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Some Results on DDCRE Class of Life Distributions

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Abstract

Asadi and Zohrevand (2007). On the dynamic cumulative residual entropy. J. Statist. Plann. Inference, 137, 1931–1941] define the decreasing dynamic cumulative residual entropy (DDCRE) class of life distributions, some properties of the DDCRE class are studied. Navarro et al. (2010). Some new results on the cumulative residual entropy. J. Statist. Plann. Inference, 140, 310–322] further investigate this class, they get some results concerning the relations between the DDCRE class and other classes of distributions. In the present paper some characterization properties of the DDCRE class are investigated, closure and reversed closure properties of this class are obtained.

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1 Introduction and Preliminaries

Let X be an absolutely continuous nonnegative random variable representing the random lifetime of a unit or a system. Assume that X has the probability density function $f_X(x)$, distribution function $F_X(x)$, survival function $\overline{F}_X(x) = 1 - F_X(x)$, respectively. The right continuous inverse function of F_X is defined by $F_X^{-1}(p) = \sup\{x|F_X(x) \leq p\}$, for all $0 \leq p \leq 1$.

The residual life of X is defined by $X_t = [X - t\,|X > t]$, for all $t \geq 0$. Then X_t has the survival function $\overline{F}_{X_t}(x) = \overline{F}_X(x+t)\big/\overline{F}_X(t)$, and the density function $f_{X_t}(x) = f_X(x+t)\big/\overline{F}_X(t)$. Thus the mean residual life of X at time $t \geq 0$ is given by

$$\mu_X(t) = E[X_t] = \frac{1}{\overline{F}_X(t)} \int_t^{+\infty} \overline{F}_X(x)\mathrm{d}x, \quad \text{for all } t \geq 0. \tag{1.1}$$

A classical measure of uncertainty for X is the Shannon differential entropy, defined by

$$H_X = -\mathrm{E}[\ln f_X(X)] = -\int_0^{+\infty} f_X(x) \ln f_X(x)\mathrm{d}x. \tag{1.2}$$

Since the classical contributions by Shannon (1948) and Wiener (1961), the properties of H_X have been investigated in detail. Subsequently, Ebrahimi and Pellerey (1995), Ebrahimi (1996), Ebrahimi and Kirmani (1996), Di Crescenzo and Longobardi (2002), and Navarro, del Aguila and Asadi (2010) investigated the differential entropy. Furthermore, numerous generalizations of (1.2) have been proposed (see, for instance, Taneja, 1990; Di Crescenzo and Longobardi, 2006; Kumar and Taneja, 2011; Khorashadizadeh, Rezaei Roknabadi and Mohtashami Borzadaran, 2013 and the references therein).

According to Ebrahimi (1996), the residual entropy of X at time $t \geq 0$ is defined as the differential entropy of X_t. Formally, for all $t \geq 0$ the residual entropy of X is given by

$$H_X(t) = H_{X_t} = -\mathrm{E}[\ln f_{X_t}(X)].$$

Di Crescenzo and Longobardi (2002) and Navarro et al. (2010) showed that

$$H_X(t) = -\int_t^{+\infty} \frac{f_X(x)}{\overline{F}_X(t)} \ln \frac{f_X(x)}{\overline{F}_X(t)} \mathrm{d}x \tag{1.3}$$

$$= \ln \overline{F}_X(t) - \frac{1}{\overline{F}_X(t)} \int_t^{+\infty} f_X(x) \ln f_X(x) \mathrm{d}x. \tag{1.4}$$

Provided that a device survived up to time t, then $H_X(t) = H_{X_t}$ measures the uncertainty of the residual life X_t. Various results concerning $H_X(t)$ have been obtained in Ebrahimi and Pellerey (1995), Ebrahimi and Kirmani (1996), Asadi and Ebrahimi (2000), and Asadi and Zohrevand (2007).

Recently, Sunoj and Sankaran (2012) and Sunoj, Sankaran and Nanda (2013) introduced a quantile version of the residual entropy. By means of the properties of $H_X\left[F_X^{-1}(p)\right]$, they proposed two new nonparametric classes of distributions and studied their properties.

As an alternative measure of uncertainty, Rao et al. (2004) defined a new uncertainty measure, the cumulative residual entropy (CRE), through

$$\mathcal{E}_X = -\int_0^{+\infty} \overline{F}_X(x) \ln \overline{F}_X(x) \mathrm{d}x. \tag{1.5}$$

Subsequently, Asadi and Zohrevand (2007) considered the corresponding dynamic measure of uncertainty, they introduced the dynamic cumulative residual entropy (DCRE), defined as the CRE of X_t. Recently, Navarro et al. (2010) investigated in detail the cumulative residual entropy, dynamic cumulative residual entropy, and dynamic cumulative past entropy based on the definition of CRE in (1.5).

The DCRE is defined by

$$\mathcal{E}_X(t) = -\int_t^{+\infty} \frac{\overline{F}_X(x)}{\overline{F}_X(t)} \ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} \mathrm{d}x. \tag{1.6}$$

That is,

$$\mathcal{E}_X(t) = \mathcal{E}_{X_t} = -\int_0^{+\infty} \overline{F}_{X_t}(x) \ln \overline{F}_{X_t}(x) \mathrm{d}x. \tag{1.7}$$

It is clear that $\mathcal{E}_X(0) = \mathcal{E}_X$. Asadi and Zohrevand (2007) showed that

$$\mathcal{E}_X(t) = \mu_X(t) \ln \overline{F}_X(t) - \frac{1}{\overline{F}_X(t)} \int_t^{+\infty} \overline{F}_X(x) \ln \overline{F}_X(x) \mathrm{d}x. \tag{1.8}$$

Based on the monotonicity of the function $\mathcal{E}_X(t)$, they proposed the following two nonparametric classes of life distributions.

DEFINITION 1.1 (Asadi and Zohrevand, 2007). *A random variable X is said to be increasing (decreasing) DCRE, denoted by IDCRE (DDCRE), if $\mathcal{E}_X(t)$ is an increasing (decreasing) function of t.*

Next we give the definitions of some aging notions of life distributions.

DEFINITION 1.2 (Shaked and Shanthikumar, 2007). *Let X be an absolutely continuous nonnegative random variable with distribution function F_X. X (or F_X) is said to be:*

(1) *IFR [increasing failure rate] (DFR [decreasing failure rate]) if $\ln\overline{F}_X(x)$ is concave (convex).*

(2) *NBU [new better than used] (NWU [new worse than used]) if*

$$\overline{F}_X(x+t) \le \overline{F}_X(t) \cdot \overline{F}_X(x), \quad \textit{for all } x, t \ge 0.$$

(3) *DMRL [decreasing mean residual life] (IMRL [increasing mean residual life]) if the mean residual life $\mu_X(t)$ of X is decreasing (increasing) in $t \ge 0$.*

In reliability theory, economics, management science, information sciences, and other related areas, many ageing concepts and lifetime distribution classes are presented and studied earlier or later, many authors did their researches in the directions of their interests. One can refer to Asadi and Ebrahimi (2000), Belzunce et al. (2004), Li and Zuo (2004), Li and Zhang (2011), Barlow and Proschan (1981), Shaked and Shanthikumar (1994, 2007) and the references therein.

The following lemma taken from Barlow and Proschan (1981) is useful in the sequel.

LEMMA 1.1. *Let W be a measure on the interval (a,b), not necessarily nonnegative, where $-\infty \le a < b \le +\infty$. Let h be a nonnegative function defined on (a,b). If $\int_t^b dW(x) \ge 0$ for all $t \in (a,b)$ and if h is increasing, then $\int_a^b h(x)dW(x) \ge 0$.*

Throughout this paper, the term *increasing* stands for monotone nondecreasing and *decreasing* stands for monotone nonincreasing. Assume that the integrals involved are finite and all ratios are well-defined whenever written.

In the present paper we mainly focus our interest on the closure properties of the DDCRE class of life distributions. The rest of the paper is organized as follows. In Section 2, we investigate some characterizations of the DDCRE class. In section 3 we deal with the closure and reversed closure properties of this class.

2 Characterizations

In this section, we investigate some characterizations of the DDCRE class. First we need a lemma taken from Asadi and Zohrevand (2007) and Navarro et al. (2010).

LEMMA 2.1. *X is DDCRE if, and only if, $\mathcal{E}_X(t) \le \mu_X(t)$ for all $t \ge 0$.*

In the following we give an important result, which will be used repeatedly in proofs of the main results.

THEOREM 2.1. *X is DDCRE if, and only if,*

$$\int_t^{+\infty} \overline{F}_X(x) \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] dx \ge 0, \quad \textit{for all}\, t \ge 0. \tag{2.1}$$

PROOF. From Lemma 2.1 we have X is DDCRE if, and only if,

$$\mathcal{E}_X(t) = -\int_t^{+\infty} \frac{\overline{F}_X(x)}{\overline{F}_X(t)} \ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} \mathrm{d}x \le \mu_X(t) = \frac{1}{\overline{F}_X(t)} \int_t^{+\infty} \overline{F}_X(x)\mathrm{d}x \tag{2.2}$$

for all $t \ge 0$. Rewriting (2.2) as (2.1), the proof is complete.

From (1.1) and (1.6) the following result is obvious.

LEMMA 2.2. *For any nonnegative random variable X, let $Y = aX + b$, where $a > 0$ and $b \ge 0$ are constants. Then for $t > b$,*

$$\mu_Y(t) = a\mu_X\left(\frac{t-b}{a}\right), \quad \mathcal{E}_Y(t) = a\mathcal{E}_X\left(\frac{t-b}{a}\right). \tag{2.3}$$

The following theorem considers the variation behavior of the DDCRE class under increasing linear transformations. By Lemma 2.1 and Lemma 2.2 the proof is obvious and hence omitted.

THEOREM 2.2. *Let X be nonnegative continuous random variable, $a > 0$ and $b \geq 0$ be constants. Then $aX + b$ is also DDCRE if X is.*

REMARK 2.1. *Theorem 2.2 indicates that the DDCRE class has closure property under the positive linear transformations.*

The following corollary is a direct consequence of Theorem 2.2.

COROLLARY 2.1. *Let X be nonnegative continuous random variable, then for $a > 0$ and $b \geq 0$,*

(1) *aX is also DDCRE if X is.*

(2) *$X + b$ is also DDCRE if X is.*

THEOREM 2.3. *Let X be an exponential random variable with mean μ_X, then X is DDCRE.*

PROOF. Suppose that $X \sim E(1/\mu_X)$, then $\overline{F}_X(x) = e^{-x/\mu_X}$, $x \geq 0$. Due to the memorylessness of the exponential random variable, we have

$$\overline{F}_{X_t}(x) = \overline{F}_X(x), \quad -\ln \overline{F}_{X_t}(x) = -\ln \overline{F}_X(x) = x\,/\mu_X$$

for all $x, t \geq 0$. From equation (1.7),

$$\mathcal{E}_X(t) = \mathcal{E}_{X_t} = -\int_0^{+\infty} \overline{F}_{X_t}(x) \ln \overline{F}_{X_t}(x) \mathrm{d}x$$

$$= -\int_0^{+\infty} \overline{F}_X(x) \ln \overline{F}_X(x) \mathrm{d}x = \int_0^{+\infty} \frac{x}{\mu_X} e^{-x/\mu_X} \mathrm{d}x = \mu_X = \mu_X(t).$$

From Lemma 2.1, X is DDCRE. As claimed.

DEFINITION 2.1. *Let X be a nonnegative random variable with distribution function F_X.*

(1) *If $\mathcal{E}_X(t) \geq \mathcal{E}_X$ for all $t \geq 0$, then X (or F_X) is said to be NBUDR [new better than used in DCRE].*

(2) *If $\mathcal{E}_X(t) \leq \mathcal{E}_X$ for all $t \geq 0$, then X (or F_X) is said to be NWUDR [new worse than used in DCRE].*

THEOREM 2.4. *Let X be an exponential random variable with mean μ_X, then X is both NBUDR and NWUDR.*

PROOF. If $X \sim E(1/\mu_X)$, then $\overline{F}_{X_t}(x) = \overline{F}_X(x)$, so $\mathcal{E}_X(t) = \mathcal{E}_X$, hence X is both NBUDR and NWUDR by Definition 2.1, as claimed.

Navarro et al. (2010) showed that the following implications hold:

$$\text{IFR} \Longrightarrow \text{DMRL} \Longrightarrow \text{DDCRE}. \tag{2.4}$$

THEOREM 2.5. *If X is DDCRE, then X is NWUDR.*

PROOF. If X is DDCRE, then $\mathcal{E}_X(t)$ is decreasing in $t \geq 0$, thus we have, for all $t \geq 0$,

$$\mathcal{E}_X(t) \leq \mathcal{E}_X(0) = \mathcal{E}_X.$$

Using Definition 2.1 (2), we see that X is NWUDR, as claimed.

So far, we have lengthened the implication chains of (2.4) established by Navarro et al. (2010):

$$\text{DMRL} \Longrightarrow \text{DDCRE} \Longrightarrow \text{NWUDR}. \tag{2.5}$$

Now we consider a proportional failure rate model. Let X be a nonnegative random variable with survival function $\overline{F}_X$. For $\theta > 0$, let $X(\theta)$ denote another random variable with survival function $\left(\overline{F}_X\right)^\theta$. Suppose that X has 0 as the left endpoint of its support. Then we have the following results.

THEOREM 2.6. *Let X and $X(\theta)$ be described as above.*

(1) *Assume $\theta \geq 1$. If $X(\theta)$ is DDCRE, then X is also DDCRE.*

(2) *Assume $0 < \theta \leq 1$. If X is DDCRE, then $X(\theta)$ is also DDCRE.*

PROOF. Since $\overline{F}_{X(\theta)}(x) = \left[\overline{F}_X(x)\right]^\theta$, from (2.1) we have that X is DDCRE if, and only if,

$$\int_t^{+\infty} \overline{F}_X(x) \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0, \tag{2.6}$$

and that $X(\theta)$ is DDCRE if, and only if, for all $t \geq 0$,

$$\int_t^{+\infty} \overline{F}_{X(\theta)}(x) \cdot \left[\ln \frac{\overline{F}_{X(\theta)}(x)}{\overline{F}_{X(\theta)}(t)} + 1\right] \mathrm{d}x = \int_t^{+\infty} \left[\overline{F}_X(x)\right]^\theta \cdot \left[\theta \ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x \geq 0. \tag{2.7}$$

(1) Assume that $X(\theta)$ is DDCRE. If $\theta \geq 1$, since $\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} \leq 0$, then

$$\int_t^{+\infty} \left[\overline{F}_X(x)\right]^\theta \cdot \left[\theta \ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x \leq \int_t^{+\infty} \left[\overline{F}_X(x)\right]^\theta \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x. \tag{2.8}$$

From (2.7) and (2.8) we have

$$\int_t^{+\infty} \left[\overline{F}_X(x)\right]^\theta \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x \geq 0. \tag{2.9}$$

Since the function $h(x) = \left[\overline{F}_X(x)\right]^{1-\theta}$ is nonnegative increasing in $x \geq 0$ when $\theta \geq 1$. From Lemma 1.1 and (2.9), we see that the inequality (2.6) is valid. That is, X is DDCRE.

(2) Assume that X is DDCRE. Since the function $h(x) = \left[\overline{F}_X(x)\right]^{\theta-1}$ is nonnegative increasing in $x \geq 0$ when $0 < \theta \leq 1$, from Lemma 1.1 and (2.6) we have

$$\int_t^{+\infty} \left[\overline{F}_X(x)\right]^\theta \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{2.10}$$

Moreover, for $0 < \theta \leq 1$,

$$\int_t^{+\infty} \left[\overline{F}_X(x)\right]^\theta \cdot \left[\theta \ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x \geq \int_t^{+\infty} \left[\overline{F}_X(x)\right]^\theta \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x. \tag{2.11}$$

By inequalities (2.10), (2.11) and (2.7) we see that $X(\theta)$ is DDCRE.

3 Closure and Reversed Closure Properties

In this section, we deal with the closure and reversed closure properties of the DDCRE class.

Let X be a nonnegative continuous random variable with distribution function F_X and survival function $\overline{F}_X$, let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) copies of X. Denote by

$$X_{1:n} = \min\{X_1, \ldots, X_n\}, \quad X_{n:n} = \max\{X_1, \ldots, X_n\}.$$

Denote the distribution function and the survival function of $X_{1:n}$ by $F_{X_{1:n}}$ and $\overline{F}_{X_{1:n}}$, respectively. Similarly, $F_{X_{n:n}}$ and $\overline{F}_{X_{n:n}}$ for $X_{n:n}$. Now we investigate the closure and the reversed closure properties of the DDCRE class. We have the following result.

THEOREM 3.1. *If X is DDCRE, then $X_{n:n}$ is also DDCRE.*

PROOF. Suppose that X is DDCRE. Then from (2.1) we have

$$\int_t^{+\infty} \overline{F}_X(x) \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{3.1}$$

It is easy to see that, for all $x \geq 0$,

$$\overline{F}_{X_{n:n}}(x) = 1 - [F_X(x)]^n = \overline{F}_X(x) \cdot \sum_{i=1}^{n} [F_X(x)]^{i-1}.$$

Since the function $h(x) = \sum_{i=1}^{n} [F_X(x)]^{i-1}$ is nonnegative increasing in $x \geq 0$, from (3.1) and Lemma 1.1 we have

$$\int_t^{+\infty} \overline{F}_X(x) \cdot \sum_{i=1}^{n} [F_X(x)]^{i-1} \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x \geq 0 \tag{3.2}$$

for all $t \geq 0$. Moreover,

$$\int_t^{+\infty} \overline{F}_{X_{n:n}}(x) \cdot \left[\ln \frac{\overline{F}_{X_{n:n}}(x)}{\overline{F}_{X_{n:n}}(t)} + 1\right] \mathrm{d}x =$$

$$\int_t^{+\infty} \overline{F}_X(x) \cdot \sum_{i=1}^{n} [F_X(x)]^{i-1} \left[\ln \left(\frac{\overline{F}_X(x)}{\overline{F}_X(t)} \cdot \frac{\sum_{i=1}^{n} [F_X(x)]^{i-1}}{\sum_{i=1}^{n} [F_X(t)]^{i-1}}\right) + 1\right] \mathrm{d}x$$

$$\geq \int_t^{+\infty} \overline{F}_X(x) \cdot \sum_{i=1}^{n} [F_X(x)]^{i-1} \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x. \tag{3.3}$$

From (3.2) and (3.3) we obtain

$$\int_t^{+\infty} \overline{F}_{X_{n:n}}(x) \cdot \left[\ln \frac{\overline{F}_{X_{n:n}}(x)}{\overline{F}_{X_{n:n}}(t)} + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0,$$

again by (2.1), which states that $X_{n:n}$ is DDCRE.

REMARK 3.1. *Theorem 3.1 indicates that the DDCRE class has the closure property under the parallel operation. Theorem 3.1 also says that the DDCRE class has the closure property with respect to the parallel systems.*

THEOREM 3.2. *If $X_{1:n}$ is DDCRE, then X is also DDCRE.*

The proof of Theorem 3.2 is similar to that of Theorem 3.1 and hence omitted.

REMARK 3.2. *Theorem 3.2 indicates that the DDCRE class has the reversed closure property under the series operation. Theorem 3.2 also says that the DDCRE class has the reversed closure property with respect to the series systems.*

Let X be a nonnegative absolutely continuous random variable having distribution function F_X such that $F_X(0) = 0$, let $X_1, X_2, \dots$, be a sequence of i.i.d. copies of X. Assume that N is a positive integer-valued random variable independent of X_i's, and N has a probability mass function $p_N(n) = P\{N = n\}$, $n = 1, 2, \dots$. Denote by

$$X_{1:N} = \min\{X_1, \dots, X_N\}, \quad X_{N:N} = \max\{X_1, \dots, X_N\}.$$

Then $X_{1:N}$ and $X_{N:N}$ have the survival functions, respectively,

$$\overline{F}_{X_{1:N}}(x) = \sum_{n=1}^{+\infty} \left[\overline{F}_X(x)\right]^n p_N(n) = \overline{F}_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\overline{F}_X(x)\right)^{n-1} p_N(n)\right] \tag{3.4}$$

and

$$\overline{F}_{X_{N:N}}(x) = 1 - \sum_{n=1}^{+\infty} \left[F_X(x)\right]^n p_N(n) = \overline{F}_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left[\sum_{i=1}^{n} \left(F_X(x)\right)^{i-1}\right] p_N(n)\right]. \tag{3.5}$$

Now we consider to extend the results in Theorem 3.1 and Theorem 3.2 from a finite number n to a random number N. The following Theorem 3.3 can be viewed as an extension of Theorem 3.1.

THEOREM 3.3. *If X is DDCRE, then $X_{N:N}$ is also DDCRE.*

The proof of Theorem 3.3 is similar to that of following Theorem 3.4 and hence omitted.

REMARK 3.3. *Theorem 3.3 indicates that the DDCRE class has the closure property under the random parallel operation. Theorem 3.3 also says that the DDCRE class has the closure property with respect to the random parallel systems.*

The following Theorem 3.4 can be viewed as an extension of Theorem 3.2.

THEOREM 3.4. *If $X_{1:N}$ is DDCRE, then X is also DDCRE.*

PROOF. Suppose that $X_{1:N}$ is DDCRE. Then from (2.1) we have

$$\int_t^{+\infty} \overline{F}_{X_{1:N}}(x) \cdot \left[\ln \frac{\overline{F}_{X_{1:N}}(x)}{\overline{F}_{X_{1:N}}(t)} + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{3.6}$$

From (3.4),

$$\overline{F}_{X_{1:N}}(x) = \overline{F}_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\overline{F}_X(x)\right)^{n-1} p_N(n)\right], \tag{3.7}$$

then

$$\int_t^{+\infty} \overline{F}_{X_{1:N}}(x) \cdot \left[\ln \frac{\overline{F}_{X_{1:N}}(x)}{\overline{F}_{X_{1:N}}(t)} + 1\right] \mathrm{d}x = \int_t^{+\infty} \overline{F}_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\overline{F}_X(x)\right)^{n-1} p_N(n)\right] \cdot$$

$$\left[\ln\left(\frac{\overline{F}_X(x)}{\overline{F}_X(t)} \cdot \frac{\left[\sum_{n=1}^{+\infty} \left(\overline{F}_X(x)\right)^{n-1} p_N(n)\right]}{\left[\sum_{n=1}^{+\infty} \left(\overline{F}_X(t)\right)^{n-1} p_N(n)\right]}\right) + 1\right] \mathrm{d}x$$

$$\leq \int_t^{+\infty} \overline{F}_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\overline{F}_X(x)\right)^{n-1} p_N(n)\right] \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x. \tag{3.8}$$

Hence, from (3.6) and (3.8) we obtain that

$$\int_t^{+\infty} \overline{F}_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\overline{F}_X(x)\right)^{n-1} p_N(n)\right] \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x \geq 0 \tag{3.9}$$

for all $t \geq 0$. Since the function $1 \Big/ \left[\sum_{n=1}^{+\infty} \left(\overline{F}_X(x)\right)^{n-1} p_N(n)\right]$ is nonnegative increasing in $x \geq 0$, from (3.9) and Lemma 1.1 we have

$$\int_t^{+\infty} \overline{F}_X(x) \cdot \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0.$$

That is, X is DDCRE.

REMARK 3.4. *Theorem 3.4 indicates that the DDCRE class has reversed closure property under the random series operation. Theorem 3.4 also says that the DDCRE class has the reversed closure property with respect to the random series systems.*

Theorem 2.2 shows that the DDCRE class is closed under the increasing linear transformation, and vice versa. A question is naturally arising. We want to know whether or not the DDCRE class is closed or reversed closed under a general increasing transform. Unfortunately, this is not true in general. The following Theorem 3.5 and Theorem 3.6 verify these conjectures. However, under adding some growth condition for the involved increasing transform, for example, convexity or concavity, the closure or reversed closure of the DDCRE class holds under such increasing transform.

THEOREM 3.5. *Let ϕ be an increasing convex function defined on $[0, +\infty)$ such that the derivative of ϕ is continuous and that $\phi(0) = 0$. If X is DD-CRE, then $\phi(X)$ is also DDCRE.*

PROOF. Suppose that X is DDCRE. Then from (2.1) we have

$$\int_t^{+\infty} \overline{F}_X(x) \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1 \right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{3.10}$$

Since $\phi(x)$ is increasing convex implies that $\phi'(x)$ is nonnegative increasing, from (3.10) and Lemma 1.1 we have

$$\int_t^{+\infty} \overline{F}_X(x)\phi'(x) \left[\ln \frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1 \right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{3.11}$$

Also from (2.1), $\phi(X)$ is DDCRE if, and only if,

$$\int_t^{+\infty} \overline{F}_{\phi(X)}(x) \left[\ln \frac{\overline{F}_{\phi(X)}(x)}{\overline{F}_{\phi(X)}(t)} + 1 \right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{3.12}$$

Moreover, $\overline{F}_{\phi(X)}(x) = \overline{F}_X(\phi^{-1}(x))$, then

$$\int_t^{+\infty} \overline{F}_{\phi(X)}(x) \left[\ln \frac{\overline{F}_{\phi(X)}(x)}{\overline{F}_{\phi(X)}(t)} + 1 \right] \mathrm{d}x = \int_t^{+\infty} \overline{F}_X\left(\phi^{-1}(x)\right) \left[\ln \frac{\overline{F}_X\left(\phi^{-1}(x)\right)}{\overline{F}_X\left(\phi^{-1}(t)\right)} + 1 \right] \mathrm{d}x$$

$$= \int_t^{+\infty} \phi'(x)\overline{F}_X(x) \ln \left[\frac{\overline{F}_X(x)}{\overline{F}_X(t)} + 1 \right] \mathrm{d}x. \tag{3.13}$$

From (3.13) and (3.11) we see that inequality (3.12) holds, which asserts that $\phi(X)$ is DDCRE.

Let X be a nonnegative continuous random variable, and ϕ be a nonnegative increasing function defined on $[0, +\infty)$ with $\phi(0) = 0$. We call $\phi(X)$ as the generalized scale transform of X. If ϕ is increasing convex with the derivative ϕ' being continuous and with $\phi(0) = 0$, then ϕ is called a risk preference function, and $\phi(X)$ is called the risk preference transform of X. If ϕ is increasing concave with the derivative ϕ' being continuous and with $\phi(0) = 0$, then ϕ is called a risk aversion function, and $\phi(X)$ is called the risk aversion transform of X.

REMARK 3.5. *Theorem 3.5 says that the DDCRE class has closure property under the convex generalized scale transforms. Theorem 3.5 also indicates that the DDCRE class has closure property under the risk preference transforms.*

Using a similar manner to Theorem 3.5 we easily have the following theorem.

THEOREM 3.6. *Let ϕ be an increasing concave function with the derivative ϕ' being continuous such that $\phi(0) = 0$. If $\phi(X)$ is DDCRE, then X is also DDCRE.*

REMARK 3.6. *Theorem 3.6 says that the DDCRE class has the reversed closure property under the concave generalized scale transforms. In other words, Theorem 3.6 also indicates that the DDCRE class has reversed closure property under the risk aversion transforms.*

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