r)$, $\langle r\rangle_j = r(r-1)\cdots(r-j+1) = r!/(r-j)!$ and $\sum_{i=a,b}^{c,d} - = \sum\{- : \max(a,b) \le i \le \min(c,d)\}$.

2 Quantile Expansions for Exponential Power Type Distributions

We shall call $X \sim F(x)$ *of exponential power type* if it can be expanded in the form

$$1 - F(x) = kx^d \exp(-x) S(x), \tag{2.1}$$

where $S(x) = \sum_{i=0}^{\infty} a_i\ x^{-ip}$, $p > 0$ and $a_0 = 1$. This may be a convergent or an asymptotic expansion as $x \to \infty$.

Example 2.1. Suppose $X \sim gamma(\gamma) = \chi_{2\gamma}^2/2$, that is $f(x) = x^{\gamma-1}\exp(-x)$ / $\Gamma(\gamma)$ on $(0,\infty)$. By (6.5.32) of Abramowitz and Stegun (1964), (2.1) holds with $p = 1$, $k = \Gamma(\gamma)^{-1}$, $d = \gamma - 1$ and $a_i = \langle\gamma - 1\rangle_i$. The expansion for its quantile $F^{-1}(u)$ depends on which of the intervals $[i, i+1)$ or $[(i+1)^{-1},\ i^{-1})$ that p lies in.

We now illustrate the method for the case $p = 1$. Define u_1, u_2 by (1.2), $k_1 = \log k$, $x_1 = \log x$ and $S_1 = \log S(x)$. Then,

$$x = u_1 + dx_1 + k_1 + S_1 \approx u_1 \text{ as } x \to \infty \text{ so } x_1 \approx u_2. \tag{2.2}$$

It follows that

$$F^{-1}(u) = x = u_1 \sum_{i=0}^{\infty} M_i\left(u_2\right) u_1^{-i}, \tag{2.3}$$

where $M_0 = 1$, and $M_i = M_i(u_2)$ is a polynomial. By (A6), (A3) of Withers and Nadarajah (2007)

$$x_1 = u_2 + \sum_{i=1}^{\infty} u_1^{-i}\ d_i\left(u_2\right), \tag{2.4}$$

where

$$d_i\left(u_2\right) = \widehat{D}_i(1,\ \mathbf{M}) = \sum_{i=1}^{r} \widehat{B}_{ri}(\mathbf{M})(-1)^{i-1}/i$$

and

$$x^{-i} = u_1^{-i} \sum_{j=0}^{\infty} u_1^{-j}\ \widehat{C}_j\left(-i,\ 1,\ \mathbf{M}\right), \tag{2.5}$$

where

$$\widehat{C}_j\left(\theta,\ \lambda,\ \mathbf{M}\right) = \sum_{i=0}^{j} \binom{\theta}{i} \widehat{B}_{ji}(\mathbf{M})\lambda^i = \widehat{C}_j \tag{2.6}$$

say. The first few such polynomials are

$$\begin{aligned}
&\widehat{D}_1(1,\ \mathbf{M}) = M_1,\\
&\widehat{D}_2(1,\ \mathbf{M}) = M_2 - M_1^2/2,\\
&\widehat{D}_3(1,\ \mathbf{M}) = M_3 - M_1 M_2 + M_1^3/3,\\
&\widehat{C}_0 = 1,\ \widehat{C}_1 = \theta\lambda M_1,\\
&\widehat{C}_2 = \theta\lambda M_2 + \langle\theta\rangle_2 \lambda^2 M_2^2/2,\\
&\widehat{C}_3 = \theta\lambda M_3 + \langle\theta\rangle_2 \lambda^2 M_1 M_2 + \langle\theta\rangle_3 \lambda^3 M_1^3/6,\\
&\widehat{C}_4 = \theta\lambda M_4 + \langle\theta\rangle_2 \lambda^2 \left(M_1 M_3 + M_2^2/2\right) + \langle\theta\rangle_3 \lambda^3 M_1^2 M_2/2 + \langle\theta\rangle_4 \lambda^4 M_1^4/24.
\end{aligned}$$

Substituting into (2.2) gives $M_1 = du_2 + k_1$ and the recurrence formula for M_{i+1}:

$$M_{i+1} = d\widehat{D}_i(1,\ \mathbf{M}) + \sum_{j=1}^{i} \widehat{D}_j(1,\ \mathbf{M})\widehat{C}_{i-j}(-j,\ 1,\ \mathbf{M})$$

for $i \geq 1$. So, $M_2 = dM_1 + a_1 = d^2u_2 + dk_1 + a_1$, $M_3 = d(M_2 - M_1^2/2) - a_1M_1 + a_2 - a_1^2/2$, $M_4 = d(M_3 - M_1M_2 + M_1^3/3) + a_1(-M_2 + M_1^2) - (2a_2 - a_1^2)M_1 + a_3 - a_1a_2 + a_1^3/3$, and

$$M_i = \sum_{j=0}^{i} \Lambda_{ij} u_2^j, \tag{2.7}$$

where $\Lambda_{ii} = 0$ for $i \geq 2$, $\Lambda_{00} = 1$, $\Lambda_{11} = d$, $\Lambda_{10} = k_1$, $\Lambda_{21} = d^2$, $\Lambda_{20} = dk_1 + a_1$, $\Lambda_{32} = -d^3/2$, $\Lambda_{31} = d^3 - d^2k_1 - da_1$, $\Lambda_{30} = d^2k_1 + d(a_1 - k_1^2/2) - a_1k_1 + a_2 - a_1^2/2$, $\Lambda_{43} = d^4/3$, $\Lambda_{42} = -3d^4/2 + d^2(k_1 + a_1)$, $\Lambda_{41} = d\Lambda_{31} - d^2\Lambda_{20}d^3k_1 + d^2k_1^2 - a_1d^2 + 2a_1k_1d + (a_1^2 - 2a_2)d$, $\Lambda_{40} = d\Lambda_{30} - dk_1\Lambda_{20} + dk_1^3/3 - a_1\Lambda_{20} + a_1k_1^2 + (a_1^2 - 2a_2)k_1 + a_3 - a_1a_2 + a_1^3/3$ and so on. By (2.3) and (A3) of Withers and Nadarajah (2007),

$$\left\{F^{-1}(u)\right\}^\theta = x^\theta = \sum_{j=0}^{\infty} u_1^{\theta-j}\ \widehat{C}_j(\theta,\ 1,\ \mathbf{M}). \tag{2.8}$$

By (2.7) the coefficient can be expanded as

$$\widehat{C}_j\ \ (\theta,\ 1,\ \mathbf{M}) = \sum_{i=0}^{j}\ u_2^i\ c_{ji}(\theta) \tag{2.9}$$

with $c_{ji} = c_{ji}(\theta)$ given by

$$\begin{aligned}
c_{rr} &= \langle\theta\rangle_r d^r/r!,\ \ c_{10} = \theta k_1,\\
c_{21} &= \langle\theta\rangle_2 dk_1 + \theta d^2,\\
c_{20} &= \langle\theta\rangle_2 k_1^2/2 + \theta\left(dk_1 + a_1\right),\\
c_{32} &= \langle\theta\rangle_3 d^2k_1/2 + \left(\theta^2 - 3\theta/2\right) d^3,\\
c_{31} &= \langle\theta\rangle_3 dk_1^2/2 + \langle\theta\rangle_2 d\left(2dk_1 + a_1\right) + \theta\Lambda_{31},\\
c_{30} &= \langle\theta\rangle_3 k_1^3/6 + \langle\theta\rangle_2 k_1\left(dk_1 + a_1\right) + \theta\Lambda_{30},\\
c_{43} &= \langle\theta\rangle_4 d^3k_1/6 + \langle\theta\rangle_3 d^2\Lambda_{21}/2 + \langle\theta\rangle_2 d\left(\Lambda_{32} + \Lambda_{21}^2/2\right) + \theta\Lambda_{43},\\
c_{42} &= \langle\theta\rangle_4 d^3k_1^2/4 + \langle\theta\rangle_3\left(d^2\Lambda_{20} + 2dk_1\Lambda_{21}\right)/2 + \langle\theta\rangle_2\Big\{k_1\left(\Lambda_{32} + \Lambda_{21}^2/2\right)\\
&\quad + d\left(\Lambda_{31} + \Lambda_{20}\Lambda_{21}\right)\Big\} + \theta\Lambda_{42},\\
c_{41} &= \langle\theta\rangle_4 dk_1^3/6 + \langle\theta\rangle_3\left(k_1^2\Lambda_{21} + 2dk_1\Lambda_{20}\right)/2 + \langle\theta\rangle_2\Big\{\Lambda_{30} + \Lambda_{20}^2/2\\
&\quad + k_1\left(\Lambda_{31} + \Lambda_{20}\Lambda_{21}\right)\Big\} + \theta\Lambda_{41},\\
c_{40} &= \langle\theta\rangle_4 k_1^4/24 + \langle\theta\rangle_3 k_1^2\Lambda_{20}/2 + \langle\theta\rangle_2 k_1\left(\Lambda_{30} + \Lambda_{20}^2/2\right) + \theta\Lambda_{40}.
\end{aligned}$$

Let

$$c_{ji} = \sum_{k=1}^{j}\ c_{jik}\ \theta^k \tag{2.10}$$

say, for $j > 0$. Now suppose that Z is a random variable of *exponential power type.* That is,

$$Z = \sigma X^{1/\alpha}, \tag{2.11}$$

where $\sigma\alpha > 0$, so that Z is an increasing function of X, and α^{-1} is an integer if X can take negative values, so that Z is real. Its distribution is

$$F_Z(z) = P(Z \leq z) = F(x) \tag{2.12}$$

of (2.1) at $x = (z/\sigma)^\alpha$.

Example 2.2. Suppose $Z = |N|$, where $N \sim N(0,\ 1)$, a unit normal random variable. Then $Z = (\chi_1^2)^{1/2} = (2X)^{1/2}$ for X of Example 2.1 with $\gamma = 1/2$. So, (2.11), (2.12) hold with $\alpha = 2$, $\sigma = 2^{1/2}$, $p = 1$, $k = \pi^{-1/2}$, $d = -1/2$, $a_i = (-1)^i (1/2)_i$.

Now take $p = 1$ so that (2.3) and (2.8) hold. The inverse of $u = F_Z(z)$ is $F_Z^{-1}(u) = z = \sigma x^{1/\alpha}$ at $x = F^{-1}(u)$. So, by (2.8), (2.9)

$$\left\{F_Z^{-1}(u)\right\}^\varphi / \sigma^\varphi = \sum_{j=0}^{\infty} u_1^{\theta-j} \sum_{i=0}^{j} u_2^i c_{ji}(\theta) \tag{2.13}$$

at $\theta = \varphi/\alpha$.

Example 2.3. Suppose $Z \sim N(0,\ 1)$, a unit normal. Then for $x > 0$, $P(Z > x) = P(|Z| > x)/2$ so (2.12) holds with parameters as in Example 2.2 except that k is halved; c.f. (26.2.12) of Abramowitz and Stegun (1964). In particular, taking $\varphi = 1$, we have the following expansions for the normal quantile:

$$\Phi^{-1}(u) = 2^{1/2} \sum_{j=0}^{\infty} u_1^{1/2-j} \sum_{i=0}^{j} u_2^i c_{ji} \tag{2.14}$$

for $1/2 < u < 1$, where $c_{ji} = c_{ji}\ (1/2)$ are given by $c_{00} = 1$, $c_{rr} = \langle 1/2 \rangle_r (-1/2)^r / r!$, $c_{10} = k_1/2$, where $k_1 = \log(\pi^{-1/2}/2)$, $c_{21} = (k_1 + 1)/8$, $c_{20} = -(k_1^2 + 2k_1 + 2)/8$, $c_{32} = (3k_1 + 4)/64$, $c_{31} = -(3k_1^2 + 8k_1 + 8)/32$, $c_{30} = (-k_1^3 + 4k_1^2 + 8k_1 + 7)/16$, and so on.

3 Main Results

Here, we obtain expansions for $Eh(M_{nr})$, in particular for the moments of M_{nr}. By (11.33) of Kendall, Stuart and Ord (1987), for F continuous $V_{nr} = F(M_{nr})$ has density $B_{nr}^{-1}\ u^{n-r}(1-u)^{r-1}$ on $0 < u < 1$, where

$$B_{nr}^{-1} = B(n-r+1,\ r)^{-1} = n^r (r-1)!^{-1} \sum_{i=0}^{\infty} n^{-i} c_i(r) \tag{3.1}$$

for $c_i = c_i(r) = e_i(-r)$ of Lemma 2.5 in Withers and Nadarajah (2007):

$$c_0 = 1,\ c_1 = -\langle r\rangle_2/2,\ c_2 = \langle r\rangle_3(3r-1)/24, \ldots. \tag{3.2}$$

Setting $\exp(-v) = u = F(x)$ and

$$H(v) = h\left(F^{-1}\left(\exp(-v)\right)\right),\ H_r(v) = \left(\exp(v)-1\right)^{r-1} H(v), \tag{3.3}$$

gives

$$Eh\left(M_{nr}\right) = Eh\left(F^{-1}\left(V_{nr}\right)\right) = B_{nr}^{-1} \int_0^\infty H_r(v) \exp(-nv) dv. \tag{3.4}$$

If $H_r(v) = \sum_i H_i v^{\alpha_i}$, $Eh(M_{nr}) = B_{nr}^{-1} \sum_i H_i J_n(\alpha_i)$, where $J_n\ (\alpha) = \int_0^\infty v^\alpha \exp(-nv)\ dv = n^{-\alpha-1}\ \Gamma(\alpha+1)$, interpreted as ∞ if Re $(\alpha+1) \leq 0$. Set

$$v_1 = -\log v,\ v_2 = \log v_1,\ t = nv,\ t_1 = -\log t. \tag{3.5}$$

If $H_r(v) = \sum_{ij} H_{ij} v^{\alpha_i} v_1^{\beta_j}$, $Eh(M_{nr}) = B_{nr}^{-1} \sum_{ij} H_{ij} J_n(\alpha_i,\ \beta_j)$, where $J_n(\alpha,\beta) = \int_0^\infty v^\alpha\ v_1^\beta \exp(-nv) dv$.

Lemma 3.1. *We have*

$$\begin{aligned} J_n(\alpha,\ \beta) &= n^{-\alpha-1}\ n_1^\beta \sum_{j=0}^\infty n_1^{-j} \binom{\beta}{j} J_1(\alpha,\ j) \\ &= n^{-\alpha} n_1^\beta \left\{\Gamma(\alpha+1) + O\left(n_1^{-1}\right)\right\} \end{aligned} \tag{3.6}$$

and, for $j = 0, 1, \ldots$,

$$J_1(\alpha,\ j) = \int_0^\infty t^\alpha\ t_1^j\ \exp(-t) dt = (-\partial/\partial\alpha)^j\, \Gamma(\alpha+1). \tag{3.7}$$

Proof. $J_n(\alpha,\ \beta) = n^{-\alpha-1} \int_0^\infty (n_1+t_1)^\beta t^\alpha \exp(-t) dt$ so (3.6) holds. □

If $H_r(v) = \sum_{ijk}\ H_{ijk}\ v^{\alpha_i}\ v_1^{\beta_j}\ v_2^{\gamma_k}$ then

$$Eh\left(M_{nr}\right) = B_{nr}^{-1} \sum_{ijk}\ H_{ijk}\ J_n\left(\alpha_i,\ \beta_j,\ \gamma_k\right), \tag{3.8}$$

where $J_n(\alpha,\ \beta,\ \gamma) = \int_0^\infty v^\alpha v_1^\beta\ v_2^\gamma \exp(-nv)\ dv$.

Lemma 3.2. *For* $n > 1$,

$$J_n\left(\alpha,\ \beta,\ \gamma\right) = n^{-\alpha-1} n_1^\beta n_2^\gamma \sum_{l=0}^\infty n_1^{-l} J_1(\alpha,\ l) \sum_{i=0}^l (-n_2)^{-i} \binom{\gamma}{i} C_{li}(\beta), \tag{3.9}$$

where

$$C_{li}(\beta) = \sum_{k=1}^l \binom{\beta}{l-k} (-1)^k \widehat{B}_{ki}(\mathbf{x}) \tag{3.10}$$

at $x_r = r^{-1}$.

PROOF. Set $x_k = k^{-1}$, $S = \sum_{k=1}^{\infty}(-t_1/n_1)^k x_k$, and

$$p_{nt} = \{\log(n_1+t_1)\}^{\gamma} = \left\{n_2\left(1-n_2^{-1}S\right)\right\}^{\gamma} = n_2^{\gamma}\sum_{k=0}^{\infty}(-t_1/n_1)^k\,\widehat{C}_k\left(\gamma, -n_2^{-1},\ \mathbf{x}\right)$$

of (2.6) by (A3) of Withers and Nadarajah (2007). Then

$$\begin{aligned} J_n(\alpha,\ \beta,\ \gamma) &= n^{-\alpha-1}\int_0^{\infty} t^{\alpha}(n_1+t_1)^{\beta}\,p_{nt}\exp(-t)dt \\ &= n^{-\alpha-1}n_1^{\beta}\sum_{j=0}^{\infty}\binom{\beta}{j}n_1^{-1}\ K_n(\alpha,\ j,\ \gamma) \end{aligned}$$

for

$$\begin{aligned} K_n(\alpha,\ j,\ \gamma) &= \int_0^{\infty} t^{\alpha}\ t_1^j p_{nt}\exp(-t)dt \\ &= n_2^{\gamma}\sum_{k=0}^{\infty}(-n_1)^{-k}\,\widehat{C}_k\left(\gamma, -n_2^{-1},\ \mathbf{x}\right)J_1(\alpha,\ j+k)\,. \end{aligned}$$

The proof is complete. □

We have $C_{l0}(\beta) = \binom{\beta}{l}$, $C_{ii}(\beta) = (-1)^i$, $C_{i+1,i}(\beta) = (-1)^i(\beta - i/2)$, and $C_{i+2,i}(\beta) = (-1)^i\{\langle\beta\rangle_2 - \beta i + 2i/3 + \langle i-2\rangle_2 I(i\geq 4)/4\}/2$, where $I(A) = 1$ or 0 if A is true or false.

To apply (3.8) with $h(x) = x^{\varphi}$ to $X \sim F_Z$ of (2.12), we need to expand $u_1^{\varphi-j}\ u_2^i$ in powers of v, v_1, v_2 of (3.5).

LEMMA 3.3. *We have*

$$u_1^{\theta} = \sum_{a=0}^{\infty}\binom{\theta}{a}v_1^{\theta-a}A_a(v) \tag{3.11}$$

for

$$A_a(v) = \sum_{c=a}^{\infty}(-v)^c A_{ca}$$

and

$$A_{ca} = \sum_{b=a}^{c}(-1)^b b!\widehat{B}_{ba}(\mathbf{x})B_{cb}(\mathbf{y}) \tag{3.12}$$

at $x_i = i^{-1}$, $y_i = (i+1)^{-1}$.

PROOF. $u = 1-\exp(-v)$ so $u_1 = v_1+T$, where $T = -\log(1+S) = \sum_{i=1}^{\infty}(-S)^i x_i$ and $1+S = (1-\exp(-v))/v = 1+\sum_{i=1}^{\infty}(-v)^i y_i$. Now use $u_1^{\theta} = v_1^{\theta}\sum_{a=0}^{\infty}\binom{\theta}{a}v_1^{-a}T^a$, $T^a = \sum_{b=a}^{\infty}(-S)^b\widehat{B}_{ba}(\mathbf{x})$ and $S^b = b!\sum_{c=b}^{\infty}(-v)^c B_{cb}(\mathbf{y})$. □

Lemma 3.4. *We have*

$$u_1^\theta u_2^i = \sum_{k=0}^{i} \binom{i}{k} v_2^k \sum_{a=i-k}^{\infty} \langle\theta\rangle_{a,i-k} v_1^{\theta-a}\ A_a(v), \tag{3.13}$$

where

$$\langle\theta\rangle_{a,k} = (\partial/\partial\theta)^k \langle\theta\rangle_a = \sum_{d=k}^{a} S_a^{(d)} \langle d\rangle_k \theta^{a-k},$$

(so $\langle\theta\rangle_{a,k} = 0$ if $k > a$) and $S_a^{(d)}$ is the Stirling number of the first kind.

Proof. $\langle\theta\rangle_a = \sum_{d=0}^{a} S_a^{(d)}\ \theta^d$ so (3.13) holds. Also (3.11), Leibniz's rule and $u_1^\theta u_2^i = (\partial/\partial\theta)^i u_1^\theta$ imply (3.13). □

For a table and recurrence formulas for the Stirling numbers, see Table 24.3 (page 833) and Section 24.1.3 (page 824) of Abramowitz and Stegun (1964).

Lemma 3.5. *We have*

$$(\exp(v) - 1)^{r-1} = \sum_{i=r-1}^{\infty} v^i d_{ri}, \tag{3.14}$$

where

$$d_{ri} = B_{i,r-1}(\mathbf{1}) = \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j}\ j^i/i!. \tag{3.15}$$

Proof. $(\exp(v)-1)^{r-1} = \sum_{j=0}^{r-1} \binom{r-1}{j} \exp(jv)(-1)^{r-1-j}$. Now expand $\exp(jv)$. Also $\exp(v) - 1 = \sum_{i=1}^{\infty} v^i/i!$ so the first equality in (3.15) holds. □

From (2.13) and Lemmas 3.4, 3.5 we have the following.

Lemma 3.6. *For F_Z of (2.12) with $p = 1$, and $\theta = \varphi/\alpha$*

$$\left\{F_Z^{-1}(u)/\sigma\right\}^\varphi = \sum_{c=0}^{\infty} v^c \sum_{\beta=0}^{\infty} v_1^{\theta-\beta} \sum_{k=0}^{\beta} v_2^k\ H_{c\beta k}(\theta),$$

where

$$H_{c\beta k}(\theta) = \sum_{i=k}^{k+c,(k+\beta)/2} \binom{i}{k} \sum_{a=i-k}^{\beta-i,c} c_{\beta-a,i}(\theta)\langle\theta-\beta+a\rangle_{a,i-k}\ A_{ca}/a!$$

for $c_{ji}(\theta)$ of (2.10) and A_{ca} of (3.12). Also for $h(x) = x^\varphi$, H_r of (3.3) is given by

$$H_r(v) = \sigma^\varphi \sum_{\alpha=r-1}^{\infty} v^\alpha \sum_{\beta=0}^{\infty} v_1^{\theta-\beta} \sum_{k=0}^{\beta} v_2^k\ H_{\alpha\beta k}^r(\theta), \tag{3.16}$$

where

$$H^r_{\alpha\beta k}(\theta) = \sum_{c=0}^{\alpha-r+1} d_{r,\alpha-c} H_{c\beta k}(\theta) \tag{3.17}$$

for d_{ri} *of (3.14).*

Finally, from (3.8), (3.9) and (3.1), replacing Z by X, we obtain our main result:

Theorem 3.1. *For* $X \sim F_Z$ *of (2.12) with* $p = 1$ *and* $\theta = \varphi/\alpha$

$$\begin{aligned} EM^{\varphi}_{nr}/\sigma^{\varphi} &= B^{-1}_{nr} \sum_{\alpha=r-1}^{\infty} n^{-\alpha-1} \sum_{b=0}^{\infty} n_1^{\theta-b} \sum_{c=0}^{b} n_2^c h^r_{\alpha bc}(\theta) \\ &= n_1^{\theta} \sum_{a=0}^{\infty} n^{-a} \sum_{b=0}^{\infty} n_1^{-b} \sum_{c=0}^{b} n_2^c \; g^r_{abc}(\theta), \end{aligned} \tag{3.18}$$

where

$$h^r_{\alpha bc}(\theta) = \sum_{l=0}^{b} J_1(\alpha,\ l) \sum_{i=0}^{l} (-1)^i \binom{c+1}{i} C_{li}(\theta - b + l) H^r_{\alpha,b-l,c+i}\ (\theta),$$

and

$$g^r_{abc}(\theta) = (r-1)!^{-1} \sum_{i=0}^{a} c_i(r) h^r_{r-1+a-i,b,c}\ (\theta)$$

for $J_1(\alpha, 1)$ *given by (3.7),* $C_{li}(\beta)$ *by (3.10),* $H^r_{\alpha\beta\gamma}$ *by (3.17), and* $c_i(r)$ *by (3.2).*

Corollary 3.1. *Under the conditions of Theorem 3.1,* $Y_{nr} = M_{nr}\ \sigma^{-1}\ n_1^{-1/\alpha}$ *satisfies*

$$EY^{\varphi}_{nr} = \sum_{b=0}^{\infty} n_1^{-b}\ S_{nb} + O\left(n^{-1}\right), \tag{3.19}$$

where

$$S_{nb} = \sum_{c=0}^{b} n_2^c g_{bc},\ \ g_{bc} = g^r_{0bc}\ (\theta) = \sum_{l=0}^{b-c} J_{rl} U_{lbc},$$

$$J_{rl} = J_1(r-1, l)/(r-1)! = (-1)^l \Gamma^{(l)}(r)/(r-1)!,$$

and

$$U_{lbc} = \sum_{i=0}^{l,b-c-l} (-1)^i \binom{c+1}{i} C_{li}(\theta - b + l) c_{b-l,c+i}.$$

Note that $J_{r0}=1$, $J_{r1}=-\psi(r)$, $J_{r2}=\psi(r)^2+\dot{\psi}(r)$, $J_{r3}=-\psi(r)^3-3\psi(r)\dot{\psi}(r)-\ddot{\psi}(r)$.

In terms of $c_{ri}=c_{ri}(\theta)$ of (2.10), $\{g_{bc}\}$ needed for $S_{n0},\ldots,S_{n4}$ are

$$\begin{aligned}
g_{bb} &= c_{bb},\\
g_{10} &= c_{10}+J_{rl}\theta,\\
g_{20} &= c_{20}+J_{rl}\left\{(\theta-1)c_{10}+c_{11}\right\}+J_{r2}\langle\theta\rangle_2/2,\\
g_{21} &= c_{21}+J_{rl}(\theta-1)c_{11},\\
g_{30} &= c_{30}+J_{rl}\left\{(\theta-2)c_{20}+c_{21}\right\}+J_{r2}\left\{\langle\theta\rangle_2c_{10}/2+(\theta-3/2)c_{11}\right\},\\
g_{31} &= c_{31}+J_{rl}\left\{(\theta-2)c_{21}+2c_{22}\right\}+J_{r2}\langle\theta-1\rangle_2c_{11}/2,\\
g_{32} &= c_{32}+J_{rl}(\theta-3)c_{22},\\
g_{40} &= c_{40}+J_{rl}\left\{(\theta-3)c_{30}+c_{31}\right\}\\
&\quad +J_{r2}\left\{\langle\theta-2\rangle_2c_{20}/2+(\theta-5/2)c_{21}+\langle\theta\rangle_2d/2\right\}\\
&\quad +J_{r3}\left\{\langle\theta\rangle_4k_1/6+\langle\theta\rangle_3d/2-\langle\theta\rangle_2d/2+1/3\right\}+J_{r4}\langle\theta\rangle_4/24,\\
g_{41} &= c_{41}+J_{rl}\left\{(\theta-3)c_{31}+2c_{32}\right\}+J_{r2}\left\{\langle\theta-2\rangle_2c_{21}/2+(2\theta-5)c_{22}\right\}\\
&\quad +J_{r3}\langle\theta-1\rangle_3c_{11}/6,\\
g_{42} &= c_{42}+J_{rl}\left\{(\theta-3)c_{32}+3c_{33}\right\}+J_{r2}\langle\theta-2\rangle_2c_{22}/2,\\
g_{43} &= c_{43}+J_{rl}(\theta-3)c_{33}.
\end{aligned}$$

The expressions for S_{n0}, S_{n1} follow from Theorem 4.4 of Withers and Nadarajah (2007):

$$\alpha n_1\left(Y_{nr}-1\right)-dn_2-k_1\overset{L}{\to}Y_r\sim G_1(y)\sum_{i=0}^{r-1}\exp(-iy)/i!$$

on R, where $G_1(y)=\exp\{-\exp(-y)\}$, the EV1 distribution.

EXAMPLE 3.1. For a sample from $N(0,1)$

$$EM_{nr}=(2n_1)^{1/2}\sum_{b=0}^{\infty}n_1^{-b}\sum_{c=0}^{b}n_2^c\ g_{bc}+O\left(n^{-1}n_1^{1/2}\right), \tag{3.20}$$

where, by Example 2.3 with $\theta=1/2$ and $k_1=\log(\pi^{-1/2}/2)$,

$$\begin{aligned}
g_{00} &= 1,\\
g_{10} &= \left(k_1+J_{r1}\right)/2,\\
g_{11} &= -1/4,\\
g_{20} &= -\left(k_1^2+2k_1+2\right)/8-J_{r1}\left(k_1+1\right)/4-J_{r2}/8,\\
g_{21} &= \left(k_1+1+J_{r1}\right)/8,\\
g_{22} &= -1/32,\\
g_{30} &= \left(-k_1^3+4k_1^2+8k_1+7\right)/16
\end{aligned}$$

$$
\begin{aligned}
& \quad +J_{r1}\left(3k_1^2+8k_1+8\right)/16 + J_{r2}\left(3k_1+4\right)/16, \\
g_{31} &= -\left(3k_1^2+8k_1+8\right)/32 - J_{r1}\left(3k_1+4\right)/16 - 3J_{r2}/32, \\
g_{32} &= \left(3k_1+4\right)/64 + 5J_{r1}/64, \\
g_{33} &= -1/128,
\end{aligned}
$$

and so on. The same holds for a sample from $|N(0,1)|$ with k_1 replaced by $k_1 = \log(\pi^{-1/2}/2)$.

Finally, we give an expression for var (M_{nr}). Some simplification is possible using the following two lemmas.

LEMMA 3.7. *Suppose* $x_i = (E \log Y)^i/i!$, *that is*

$$EY^{\varphi} = \sum_{i=0}^{\infty} x_i \varphi^i. \tag{3.21}$$

Then

$$var\,(Y^{\varphi}) = \sum_{i=2}^{\infty} v_i(\mathbf{x})\varphi^i,$$

where

$$v_i(\mathbf{x}) = x_i(2^i-2) - B_{2i}(\mathbf{x}) \text{ and } B_{2i}(\mathbf{x}) = \sum_{j=1}^{i-1} x_j x_{i-j}. \tag{3.22}$$

PROOF. Let $S = \sum_{i=1}^{\infty} x_i\varphi^i$. Then $S^2 = \sum_{i=2}^{\infty} B_{2i}(\mathbf{x})\varphi^i$ and var $(Y^{\varphi}) = \sum_{i=0}^{\infty} x_i(2\varphi)^i - 1 - 2S - S^2$. □

LEMMA 3.8. *Suppose (3.21) holds with* $x_i = \sum_{j=i}^{\infty} \varepsilon^j\, x_{ij}$. *Then* $v_i(\mathbf{x})$ *of (3.22) is given by* $v_i(\mathbf{x}) = \varepsilon^i \sum_{k=0}^{\infty} \varepsilon^k v_{ik}(\mathbf{X})$ *for* $v_{ik}(\mathbf{X}) = X_{ik}(2^i-2) - B_{2ik}(\mathbf{X})$, $X_{ik} = x_{i,i+k}$ *and* $B_{2ik}(\mathbf{X}) = \sum_{j=1}^{r-1} \sum_{a=0}^{k} X_{ja}X_{i-j,k-a}$. *So,* $var(Y^{\varphi}) = \sum_{j=2}^{\infty} \varepsilon^j V_j(\varphi)$, *where* $V_j(\varphi) = \sum_{i=2}^{j} \varphi^i v_{i,j-i}(\mathbf{X})$. *In particular,* $V_2(\varphi) = 2\varphi^2(2x_{22} - x_{11}^2)$ *and* $V_3(\varphi) = 2\varphi^2(x_{23} - x_{11}x_{12}) + \varphi^3(3x_{33} - x_{11}x_{22})$.

PROOF. $x_i = \varepsilon^i \sum_{k=0}^{\infty} \varepsilon^k\, X_{ik}$ and $B_{2i}(\mathbf{x}) = \varepsilon^i \sum_{k=0}^{\infty} \varepsilon^k B_{2ik}(\mathbf{X})$. □

Applying this with $Y = Y_{nr}$ of Corollary 3.1, $\varepsilon = n_1^{-1}$, $x_i = x_i^{'}\, \alpha^{-i}$, and $x_i^{'} =$ coefficient of θ^i in (3.18), $x_{ij} = x_{ij}^{'}\, \alpha^{-i}$, $x_{ij}' =$ coefficient of θ^i in S_{nj}. So,

$$
\begin{aligned}
x_{11}^{'} &= \alpha_n + J_{rl}, \\
x_{22}^{'} &= \alpha_n^2/2 + J_{rl}\alpha_n + J_{r1}\alpha_n + J_{r2}/2, \\
x_{12}^{'} &= -k_1^2/2 + \Lambda_{20} + J_{r1}\left(-k_1+d\right) - J_{r2}/2 + n_2\left(d^2 - dk_1 - dJ_{r1}\right) - n_2^2 d^2/2, \\
x_{23}^{'} &= -k_1^3/2 + k_1\Lambda_{20} + J_{r1}\left(-3k_1^2/2 + \Lambda_{20} + dk_1\right) + J_{r2}\left(-3k_1/2 + d\right)
\end{aligned}
$$

$$
\begin{aligned}
&+n_2\left\{-3dk_1^2/2+2d^2k_1+dk_1+J_{r1}\left(-3dk_1+2d^2\right)-3J_{r2}d/2\right\}\\
&+n_2^2\left(d^3-3d^2k_1/2-2J_{r1}d^2\right)-n_2^3d^3/2,\\
x_{33}^{'} \;=\;& d^3/6+J_{r1}d^2/2+J_{r2}d/2+n_2\left(dk_1^2/2+J_{r1}dk_1+J_{r2}d/2\right)\\
&+n_2^2d^2\left(k_1+J_{r1}\right)/2+n_2^3d^3/6,
\end{aligned}
$$

where $\alpha_n = k_1 + dn_2$.

So, one obtains the following.

COROLLARY 3.2. *For Y_{nr} of Corollary 3.1,*

$$
\begin{aligned}
var\left(Y_{nr}^{\varphi}\right) \;=\;& n_1^{-2}\theta^2\dot{\psi}(r)+n_1^{-3}\Big[2\theta^2\Big\{J_{r2}\left(d-k_1\right)+J_{r1}^2\left(k_1^2-k_1d\right)+J_{r1}J_{r2}/2\\
&-n_2d\dot{\psi}(r)-n_2^2J_{r1}d^2/2\Big\}+\theta^3\Big\{d^3/2+3J_{r1}d^2/2+3J_{r2}d/2\\
&-\left(k_1+J_{r1}\right)\left(k_1^2/2+J_{r1}k_1+J_{r2}/2\right)+n_2d\dot{\psi}(r)\Big\}\Big]\\
&+O\left(n_1^{-4}\right). \qquad (3.23)
\end{aligned}
$$

For $\theta = 1$ the leading term in (3.23) follows from Theorem 4.4 of Withers and Nadarajah (2007).

For Example 2.3, the unit normal, we only use the asymptotic expansion for $x > 0$, that is $0 < u < 1$, $0 < v < \log 2 =: \gamma$ say. So, the integral in (3.4) is broken into say

$$
\begin{aligned}
&\left\{\int_0^{\gamma} H_r(v)+\int_{\gamma}^{\infty} H_r(v)\right\}\exp(-nv)dv\\
=\;& \int_0^{\gamma} H_r(v)\exp(-nv)dv+\delta_{rn}n_2^{-r}2^{-n}, \qquad (3.24)
\end{aligned}
$$

where δ_{rn} is bounded as $n \to \infty$. So, (3.16) neglects a term of magnitude 2^{-n}.

More generally suppose $p = P(X > 0) > 0$ and $Z = |X|$ has distributions F_Z given by (2.12). Then for $x > 0$, $1 - F_X(x) = P(X > x) = p(1 - F_Z(x))$ so that F_X satisfies (2.12) with the same parameters as F_Z except that k is replaced by pk. So, the results of this section hold for F_X with this change. As in (3.24), (3.17) neglects a term of magnitude $O(2^{-n})$.

4 Discussion

For most businesses, the reward given a random sample $X_1, X_2, \ldots, X_n$ can be expressed as $M_{n1} - cn$ for some constant c. However, for most distributions, the optimal fixed sample size, $n = n_0$ say, for maximizing the expected reward, $E(M_{n1} - cn)$, cannot be found analytically. This has led to the development of various approximations for the optimal fixed sample size expected reward, $E(M_{n_01} - cn_0)$. Unfortunately, all of the known approximations are sample based and can be computationally expensive. See Liu (2001) for a recent work, where $X_1, X_2, \ldots, X_n$ are taken to be gamma random variables.

The results in this note can be used to derive explicit approximations for the optimal fixed sample size. Note that exponential power type distributions and in particular gamma distributions are popular models for optimal stopping problems of the above kind (Gupta and Groll, 1961). Clearly, the results of this note hold for gamma distributions. See Example 2.1.

The results of this note can be extended using Lemma 2.1 of Withers and Nadarajah (2007) to obtain similar expansions for the multivarite moments of $(M_{n1}, \ldots, M_{nr})$. However, the level of complexity is one step higher than in Withers and Nadarajah (2007) as we choose to work not with the variable u in (1.2), but with the variable $v = \log u$ in order to obtain all the terms in the expansions, and not just the leading terms given by the asymptotic saddlepoint expansion one would use if working with the variable u. This does perhaps make the formula for the coefficients in (1.3) more cumbersome.

Throughout Sections 2 and 3, we have used various infinite series and iterated infinite sums without checking their convergence. We now discuss some conditions for convergence for the main infinite sums (excluding those appearing in the lemmas and their proofs). By the Cauchy Hadamard theorem, we have the following:

- (2.3) converges absolutely for $1/u_1 < \omega = \lim_{i\to\infty} \mid M_i(u_2)/M_{i+1}(u_2) \mid$, where the limit is assumed to exist;
- (2.4) converges absolutely for $1/u_1 < \omega = \lim_{i\to\infty} \mid d_i(u_2)/d_{i+1}(u_2) \mid$, where the limit is assumed to exist;
- (2.5) converges absolutely for
$$1/u_1 < \omega = \lim_{j\to\infty} \left| \widehat{C}_j\left(-i, 1, \mathbf{M}\right) / \widehat{C}_{j+1}\left(-i, 1, \mathbf{M}\right) \right|,$$
where the limit is assumed to exist;
- (2.8) converges absolutely for
$$1/u_1 < \omega = \lim_{j\to\infty} \left| \widehat{C}_j\left(\theta, 1, \mathbf{M}\right) / \widehat{C}_{j+1}\left(\theta, 1, \mathbf{M}\right) \right|,$$
where the limit is assumed to exist;
- (2.13) converges absolutely for
$$1/u_1 < \omega = \lim_{j\to\infty} \left| \sum_{i=0}^{j} u_2^i c_{ji}(\theta) / \sum_{i=0}^{j+1} u_2^i c_{j+1,i}(\theta) \right|,$$
where the limit is assumed to exist;

- (2.14) converges absolutely everywhere if $c_{j,j+1} \neq 0$ for all j;
- (3.1) converges absolutely for $1/n < \omega = \lim_{i\to\infty} \mid c_i(r)/c_{i+1}(r) \mid$, where the limit is assumed to exist;
- (3.19) converges absolutely for $1/n_1 < \omega = \lim_{b\to\infty} \mid S_{nb}/S_{n,b+1} \mid$, where the limit is assumed to exist;
- (3.20) converges absolutely everywhere if $g_{b,b+1} \neq 0$ for all b.

We were unable to obtain the conditions for the iterated infinite sum given by (3.18).

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A Some Asymptotes

Suppose $F(x) = 1 - \exp\{-G(x)\}$, where $G(x) \uparrow \infty$ as $x \uparrow \infty$ with right-continuous inverse, G continuous and $g(u) = G^{-1}(u)$. Let M_n be the maximum of a random sample of size n from F. Then $EM_n^\theta = \int x^\theta dF(x)^n = \int g(u_1)^\theta d\exp(-t)$, where $t = -\log u$, $u = F(x)$, $u_1 = -\log(1-u) = -\log(1-\exp(-t/n)) = n_1 + O(t_1)$ as $u \to 1$ and $t_1 = -\log t \to \infty$. So, if

$$g\left(n_1 + O\left(t_1\right)\right)/g\left(n_1\right) = 1 + O\left(\varepsilon_n a\left(t_1\right)\right) \tag{A.1}$$

as $n \to \infty$, $t_1 \to \infty$ with

$$\int_0^\infty \left|a\left(t_1\right)\right| \exp(-t)dt < \infty, \text{ and } \varepsilon_n \to 0 \tag{A.2}$$

then $EM_n^\theta = g(n_1)^\theta\{1 + O(\varepsilon_n)\}$.

Suppose $G(x) = (x/\sigma)^\alpha$, where $\sigma > 0$, $\alpha > 0$. Then $g(u) = \sigma u^{1/\alpha}$, $g(n_1) = \sigma n_1^{1/\alpha}$, so the left hand side of (A.1) is equal to $\{1 + O(t_1/n_1)\}^{1/\alpha}$, and (A.1), (A.2) hold with $\varepsilon_n = n_1^{-1}$, $a(t_1) = t_1$.

Suppose $G(x) = \exp(x)$. Then $g(u) = \log u$, $g(n_1) = n_2$ and (A.1), (A.2) hold with $a(t_1) = t_1$, $\varepsilon_n = n_1^{-1} n_2^{-1}$.

Suppose $G(x) = \exp\{\exp(x)\}$. Then $g(u) = \log\log u$, $g(n_1) = \log n_2 = n_3$ say and (A.1), (A.2) hold with $\varepsilon_n = n_1^{-1} n_2^{-1} n_3^{-1}$, $a(t_1) = t_1$.

Similarly, by induction if $f \circ g(x) = f(g(x))$ and $G(x) = \exp \circ \cdots \circ \exp(x)$ with exp appearing $k \geq 1$ times, then $g(u) = \log \circ \cdots \circ \log(u) = g_k(u)$ say, $g(n_1) = n_{k+1}$ say, and (A.1), (A.2) hold with $\varepsilon_n = (n_1 \cdots n_{k+1})^{-1} = \varepsilon_{nk}$ say, and $a(t_1) = t_1$. So, for n large M_n may behave as n_{k+1} for any k. So, if $G(x) = \exp \circ \cdots \circ \exp(x/\sigma)^\alpha$, where $\sigma > 0$, $\alpha > 0$ and exp appears k times, then (A.1), (A.2) hold with $a(t_1) = t_1$, $\varepsilon_n = \varepsilon_{nk}^{1/\alpha}$ and $g(u) = \sigma g_k(u)^{1/\alpha}$, so M_n behaves like $\sigma n_{k+1}^{1/\alpha}$. A similar conclusion holds for M_{nr}.

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