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# Some Aging Properties Involved with Compound Geometric Distributions

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**Abstract**

In this article we study stochastic monotone properties of the deficit at ruin in terms of the increasing convex (concave) order. Also, we conduct comparisons on the extended deficit at ruin in the sense of the usual stochastic order and expectation. Additionally, the increasing convex (concave) order between the deficit at ruin and the amount of every drop in surplus is presented as well.

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## 1 Introduction

Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. copies of the nonnegative random variable $X$ with distribution function $F(x) = 1 - \bar{F}(x)$ satisfying $F(0) = 0$, and $N$, independent of $\{X_i, i \geq 1\}$, is a geometrically distributed counting random variable with probability mass function

$$\mathrm{P}(N = n) = \alpha^n (1 - \alpha), \qquad \alpha \in (0, 1),\, n = 0, 1, 2, \cdots .$$

Then, the *geometric convolution* $S_N = X_1 + X_2 + \cdots + X_N$ has distribution function

$$G(x) = (1 - \alpha) \sum_{n=0}^{\infty} \alpha^n F^{(n)}(x), \qquad x \geq 0,$$

and $S_N$ has survival function

$$\bar{G}(x) = (1 - \alpha) \sum_{n=1}^{\infty} \alpha^n \bar{F}^{(n)}(x), \qquad x \geq 0,$$

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where $F^{(n)}(x)$ is the $n$-fold convolution of $F$ and $F^{(0)}(x) \equiv 1$ for $x \geq 0$. The compound geometric distribution has significant applications in many fields. For example, this distribution characterizes the maximal aggregate loss of the surplus process in ruin theory (see Kaas et al., 2008) and the limiting distribution of the waiting time in a classic single-server queueing model with infinite waiting room (see Kalashnikov, 1997).

Let $F : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a generalized distribution function, that is, $F(x) = cV(x)$ where $c \in (0, \infty)$ and $V$ is the distribution function of a proper nonnegative real random variable satisfying $V(\infty) = 1$. According to Brémaud (2014), for $\upsilon : \mathbb{R}_+ \mapsto \mathbb{R}$, the renewal function $m(x)$ satisfies the *extended version of the classical renewal equation*

$$m(x) = \int_0^x m(x-t)\,\mathrm{d}F(t) + \upsilon(x), \qquad x \geq 0.$$

which is said to be *proper* when $F(\infty) = 1$, *defective* when $F(\infty) < 1$ and *excessive* when $F(\infty) > 1$. In age-dependent branching process, the expected number of organisms alive at a certain time satisfies the renewal equation (see Ross, 1996), and in actuarial science, Lin and Willmot (1999) studied a defective renewal equation concerned about the time to ruin, the immediate surplus before ruin, and the deficit at the ruin.

Noted that a compound geometric distribution may be viewed as the solution of a special defective renewal equation, Willmot and Lin (2001) explored a close connection between defective renewal equations and compound geometric distributions $G$:

$$\bar{G}(x) = \alpha \int_0^x \bar{G}(x-t)\,\mathrm{d}F(t) + \alpha \bar{F}(x), \qquad x \geq 0. \tag{1.1}$$

To derive reliability properties of the compound geometric distribution $G$, Willmot (2002) introduced $\bar{G}(x,y)$ with $\bar{G}(x,0) = \bar{G}(x)$ to satisfy the defective renewal equation

$$\bar{G}(x,y) = \alpha \int_0^x \bar{G}(x-t,y)\,\mathrm{d}F(t) + \alpha \bar{F}(x+y), \qquad x \geq 0,\ y \geq 0. \tag{1.2}$$

Define $G(x,y) = \bar{G}(x) - \bar{G}(x,y)$. Furthermore, for any $x \geq 0$,

$$\bar{G}_x(y) = 1 - G_x(y) = \frac{\bar{G}(x,y)}{\bar{G}(x)}$$

is a proper survival function of some nonnegative random variable, say $S_{N,x}$. Let $H(x) = 1 - \bar{H}(x)$ with $H(0) = 0$ be the distribution function of another nonnegative random variable $Y$, independent of $S_N$. Then, the *compound geometric convolution* $S_W = S_N + Y$ has the survival function

$$\bar{W}(x) = \bar{H}(x) + \int_0^x \bar{G}(x-t)\,\mathrm{d}H(t), \qquad x \geq 0. \tag{1.3}$$

Due to nice applications in various areas, the compound geometric convolution has attracted wide attention in the literature. For example, in the M/G/c queue, the approximate equilibrium waiting time has a compound geometric convolution distribution given that there is indeed a wait (See Van Hoorn, 1984), and in the classical model of collective risk theory, adding a diffusion process to the surplus process also makes the probability of ultimate survival have a compound geometric convolution distribution (See Dufresne and Gerber, 1991). As one extension, Willmot and Cai (2004) introduced

$$\bar{K}(x,y) = \bar{H}(x+y) + \int_0^x \bar{G}(x-t,y)\,\mathrm{d}H(t), \qquad x \geq 0,\ y \geq 0, \tag{1.4}$$

and proved that $\bar{K}(x,y)$ satisfies

$$\bar{K}(x,y) = \bar{H}(x+y) + \frac{\alpha}{1-\alpha}\int_0^x \bar{F}(x+y-t)\,\mathrm{d}W(t), \tag{1.5}$$

and

$$(1-\alpha)\bar{K}(x,y) = \bar{W}(x+y) - \alpha\bar{W}(x)\bar{F}(y) - \alpha\int_0^y \bar{W}(x+y-t)\,\mathrm{d}F(t). \tag{1.6}$$

For any $x \geq 0$,

$$\bar{K}_x(y) = 1 - K_x(y) = \frac{\bar{K}(x,y)}{\bar{W}(x)}$$

is also a proper survival function of some nonnegative random variable, say $S_{W,x}$. In view of (1.1), (1.2), (1.3) and (1.4), setting $\bar{H}(x) = \bar{F}(x)$ yields $\bar{W}(x) = \bar{G}(x)/\alpha$, $\bar{K}(x,y) = \bar{G}(x,y)/\alpha$ and $\bar{K}_x(y) = \bar{G}_x(y)$. That is, the distribution function $K_x(y)$ is an extension of $G_x(y)$.

Now, let us review some stochastic orders and aging notions, for more, please refer to Abouammoh et al. (2000), Shaked and Shanthikumar (2007), Marshall and Olkin (2007), and Li and Li (2013). Let $X$ and $Y$ be two random variables with distribution functions $F$, $G$ and survival functions $\bar{F} = 1 - F$, $\bar{G} = 1 - G$, respectively. When $X$ is nonnegative, we denote $\tilde{X}$ the random variable with the equilibrium distribution of $X$, that is, $\tilde{X}$ has density function $\frac{\bar{F}(x)}{\mathsf{E}[X]}$ for $x \geq 0$.

Definition 1.1. *$X$ is said to be smaller than $Y$ in the*

*(i) usual stochastic order (denoted by $X \leq_{\mathrm{st}} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x$;*

*(ii) increasing concave order (denoted by $X \leq_{\mathrm{icv}} Y$) if $\int_{-\infty}^{x} F(u)\,\mathrm{d}u \geq \int_{-\infty}^{x} G(u)\,\mathrm{d}u$ for all $x$;*

*(iii) increasing convex order (denoted by $X \leq_{\mathrm{icx}} Y$) if $\int_{x}^{\infty} \bar{F}(u)\,\mathrm{d}u \leq \int_{x}^{\infty} \bar{G}(u)\,\mathrm{d}u$ for all $x$.*

Definition 1.2. *A nonnegative random variable $X$ or its distribution is said to be*

*(i) decreasing (increasing) failure rate (DFR (IFR)) if $-\log \bar{F}(x)$ is concave (convex) in $x \in [0, \infty)$ ($x \in \mathbb{R}$);*

*(ii) increasing (decreasing) mean residual life (IMRL (DMRL)) if $\mathsf{E}[X - t|X > t]$ is increasing (decreasing) in $t \geq 0$;*

*(iii) new worse (better) than used (NWU (NBU)) if $X \leq_{\mathrm{st}} (\geq_{\mathrm{st}})\, [X - t|X > t]$ for all $t \geq 0$;*

*(iv) new worse (better) than used [2] (NWU[2] (NBU[2])) if $X \leq_{\mathrm{icv}} (\geq_{\mathrm{icv}})\, [X - t|X > t]$ for all $t \geq 0$;*

*(v) new worse (better) than used in convex ordering (NWUC (NBUC)) if $X \leq_{\mathrm{icx}} (\geq_{\mathrm{icx}})\, [X - t|X > t]$ for all $t \geq 0$;*

*(vi) new worse (better) than used in expectation (NWUE (NBUE)) if $\mathsf{E}[X] \leq (\geq) \mathsf{E}[X - t|X > t]$ for all $t \geq 0$;*

*(vii) new renewal worse (better) than used in expectation (NRWUE (NRBUE)) if $\mathsf{E}[X - t|X > t] \geq (\leq)\, \mathsf{E}[\tilde{X}]$ for all $t \geq 0$.*

The rest of this paper is rolled out as follows: In Section 2, the increasing convex (concave) order on $S_{N,x}$ is proved to imply the NWU[2] (NWUC) aging properties of $X$, and the compound geometric distribution is shown not to be IFR. In Section 3, we derive the usual stochastic order and the order by expectation on $S_{W,x}$ and conduct comparison on $S_{W,x}$, $\tilde{X}$ and $\tilde{Y}$ in terms of the expectation. Also, the increasing convex (concave) order between $S_{N,x}$ and $X$ is presented as well.

Throughout this note, denote $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. All random variables are nonnegative, and expectations are implicitly assumed to be finite whenever they appear. The terms *increasing* and *decreasing* stand for non-decreasing and non-increasing, respectively.

## 2 Aging Properties

According to the proofs of Theorems 2.1 and 2.2 of Psarrakos (2010),

- $F$ is NWU (NBU) if $\bar{G}(x,y)/\bar{G}(x+y)$ and $\bar{G}(x+y)/\bar{G}(x)$ are increasing (decreasing) in $x$ for any $y \geq 0$, and
- $F$ is NWUE (NBUE) if $\mathsf{E}[S_{N,x_1}] \leq (\geq)\, \mathsf{E}[S_{N,x_2}]$ for all $x_2 \geq x_1 \geq 0$.

Naturally, one may wonder whether the monotone properties of $S_{N,x}$ with respect to $x$ in the sense of the increasing convex/concave order imply some aging properties of $F$ also. Our first two theorems serve as the positive answer.

THEOREM 2.1. *If $S_{N,x_1} \leq_{\text{icv}} (\geq_{\text{icv}})\, S_{N,x_2}$ for all $x_2 \geq x_1 \geq 0$, then $F$ is NWU[2] (NBU[2]).*

PROOF. Let us consider only NWU[2], NBU[2] property may be obtained in a similar manner.

From (1.1) and (1.2), it follows that, for $x, y \geq 0$,

$$\bar{G}(x) - \bar{G}(x,y) = \alpha \int_0^x \left[\bar{G}(x-t) - \bar{G}(x-t,y)\right] \mathrm{d}F(t) + \alpha\left[\bar{F}(x) - \bar{F}(x+y)\right].$$

By integrating with respect to $y \in [0,u]$ on both sides, we obtain, for $x, u \geq 0$,

$$\begin{aligned} \bar{G}(x)\int_0^u G_x(y)\,\mathrm{d}y &= \alpha \int_0^u \int_0^x \bar{G}(x-t) G_{x-t}(y)\,\mathrm{d}F(t)\,\mathrm{d}y \\ &\quad + \alpha \bar{F}(x) \int_0^u \left[1 - \frac{\bar{F}(x+y)}{\bar{F}(x)}\right] \mathrm{d}y. \end{aligned}$$

According to the assumption, we have $\int_0^u G_x(y)\,\mathrm{d}y$ is decreasing in $x \geq 0$ for any $u \geq 0$. So, it holds that, for $x, u \geq 0$,

$$\begin{aligned} \bar{G}(x)\int_0^u G_x(y)\,\mathrm{d}y &= \alpha \int_0^x \bar{G}(x-t) \int_0^u G_{x-t}(y)\,\mathrm{d}y\,\mathrm{d}F(t) \\ &\quad + \alpha \bar{F}(x) \int_0^u \left[1 - \frac{\bar{F}(x+y)}{\bar{F}(x)}\right] \mathrm{d}y \\ &\geq \alpha \int_0^x \bar{G}(x-t)\,\mathrm{d}F(t) \int_0^u G_x(y)\,\mathrm{d}y \\ &\quad + \alpha \bar{F}(x) \int_0^u \left[1 - \frac{\bar{F}(x+y)}{\bar{F}(x)}\right] \mathrm{d}y, \end{aligned}$$

and hence

$$\left[\bar{G}(x) - \alpha \int_0^x \bar{G}(x-t)\,\mathrm{d}F(t)\right] \int_0^u G_x(y)\,\mathrm{d}y \geq \alpha \bar{F}(x) \int_0^u \left[1 - \frac{\bar{F}(x+y)}{\bar{F}(x)}\right] \mathrm{d}y.$$

Taking (1.1) into account, we have, for $x, u \geq 0$,

$$\int_0^u G_x(y)\,\mathrm{d}y \geq \int_0^u \left[1 - \frac{\bar{F}(x+y)}{\bar{F}(x)}\right] \mathrm{d}y.$$

By the assumption once again, it holds that, for $x, u \geq 0$,

$$\int_0^u F(y)\,\mathrm{d}y = \int_0^u G_0(y)\,\mathrm{d}y \geq \int_0^u G_x(y)\,\mathrm{d}y \geq \int_0^u \left[1 - \frac{\bar{F}(x+y)}{\bar{F}(x)}\right] \mathrm{d}y.$$

As a consequence, it holds that

$$\int_{-\infty}^u F(y)\,\mathrm{d}y \geq \int_{-\infty}^u \mathrm{P}(X - x \leq y \mid X > x)\,\mathrm{d}y, \qquad \text{for all } u \text{ and } x \geq 0.$$

That is, $F$ is NWU[2].

Theorem 2.2. *If $S_{N,x_1} \leq_{\mathrm{icx}} (\geq_{\mathrm{icx}})\, S_{N,x_2}$ for all $x_2 \geq x_1 \geq 0$, then $F$ is NWUC (NBUC).*

Proof. We only prove the NWUC property, and the NBUC property may be built in a similar manner.

Divided by $\bar{G}(x)$ on both sides of (1.2), we have, for $x, y \geq 0$,

$$\bar{G}_x(y) = \alpha \int_0^x \frac{\bar{G}(x-t, y)}{\bar{G}(x)}\,\mathrm{d}F(t) + \alpha \frac{\bar{F}(x+y)}{\bar{G}(x)}.$$

By integrating on $y \in [u, \infty)$ on both sides, we obtain, for $x, u \geq 0$,

$$\int_u^\infty \bar{G}_x(y)\,\mathrm{d}y = \alpha \int_u^\infty \int_0^x \frac{\bar{G}(x-t, y)}{\bar{G}(x)}\,\mathrm{d}F(t)\,\mathrm{d}y + \alpha \int_u^\infty \frac{\bar{F}(x+y)}{\bar{G}(x)}\,\mathrm{d}y.$$

By the assumption, we have $\int_u^\infty \bar{G}_x(y)\,\mathrm{d}y$ is increasing in $x \geq 0$ for any $u \geq 0$. So, it holds that, for $x, u \geq 0$,

$$\begin{aligned}\int_u^\infty \bar{G}_x(y)\,\mathrm{d}y &= \alpha \int_0^x \int_u^\infty \bar{G}_{x-t}(y) \cdot \frac{\bar{G}(x-t)}{\bar{G}(x)}\,\mathrm{d}y\,\mathrm{d}F(t) + \alpha \int_u^\infty \frac{\bar{F}(x+y)}{\bar{G}(x)}\,\mathrm{d}y \\ &\leq \alpha \int_0^x \frac{\bar{G}(x-t)}{\bar{G}(x)}\,\mathrm{d}F(t) \int_u^\infty \bar{G}_x(y)\,\mathrm{d}y + \alpha \int_u^\infty \frac{\bar{F}(x+y)}{\bar{G}(x)}\,\mathrm{d}y,\end{aligned}$$

and hence

$$\left[1 - \alpha \int_0^x \frac{\bar{G}(x-t)}{\bar{G}(x)}\,\mathrm{d}F(t)\right] \int_u^\infty \bar{G}_x(y)\,\mathrm{d}y \leq \alpha \int_u^\infty \frac{\bar{F}(x+y)}{\bar{G}(x)}\,\mathrm{d}y.$$

In view of (1.1), we have, for $x, u \geq 0$,

$$\int_u^\infty \bar{G}_x(y)\,\mathrm{d}y \leq \int_u^\infty \frac{\bar{F}(x+y)}{\bar{F}(x)}\,\mathrm{d}y.$$

By the assumption once again, it holds that, for $x, u \geq 0$,

$$\int_u^\infty \bar{F}(y)\,\mathrm{d}y = \int_u^\infty \bar{G}_0(y)\,\mathrm{d}y \leq \int_u^\infty \bar{G}_x(y)\,\mathrm{d}y \leq \int_u^\infty \frac{\bar{F}(x+y)}{\bar{F}(x)}\,\mathrm{d}y.$$

As a consequence, it holds that

$$\int_u^\infty \bar{F}(y)\,\mathrm{d}y \leq \int_u^\infty \mathrm{P}(X-x>y \mid X>x)\,\mathrm{d}y, \qquad \text{for all } u \text{ and } x \geq 0.$$

That is, $F$ is NWUC.

In ruin theory, suppose the initial surplus is $x$, $F$ is the common distribution function of the amount of every drop given that the drop occurs, and the probability of ultimate ruin is $\bar{G}(x)$, which is the survival function of a compound geometric distribution, then $G(x, y)$ may be interpreted as the defective distribution of the deficit at ruin. So $G_x(y)$ is the distribution of the deficit at ruin given the occurrence of the ruin (See Willmot, 2002). According to the above two theorems, given that the ruin occurs, the stochastic increasing/decreasing property of the deficit at ruin with respect to the initial surplus $x$ in terms of the increasing concave (convex) order implies the NWU[2]/NBU[2] (NWUC/NBUC) aging property of the amount of every drop given that the drop occurs.

To close this section, we address here a supplement to the discussion on the form of $G$ with monotone failure rate in Section 4 of Psarrakos (2010).

Let $S_N^*$ be a nonnegative random variable with distribution function, for $x \geq 0$,

$$G^*(x) = \sum_{n=1}^\infty (1-\alpha)\alpha^{n-1}F^{(n)}(x).$$

According to Bhattacharjee et al. (2003), $S_N$ is a mixture of $S_N^*$ and 0 with probabilities $\alpha$ and $1-\alpha$, respectively. Actually, for $x \geq 0$,

$$G(x) = (1-\alpha) + \alpha \sum_{n=1}^\infty (1-\alpha)\alpha^{n-1}F^{(n)}(x) = (1-\alpha) + \alpha G^*(x).$$

Thus, it holds that, for $x \geq 0$,

$$-\log \bar{G}(x) \quad = \quad -\log\left[1 - \big((1-\alpha) + \alpha G^*(x)\big)\right]$$

$$= \quad -\log\left[\alpha\big(1-G^*(x)\big)\right] = -\log\alpha - \log\bar{G}^*(x).$$

According to Proposition C.1.b of Marshall and Olkin (2007, Chapter 4), a distribution function $G$, satisfying $G(x)=0$ for $x<0$, is DFR if and only if the hazard function $-\log\bar{G}$ is concave on $[0,\infty)$. As a consequence, $G$ is DFR if and only if $G^*$ is DFR.

Psarrakos (2010, Example 4.1) employed

$$\bar{G}(x) = \begin{cases} 1, & x\in(-\infty,0), \\ 0.8841\,e^{-0.1818x} - 0.0109\,e^{-2.7892x}, & x\in[0,\infty), \end{cases}$$

to illustrate that $G$ can be IFR. By direct calculation, we have

$$-\log\bar{G}(x) = \begin{cases} 0, & x\in(-\infty,0), \\ -\log\big(0.8841\,e^{-0.1818x} - 0.0109\,e^{-2.7892x}\big), & x\in[0,\infty). \end{cases}$$

According to Proposition C.1.b of Marshall and Olkin (2007, Chapter 4), a distribution function $G$, satisfying $G(x)=0$ for $x<0$, is IFR if and only if the hazard function $-\log\bar{G}$ is convex where finite. Unfortunately, as is seen in Fig. 1, $-\log\bar{G}(0) = -\log 0.8732 \neq 0 = -\log\bar{G}(0-)$ implies that $-\log\bar{G}$ is not convex on $\mathbb{R}$. That is, $G$ is not IFR.

Actually, the compound geometric distribution $G$ has a mass point at 0, and thus, as an elevated version of $-\log\bar{G}^*$, $-\log\bar{G}$ gets the height $-\log\alpha$ at the origin. Since $-\log\bar{G}(x)=0$ for $x<0$, $-\log\bar{G}$ can't be convex on $\mathbb{R}$. So, the compound geometric distribution $G$ can not be IFR whatever $G^*$ is.

## 3 Monotone Properties of $S_{W,x}$

In this section, we investigate stochastic monotone properties of $S_{W,x}$ with respect to $x$. The first results tells that the DFR (IFR) aging property of

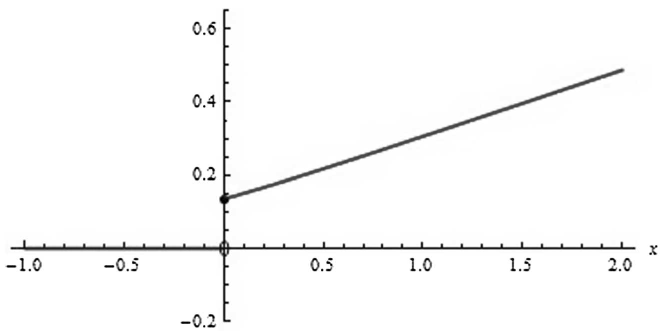


Figure 1: The hazard function $-\log\bar{G}(x)$

$W$ implies the monotone property of $S_{W,x}$ in the sense of the usual stochastic order.

THEOREM 3.1. *$S_{W,x_1} \leq_{\mathrm{st}} (\geq_{\mathrm{st}})\, S_{W,x_2}$ for all $x_2 \geq x_1 \geq 0$ if the compound geometric convolution $W$ is DFR (IFR).*

PROOF. Since $W$ is DFR (IFR), by Lemma 3.1 of Psarrakos (2010), it holds that $\bar{K}_x(y)$ is increasing (decreasing) in $x$ for all $y \geq 0$. That is, $S_{W,x_1} \leq_{\mathrm{st}} (\geq_{\mathrm{st}})\, S_{W,x_2}$ for all $x_2 \geq x_1 \geq 0$.

IMRL (DMRL) property is a comparison of the mean of residual life. Assume that $W$ is IMRL (DMRL), the following theorem give a comparison of the expectation of $S_{W,x}$.

THEOREM 3.2. *$\mathsf{E}[S_{W,x_1}] \leq (\geq)\, \mathsf{E}[S_{W,x_2}]$ for all $x_2 \geq x_1 \geq 0$ if the compound geometric convolution $W$ is IMRL (DMRL).*

PROOF. We consider only the IMRL case, the DMRL case may be completed in a similar manner.

The IMRL property of $W$ implies that $\int_0^\infty \frac{\bar{W}(x+y)}{\bar{W}(x)}\,\mathrm{d}y$ is increasing in $x \geq 0$. Then, according to (1.6), we have, for $x \geq 0$,

$$\begin{aligned}
(1-\alpha)\mathsf{E}[S_{W,x}] &= (1-\alpha)\int_0^\infty \bar{K}_x(y)\,\mathrm{d}y \\
&= \int_0^\infty \frac{\bar{W}(x+y)}{\bar{W}(x)}\,\mathrm{d}y - \alpha\int_0^\infty \bar{F}(y)\,\mathrm{d}y \\
&\quad -\alpha\int_0^\infty\int_0^y \frac{\bar{W}(x+y-t)}{\bar{W}(x)}\,\mathrm{d}F(t)\,\mathrm{d}y \\
&= \int_0^\infty \frac{\bar{W}(x+y)}{\bar{W}(x)}\,\mathrm{d}y - \alpha\int_0^\infty \bar{F}(y)\,\mathrm{d}y \\
&\quad -\alpha\int_0^\infty \mathrm{d}F(t)\int_0^\infty \frac{\bar{W}(x+y)}{\bar{W}(x)}\,\mathrm{d}y \\
&= \int_0^\infty \frac{\bar{W}(x+y)}{\bar{W}(x)}\,\mathrm{d}y - \alpha\int_0^\infty \bar{F}(y)\,\mathrm{d}y \\
&\quad -\alpha\int_0^\infty \frac{\bar{W}(x+y)}{\bar{W}(x)}\,\mathrm{d}y \\
&= (1-\alpha)\int_0^\infty \frac{\bar{W}(x+y)}{\bar{W}(x)}\,\mathrm{d}y - \alpha\int_0^\infty \bar{F}(y)\,\mathrm{d}y,
\end{aligned}$$

which is increasing in $x \geq 0$. This completes the proof.

According to Theorem 2.1 of Psarrakos (2009),

- $\mathsf{E}[S_{W,x} - t | S_{W,x} > t] \le \max\{\mathsf{E}[X],\, \mathsf{E}[Y]\}$ for all $x, t \ge 0$ if $X$ and $Y$ are both NBUE,
- $\mathsf{E}[S_{W,x} - t | S_{W,x} > t] \ge \min\{\mathsf{E}[X],\, \mathsf{E}[Y]\}$ for all $x, t \ge 0$ if $X$ and $Y$ are both NWUE.

In what follows, we obtain parallel results for NRBUE, NBUC and NBU[2].

Being defined through a comparison between the mean residual lifetime and the mean lifetime of the equilibrium distribution, NRBUE and NRWUE aging properties are useful in maintenance polices. For $X$ and $Y$ both NRBUE (NRWUE), the following theorem present comparison on the expectations of $S_{W,x}$, $\tilde{X}$ and $\tilde{Y}$.

THEOREM 3.3. *If $X$ and $Y$ are both*

*(i) NRBUE, then* $\mathsf{E}[S_{W,x}] \le \max\{\mathsf{E}[\tilde{X}],\, \mathsf{E}[\tilde{Y}]\}$ *for all* $x \ge 0$*;*

*(ii) NRWUE, then* $\mathsf{E}[S_{W,x}] \ge \min\{\mathsf{E}[\tilde{X}],\, \mathsf{E}[\tilde{Y}]\}$ *for all* $x \ge 0$.

PROOF. (i) Since $X$ and $Y$ are both NRBUE, it holds that, for $x \ge 0$,

$$\int_0^\infty \bar{F}(x+y)\,\mathrm{d}y \le \bar{F}(x)\mathsf{E}[\tilde{X}], \qquad \int_0^\infty \bar{H}(x+y)\,\mathrm{d}y \le \bar{H}(x)\mathsf{E}[\tilde{Y}]. \tag{3.1}$$

In view of $\bar{G}(x,0) = \bar{G}(x)$ and from (1.3), (1.4), we obtain $\bar{K}(x,0) = \bar{W}(x)$ for all $x \ge 0$. Then, by (1.5), we have, for $x \ge 0$,

$$\bar{W}(x) = \bar{H}(x) + \frac{\alpha}{1-\alpha}\int_0^x \bar{F}(x-t)\,\mathrm{d}W(t). \tag{3.2}$$

By integrating both sides of (1.5) on $y \in [0,\infty)$, we have, for $x \ge 0$,

$$\begin{aligned}
\int_0^\infty \bar{K}(x,y)\,\mathrm{d}y &= \int_0^\infty \bar{H}(x+y)\,\mathrm{d}y + \frac{\alpha}{1-\alpha}\int_0^\infty \int_0^x \bar{F}(x+y-t)\,\mathrm{d}W(t)\,\mathrm{d}y \\
&= \int_0^\infty \bar{H}(x+y)\,\mathrm{d}y + \frac{\alpha}{1-\alpha}\int_0^x \int_0^\infty \bar{F}(x+y-t)\,\mathrm{d}y\,\mathrm{d}W(t) \\
&\le \bar{H}(x)\mathsf{E}[\tilde{Y}] + \frac{\alpha}{1-\alpha}\int_0^x \bar{F}(x-t)\mathsf{E}[\tilde{X}]\,\mathrm{d}W(t) \\
&\le \max\{\mathsf{E}[\tilde{X}],\, \mathsf{E}[\tilde{Y}]\}\left[\bar{H}(x) + \frac{\alpha}{1-\alpha}\int_0^x \bar{F}(x-t)\,\mathrm{d}W(t)\right] \\
&= \max\{\mathsf{E}[\tilde{X}],\, \mathsf{E}[\tilde{Y}]\}\bar{W}(x),
\end{aligned}$$

where the first inequality follows from (3.1) and the last equality follows from (3.2). So, it holds that

$$\mathsf{E}[S_{W,x}] = \int_0^\infty \bar{K}_x(y)\,\mathrm{d}y \le \max\left\{\mathsf{E}[\tilde{X}],\, \mathsf{E}[\tilde{Y}]\right\}, \qquad \text{for all } x \ge 0.$$

(ii) The proof may be completed in a similar manner and hence is omitted.

THEOREM 3.4. *If both $X$ and $Y$ are*

*(i) NBUC, then, for all $x, u \ge 0$,*

$$\int_u^\infty \bar{K}_x(y)\,\mathrm{d}y \le \max\left\{\int_u^\infty \bar{H}(y)\,\mathrm{d}y,\, \int_u^\infty \bar{F}(y)\,\mathrm{d}y\right\};$$

*(ii) NWUC, then, for all $x, u \ge 0$,*

$$\int_u^\infty \bar{K}_x(y)\,\mathrm{d}y \ge \min\left\{\int_u^\infty \bar{H}(y)\,\mathrm{d}y,\, \int_u^\infty \bar{F}(y)\,\mathrm{d}y\right\}.$$

PROOF. (i) Since $X$ and $Y$ are both NBUC, it holds that, for $x, u \ge 0$,

$$\begin{aligned}\int_u^\infty \bar{F}(x+y)\,\mathrm{d}y &\le \bar{F}(x)\int_u^\infty \bar{F}(y)\,\mathrm{d}y,\\ \int_u^\infty \bar{H}(x+y)\,\mathrm{d}y &\le \bar{H}(x)\int_u^\infty \bar{H}(y)\,\mathrm{d}y.\end{aligned} \tag{3.3}$$

By integrating both sides of (1.5) on $y \in [u, \infty)$, we have, for $x, u \ge 0$,

$$\begin{aligned}\int_u^\infty \bar{K}(x,y)\,\mathrm{d}y &= \int_u^\infty \bar{H}(x+y)\,\mathrm{d}y + \frac{\alpha}{1-\alpha}\int_u^\infty \int_0^x \bar{F}(x+y-t)\,\mathrm{d}W(t)\,\mathrm{d}y\\ &= \int_u^\infty \bar{H}(x+y)\,\mathrm{d}y + \frac{\alpha}{1-\alpha}\int_0^x \int_u^\infty \bar{F}(x+y-t)\,\mathrm{d}y\,\mathrm{d}W(t)\\ &\le \bar{H}(x)\int_u^\infty \bar{H}(y)\,\mathrm{d}y + \frac{\alpha}{1-\alpha}\int_0^x \bar{F}(x-t)\int_u^\infty \bar{F}(y)\,\mathrm{d}y\,\mathrm{d}W(t)\\ &\le \max\left\{\int_u^\infty \bar{H}(y)\,\mathrm{d}y,\, \int_u^\infty \bar{F}(y)\,\mathrm{d}y\right\}\Big[\bar{H}(x)\\ &\quad + \frac{\alpha}{1-\alpha}\int_0^x \bar{F}(x-t)\,\mathrm{d}W(t)\Big]\\ &= \max\left\{\int_u^\infty \bar{H}(y)\,\mathrm{d}y,\, \int_u^\infty \bar{F}(y)\,\mathrm{d}y\right\}\bar{W}(x),\end{aligned}$$

where the first inequality follows from (3.3) and the last equality follows from (3.2). So, it concludes that

$$\int_u^\infty \bar{K}_x(y)\,\mathrm{d}y \le \max\left\{\int_u^\infty \bar{H}(y)\,\mathrm{d}y,\ \int_u^\infty \bar{F}(y)\,\mathrm{d}y\right\}, \qquad \text{for all } x,u \ge 0.$$

(ii) The proof may be completed in a similar manner and hence is omitted.

THEOREM 3.5. *If both $X$ and $Y$ are*

*(i) NBU[2], then, for all $x,u \ge 0$,*

$$\int_0^u \bar{K}_x(y)\,\mathrm{d}y \le \max\left\{\int_0^u \bar{H}(y)\,\mathrm{d}y,\ \int_0^u \bar{F}(y)\,\mathrm{d}y\right\};$$

*(ii) NWU[2], then, for all $x,u \ge 0$,*

$$\int_0^u \bar{K}_x(y)\,\mathrm{d}y \ge \min\left\{\int_0^u \bar{H}(y)\,\mathrm{d}y,\ \int_0^u \bar{F}(y)\,\mathrm{d}y\right\}.$$

PROOF. (i) Since $X$ and $Y$ are both NBU[2], it holds that, for $x,u \ge 0$,

$$\begin{aligned}\int_0^u \bar{F}(x+y)\,\mathrm{d}y &\le \bar{F}(x)\int_0^u \bar{F}(y)\,\mathrm{d}y,\\ \int_0^u \bar{H}(x+y)\,\mathrm{d}y &\le \bar{H}(x)\int_0^u \bar{H}(y)\,\mathrm{d}y.\end{aligned} \tag{3.4}$$

By integrating both sides of (1.5) on $y \in [0,u]$, we have, for $x,u \ge 0$,

$$\begin{aligned}\int_0^u \bar{K}(x,y)\,\mathrm{d}y &= \int_0^u \bar{H}(x+y)\,\mathrm{d}y + \frac{\alpha}{1-\alpha}\int_0^u\int_0^x \bar{F}(x+y-t)\,\mathrm{d}W(t)\,\mathrm{d}y\\ &= \int_0^u \bar{H}(x+y)\,\mathrm{d}y + \frac{\alpha}{1-\alpha}\int_0^x\int_0^u \bar{F}(x+y-t)\,\mathrm{d}y\,\mathrm{d}W(t)\\ &\le \bar{H}(x)\int_0^u \bar{H}(y)\,\mathrm{d}y + \frac{\alpha}{1-\alpha}\int_0^x \bar{F}(x-t)\int_0^u \bar{F}(y)\,\mathrm{d}y\,\mathrm{d}W(t)\\ &\le \max\left\{\int_0^u \bar{H}(y)\,\mathrm{d}y,\ \int_0^u \bar{F}(y)\,\mathrm{d}y\right\}\Big[\bar{H}(x)\\ &\quad + \frac{\alpha}{1-\alpha}\int_0^x \bar{F}(x-t)\,\mathrm{d}W(t)\Big]\\ &= \max\left\{\int_0^u \bar{H}(y)\,\mathrm{d}y,\ \int_0^u \bar{F}(y)\,\mathrm{d}y\right\}\bar{W}(x),\end{aligned}$$

where the first inequality follows from (3.4) and the last equality follows from (3.2). So, it concludes that

$$\int_0^u \bar{K}_x(y)\,\mathrm{d}y \le \max\left\{\int_0^u \bar{H}(y)\,\mathrm{d}y,\ \int_0^u \bar{F}(y)\,\mathrm{d}y\right\}, \qquad \text{for all } x, u \ge 0.$$

(ii) The proof may be completed in a similar manner and hence is omitted.

Setting $\bar{H}(x) = \bar{F}(x)$ in (1.3) and (1.4) yields

$$\bar{W}(x) = \bar{G}(x)/\alpha, \qquad \bar{K}(x,y) = \bar{G}(x,y)/\alpha, \qquad \bar{K}_x(y) = \bar{G}_x(y).$$

Consequently, Corollary 3.6 follows directly from Theorems 3.3, 3.4 and 3.5.

COROLLARY 3.6. *(i) If $X$ is NRBUE (NRWUE), then* $\mathsf{E}[S_{N,x}] \le (\ge)\mathsf{E}[\tilde{X}]$ *for all* $x \ge 0$*;*

*(ii) If $X$ is NBUC (NWUC), then* $S_{N,x} \le_{\text{icx}} (\ge_{\text{icx}})\, X$ *for all* $x \ge 0$*;*

*(iii) If $X$ is NBU[2] (NWU[2]), then* $S_{N,x} \le_{\text{icv}} (\ge_{\text{icv}})\, X$ *for all* $x \ge 0$.

According to this corollary, given that the ruin occurs, the distribution of the deficit at ruin is bounded above by the equilibrium distribution of the amount of every drop given that the drop occurs (the amount of every drop given that the drop occurs, the amount of every drop given that the drop occurs) in the sense of mean (increasing convex order, increasing concave order) when the amount of every drop itself is NRBUE (NBUC, NBU[2]) given that the drop occurs, and the distribution of the deficit at ruin is bounded below by the equilibrium distribution of the amount of every drop given that the drop occurs (the amount of every drop given that the drop occurs, the amount of every drop given that the drop occurs) in the sense of mean (increasing convex order, increasing concave order) when the amount of every drop itself is NRWUE (NWUC, NWU[2]) given that the drop occurs.

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