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Limit Theorems for Occupation Rates of Local Empirical Processes

Davit Varron
Université de Franche-Comté, Besançon, France

Abstract

Given a continuous probability measure μ on a Borel set $H \subset \mathbb{R}^d$, we prove a limit theorem for occupation rates of the form

$$\mu\left(\{z \in H,\ \Delta_n(\cdot, h, z) \in F\}\right),$$

where the $\Delta_n(\cdot, h, z)$ are normalized versions of local empirical processes indexed by a class of functions $\mathcal{G}$. Under standard structural conditions upon $\mathcal{G}$, and under some regularity conditions upon the law of the sample, we show that, almost surely, those occupation rates converge to those of a Gaussian process, uniformly in $h \in [h_n, \mathfrak{h}_n]$, where h_n and $\mathfrak{h}_n$ are two deterministic bandwidthsequences, upon which mild assumptions are made.

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1 Introduction and Statement of Main Results

The uniform empirical process and its increments have been intensively investigated during the past half-century, and it is well known that such processes have limit properties (as the sample size increases) that are similar to those of a Wiener process W or a Brownian bridge B (for an overview on these results, see Shorack and Wellner (1986), Csőrgő and Révész (1981), Einmahl (1986), and Del Barrio, Deheuvels and Van De Geer (2007) and the references therein). Among those results, Varron (2011) showed that, in the usual space $\big(B[0,1], ||\cdot||\big)$ of cdlg functions on $[0,1]$ endowed with the supremum norm, the occupation rates of the functional increments of the uniform empirical process are asymptotically the same as those of W. Its unidimensional version can be stated as follows, writing λ^* for the outer Lebesgue measure on $[0,1]$, writing

$$\boldsymbol{\Delta}_n(s,h,z) := (nh)^{-1/2}\Big(\sum_{i=1}^{n}\big(\mathbb{1}_{[z,z+hs]}(U_i) - hs\big)\Big),$$

and denoting by $(U_i)_{i\geq 1}$ an independent, identically distributed [i.i.d.] sample having the uniform law on $[0,1]$.

THEOREM 1. *[from Varron (2011)] Let* $(h_n)_{n\geq 1}$ *be a sequence of nonrandom positive numbers such that* $h_n \downarrow 0$, $nh_n \uparrow +\infty$, $\underline{\lim}\log(1/h_n)/\log\log(n) > 1$. *Then, for any closed interval* $I = [a,b]$ *with* $0 \leq a < b < 1$ *the following property holds with probability one : for each closed set* F *of* $\big(B([0,1]), ||\cdot||\big)$, *we have :*

$$\overline{\lim_{n\to\infty}} \frac{\lambda^*\big(A_{F,n}\big)}{\lambda(I)} \leq \mathbb{P}(W\in F), \text{ where} \tag{1.1}$$
$$A_{F,n} := \big\{z\in I,\ \mathbf{\Delta}_n(\cdot,h_n,z)\in F\big\}$$

REMARK. The original statement of Theorem 1 was misleading, as it involved a subspace of $B([0,1])$ that fails to be closed under the sup norm. We bring here a correction to this error. Note that the above lim sup is a limit for any Borel set F for which the boundary ∂F satisfies $\mathbb{P}\big(W\in\partial F\big)=0$.

Theorem 1 can be heuristically explained as follows :

– *(Expectation argument)* Assume that F makes the sets $A_{F,n}$ Borel with probability one. First, Tonelli's theorem yields, as soon as $b+h_n\leq 1$:

$$\begin{aligned}\mathbb{E}\Big(\lambda(A_{F,n})\Big) &= \int_I \mathbb{P}\Big(\mathbf{\Delta}_n(\cdot,h_n,z)\in F\Big)dz\\ &= \lambda(I)\times\mathbb{P}\Big(\mathbf{\Delta}_n(\cdot,h_n,0)\in F\Big),\end{aligned}$$

by stationarity of the increments of the uniform empirical process. Hence those expectations have a lim sup less than $\mathbb{P}\big(W\in F\big)$, since $h_n\to 0$ and $nh_n\to\infty$, by Donsker's theorem for the increments of the uniform empirical process.

– *(Variance argument)* Second, we have

$$\mathrm{Var}\Big(\lambda(A_F)\Big) = \int_{I^2}\mathrm{Cov}\bigg(\mathbb{1}_F\Big(\mathbf{\Delta}_n(\cdot,h_n,z)\Big),\mathbb{1}_F\Big(\mathbf{\Delta}_n(\cdot,h_n,z')\Big)\bigg)dzdz'.$$

Now for n large enough and $|z-z'|>h_n$, we can expect that

$$\text{"}\mathbf{\Delta}_n(\cdot,h_n,z) \text{ and } \mathbf{\Delta}_n(\cdot,h_n,z') \text{ are almost independent"}, \tag{1.2}$$

and hence the considered variance is of order $\lambda^{\otimes 2}(D(h_n))\sim h_n$, where $D(h) := \big\{(z,z')\in I^2,\ |z-z'|\leq h\big\}$. In view of those two arguments, one can already see that $\limsup \lambda(I)^{-1}\lambda(A_{F,n})\leq\mathbb{P}\big(W\in F\big)$ in probability, for fixed F.

The aim of the present article is to generalize Theorem 1 to a more general and abstract setup, by addressing two natural questions.

1.1. First Direction. A first question of interest is to further investigate those occupation rates when substituting λ by a general probability measure μ. Wether the condition $\mu^{\otimes 2}(D(h)) \to 0$ (as $h \to 0$) is necessary or not can be partially answered by the following remark : if we assume that μ is discrete with a finite number K of atoms, then a generalization of Theorem 1 cannot hold, because the set of all the values

$$\{\mu^*(A),\ A \subset [0,1]\},$$

has cardinality 2^K and hence its closure is certainly not equal to the expected range of values $[0,1]$. In this article we shall focus on the case where $\mu^{\otimes 2}(D(h)) \to 0$, which includes all continuous measures.

1.2. Second Direction. A second question of interest would be to raise Theorem 1 to the abstraction level of (local) empirical processes indexed by functions, and confront that theorem to the two usual *bracketing* and *uniform entropy* conditions that arise in this theory (see, e.g. Van der Vaart and Wellner, 1996, Part 2). Such extensions to have already been achieved for Donsker type theorems and functional limit laws of Strassen type (see Mason, 2004, Einmahl and Mason, 1997). The local empirical process indexed by functions, as introduced by Mason (2004), is defined as follows : consider a sequence $(Z_i)_{i\geq 1}$ taking values in $\mathbb{R}^d$, admitting a (version of) Lebesgue density f on an open set $\mathfrak{O}$. Then, for any real Borel function g, $h > 0$ (usually called a bandwidth), and $z \in \mathfrak{O}$, define, whenever $f(z) > 0$:

$$\begin{aligned}
\Delta_n(g,h,z) &:= \frac{T_n(g,h,z)}{\sqrt{nh^d f(z)}},\ g \in \mathcal{G},\ \text{where} \\
T_n(g,h,z) &:= \sum_{i=1}^{n} g_{h,z}(Z_i) - \mathbb{E}\Big(g_{h,z}(Z_i)\Big),\ \text{and where} \\
g_{h,z}(\mathbf{z}) &:= g\Big(h^{-1}\big(\mathbf{z} - z\big)\Big),\ \mathbf{z} \in \mathbb{R}^d.
\end{aligned} \tag{1.3}$$

The notion of general empirical process then arises by making g vary into a class of functions $\mathcal{G}$, for which we make thefollowing usual assumption, in order to avoid measurability problems (see, e.g. Van der Vaart and Wellner, 1996, p.47).

$(HG1)$ The class $\mathcal{G}$ is pointwise separable.

Denote by $\mathcal{B}(\mathcal{G})$ the space of all real bounded functions on $\mathcal{G}$ that are continuous with respect to the topology on $\mathcal{G}$ spanned by the maps $\{\psi \to \psi(g),\ g \in \mathcal{G}\}$. Assumption $(HG1)$ then makes the $T_n(\cdot,h,z),\ z \in \mathfrak{O},\ h > 0,\ n \geq 1$

take their values in $\mathcal{B}(\mathcal{G})$. We endow that space with the usual supremum norm

$$|| \psi ||_{\mathcal{G}} := \sup_{g \in \mathcal{G}} | \psi(g) | . \tag{1.4}$$

Note that, whereas assuming that Z_1 admits a density is quite natural (if z is an atom, then the asymptotic behaviors of $T_n(\cdot, h, z)$, as $n \to \infty$ and $h \to 0$, become degenerate), definition (1.3) depends on the choice of the representative f. As already shown by Mason (2004), a very useful way to give a rigorous sense to the heuristic (1.2) is to make the assumption that $\mathcal{G}$ is a strongly localizing class, namely

$(HG2)$ $\mathcal{G}$ admits an envelope $G \geq 1$ having compact support J and fulfilling $\int_J G^2(z)dz < \infty$,

and then use the fact that , when $\{z + hJ\} \cap \{z' + hJ\} = \emptyset$, then two arbitrary $g_{h,z}$ and $g_{h,z'}$ have disjoint supports, and hence (1.2) is true when replacing the $\Delta_n(\cdot, h, z)$ by their "poissonized" version.

We will now consider a fixed measurable set H having closure $\overline{H}$ strictly included in $\mathfrak{O}$, and endow H with a continuous probability measure μ. Such a set will play the same role as I in Theorem 1, while μ will play the role of $\lambda(\cdot \cap I)/\lambda(I)$.

For a measure Q and $r > 0$, denote by $|| \cdot ||_{Q,r}$ the associated $L^r(Q)$ norm. Looking at the *expectation argument*, we see that it seems natural to at least require, for μ-almost every point z, the following convergence in law

$(Donsker\ at\ z)$ $\Delta_n(\cdot, h_n, z) \to_{\mathcal{L}} \mathcal{W}_{\mathcal{G}}$,

to a limit process (here $\to_{\mathcal{L}}$ denotes, as usual, the convergence of stochastic processes in the sense of Hoffman-Jørgensen). For fixed z, a close look at a remark made in Einmahl and Mason (1997, Example 2) leads to the conclusion that $(Donsker\ at\ z)$ is true, with limit Gaussian process $\mathcal{W}_{\mathcal{G}}$ being the $|| \cdot ||_{\lambda,2}$ isonormal process on $\mathcal{G}$, when

– The sequence h_n tends to zero at the rate of weak invariance principle:

(HV) $nh_n^d \uparrow \infty,\ h_n \downarrow 0$ as $n \to \infty$.

– The class $\mathcal{G}$ satisfies the usual uniform entropy condition :

$(Unif.\ entropy)$ $\int_0^\infty \sup_{Q \text{ probab.}} \sqrt{\log N(\epsilon \,||\, G \,||_{Q,2}, \mathcal{G}, || \cdot ||_{Q,2})}d\epsilon < \infty.$

– The (local) covariance function of $g_{h,z}(Z_1), g \in \mathcal{G}$ converges to that of $\mathcal{W}_{\mathcal{G}}(g),\ g \in \mathcal{G}$ in the following sense

$$\lim_{h\to 0} \sup_{(g,g')\in\mathcal{G}^2} \Big| h^{-d}\mathbb{E}\Big(g_{h,z}(Z_1)g'_{h,z}(Z_1)\Big) - f(z)\int_{\mathbb{R}^d} g(u)g'(u)du \Big| = 0, \tag{1.5}$$

with $f(z) > 0$.

In view of such a result, it then seems natural to require that, on a subset $H_0 \subset H$ fulfilling $\lambda(H - H_0) = 0$, and for a version of f, we simultaneously have

($Hf1$) $0 < f(z) < \infty$ for each $z \in H_0$;

($Hf2$) Assertion (1.5) holds for each $z \in H_0$;

and we shall take that version f in definition (1.3).

REMARK. Assumptions $(Hf1)$ and $(Hf2)$ satisfied in several cases. The simplest case is when H is bounded and Z_1 admits a version of Lebesgue density f that is continuous on the closure of H, and for which $\inf_H f > 0$, as assumed in Mason (2004). In that case we can take $H_0 := H$. Another interesting case is when Z_1 has a density fulfilling $(Hf1)$ and, for each $M > 0$, the class $\mathcal{G} \odot \mathcal{G}_M := \left\{gg'\mathbb{1}_{\{G\le M\}},\ (g,g') \in \mathcal{G}^2\right\}$ satisfies $N_{[]}(\epsilon, \mathcal{G}\odot\mathcal{G}_M, ||\cdot||_{\lambda,2}) < \infty$ for each$0 < \epsilon < 1$ (see e.g. Van der Vaart and Wellner, 1996, p. 83 for a definition of bracketing numbers). In that case, it is possible to prove the validity of $(Hf2)$ by making use of the of an extension of the Lebesgue density theorem (see, e.g., Devroye and Lugosi, 2001, Exercise 5.9, p. 45), repeatedly for each element defining a bracket.

1.3. Results. We have now set up almost all the basic assumptions that will permit to generalize Theorem 1. We shall go one step further by enriching our result with a uniformity in $[h_n, \mathfrak{h}_n]$, for two deterministic bandwidth sequences, in order to handle random bandwidths sequences as a byproduct (see Corollary 1). Our first result is stated as follows.

THEOREM 2. *Assume that there exists $\rho_0 > 1$ such that $(Hf1)$, $(Hf2)$, $(HG1)$, $(HG2)$ and $(Unif.\ entropy)$ are fulfilled with $\mathcal{G}$ replaced by the larger class*

$$\mathcal{G}_{\rho_0} := \left\{g\big(\rho^{-1}\,\cdot\big),\ g \in \mathcal{G},\ \rho \in [1, \rho_0]\right\}, \tag{1.6}$$

and that $\mathcal{G}$ satisfies the following regularity condition :

($HG3$) $\lim_{\rho\downarrow 1} \sup_{g\in\mathcal{G}} \ || \ g(\cdot) - g(\rho^{-1}\cdot) \ ||_{\lambda,2} = 0.$

Also assume that there exists $\alpha > 0$ for which, when $h \to 0$:

$$\delta(h) := \mu\Big(\big\{(z,z') \in H^2,\ \{z + hJ\} \cap \{z' + hJ\} \neq \emptyset\big\} \Big) = O(h^\alpha). \quad (1.7)$$

Then, for each couple of sequences $h_n \leq \mathfrak{h}_n$ both fulfilling (HV) and

$$\underline{\lim}_{n\to\infty} \log(1/\mathfrak{h}_n)/\log\log(n) > 1/\alpha, \quad (1.8)$$

the following assertion holds with probability one : for each closed set F of $\big(\mathcal{B}(\mathcal{G}), ||\cdot||_{\mathcal{G}}\,\big)$, we have

$$\overline{\lim_{n\to\infty}} \sup_{h_n \leq h \leq \mathfrak{h}_n} \mu^*\Big(\big\{z \in H,\ \Delta_n(\cdot,h,z) \in F\big\}\Big) \leq \mathbb{P}\big(\mathcal{W}_{\mathcal{G}} \in F\big). \quad (1.9)$$

REMARK. Assumption $(HG3)$ should not be surprising, since Theorem 2 is an almost sure result. Achieving such a strength is made by using the usual blocking arguments along subsequences n_k of exponential kind, and then interpolating the (small) variations of h_n between successive n_k (for example, see Mason, 2004). Also note that (1.7) encompasses a large class of continuous measures. For example, if $\frac{d\mu}{d\lambda} \in L^p(\lambda)$ for some $p > 1$, then (1.7) holds with $\alpha := 1 - 1/p$.

Our next result shows that the preceding theorem remains true if the uniform entropy condition is replaced by the usual bracketing condition under the uniform distribution $\mathbf{P}_0$ on J (see, e.g., Van der Vaart and Wellner, 1996, p. 83)

$(Bracketing)$ $\int_0^\infty \sqrt{\log N_{[]}(\epsilon, \mathcal{G}, ||\cdot||_{\mathbf{P}_0,2})}d\epsilon < \infty,$

at a quite small price on the regularity of f.

THEOREM 3. *Theorem 2 still holds if (HGE) is replaced by (HGB) and $(Hf1)$ is replaced by*

$(Hf1')$ *For each $\tau > 0$ there exists $H' \subset H$ open such that $\mu(H - H') \leq \tau$ and f is bounded away from zero and infinity on $H' \cap H_0$.*

REMARK. Of course, $(Hf1')$ is true if f is bounded away from zero and infinity on H, or if $0 < f < \infty$ on H and f is piecewise continuous on $\overline{H}$, except at a finite number of points (since μ has no atom).

As mentioned earlier, the in bandwidth uniformity of Theorems 2 and 3 has the following immediate consequence, which may have an impact on statistical procedures that rely on occupations rates of the $T_n(\cdot,h,z)$, and for which the choice of bandwidth is data driven. Such procedures have never been investigated yet.

Corollary 1. *Under the assumptions of either Theorem 2 or 3, if* $(h_n^*)_{n\geq 1}$ *is a sequence of positive random variables on the same probability space as the* $(Z_i)_{i\geq 1}$ *for which, almost surely*

$$\underline{\lim}_{n\to\infty} \log(1/h_n^*)/\log\log(n) > 1/\alpha, \tag{1.10}$$

$$\overline{\lim}_{n\to\infty} \log(1/h_n^*)/\log(n) < 1/d. \tag{1.11}$$

Then the following assertion holds with probability one : for each closed set F *of* $\big(\mathcal{B}(\mathcal{G}), ||\cdot||_{\mathcal{G}}\big)$, *we have*

$$\overline{\lim}_{n\to\infty} \mu^*\Big(\big\{z\in H,\ \Delta_n(\cdot,h_n^*,z)\in F\big\}\Big) \leq \mathbb{P}\big(\mathcal{W}_{\mathcal{G}}\in F\big). \tag{1.12}$$

1.4. Overview of the Main Arguments of the Proofs. The proofs of theorems 2 and 3 can be roughly described as follows

1. The pointwise convergence in distribution, for all $z\in H_0$, of $\Delta_n(\cdot,h_n,z)$ to $\mathcal{W}_{\mathcal{G}}$ suffices to derive, for fixed ϕ bounded and Lipschitz on $\big(\mathcal{B}(\mathcal{G}), ||\cdot||_{\mathcal{G}}\big)$:
$$\int_H \mathbb{E}\Big(\phi(\Delta_n(\cdot,h_n,z))\Big)d\mu(z) \to \mathbb{E}\Big(\phi(\mathcal{W}_{\mathcal{G}})\Big), \tag{1.13}$$
that convergence playing the role of an *expectation argument.*

2. A *variance argument* is achieved by an abstract Poissonization technique due to Giné, Mason, and Zaitsev (2003). The variances of interest are of order at most $\delta(h)\leq Ch^\alpha$ for a universal constant C.

3. The key to obtain uniformity in $h\in[h_n,\mathfrak{h}_n]$ is that, if we discretize those intervals into grids of the form $h_{n,\ell}:=\rho^\ell h_n,\ \ell\leq R_n\sim\log(\mathfrak{h}_n/h_n)/\log(\rho)$, with $\rho>1$ small, then, by properties of geometric sums
$$\sum_{\ell=0}^{R_n}\delta(h_{n,\ell})\leq C\sum_{\ell=0}^{R_n}(\rho^\ell h_n)^\alpha\leq C_{\rho,\alpha}\mathfrak{h}_n^\alpha. \tag{1.14}$$
Hence taking the maximum along those grids leads to a sum of variance of the same order as if we hae taken $\mathfrak{h}_n$ only. To obtain bounds such as (1.14), we need to refine tools in local empirical processes theory to gain a uniformity in h (this is the core of Proposition 4). We naturally borrow arguments from Sheehy and Wellner (1992).

4. To obtain an almost sure result, we have to use technical tools that are usual in empirical processes theory : we introduce subsequences n_k of

exponential type, then we control the almost sure oscillations of our empirical processes between consecutive n_k by using maximal inequalities as well as a regularity in h inherited by $(HG3)$. To successfully invoke a variance argument, we make use of a general poissonization principle, due to Varron (2011), which is naturally adapted to maximal inequalities.

The remainder of this article is organized as follows : we first prove Theorem 2 in Section 2. We then prove Theorem 3 in Section 3. Some minor proofs are postponed to the Appendix.

2 Proof of Theorem 2

According to an extension of the Lebesgue density theorem (see, e.g., Devroye and Lugosi, 2001, Exercise 5.9, p. 45), we can assume without loss of generality that G belongs to $\mathcal{G}$ and that, for each $z \in H_0$ and $K \in \mathbb{N}$:

$$\lim_{h\to 0} h^{-d}\mathbb{E}\Big(G^2_{h,z}(Z_1)\mathbb{1}_{\{G^2_{h,z}(Z_1)\geq K\}}\Big) = f(z)\int_{\mathbb{R}^d} G^2(u)\mathbb{1}_{\{G^2(u)\geq K\}}du, \quad (2.1)$$

$$\lim_{h\to 0}\Big|\frac{a(h,z)}{f(z)h^d\lambda(J)} - 1\Big| = 0, \text{ with} \quad (2.2)$$

$$a(h,z) := \mathbb{P}\Big(h^{-1}\big(Z_1 - z\big)\in J\Big). \quad (2.3)$$

The proof shall be divided into several steps.

2.1. Step 1 : Preliminary Tools. We first take $\rho_0 > 1$ such that $\mathcal{G}_{\rho_0}$ satisfies $(Unif.\ entropy)$. Theorem 2 states that, on the same underlying event Ω_0 with $\mathbb{P}\big(\Omega_0\big) = 1$, we have, for all $\phi = \mathbb{1}_F$, with F closed

$$\overline{\lim_{n\to\infty}} \sup_{h\in[h_n,\mathfrak{h}_n]} \mu_n(\phi,h) \leq \mathbb{E}\big(\phi(\mathcal{W}_{\mathcal{G}})\big), \text{ with} \quad (2.4)$$

$$\mu_n(\phi,h) := \textstyle\int_H \phi\Big(\Delta_n(\cdot,h,z)\Big)d\mu(z), \quad (2.5)$$

or, equivalently, an exact convergence (uniformly in h) for each bounded Lipschitz ϕ.

In this subsection, we show that is will be sufficient to separately derive the almost sure convergences (or limsup) of a *countable* number of functions ϕ which will together characterize (2.4). Note that we index our processes by $\mathcal{G}_{\rho_0} \supset \mathcal{G}$ in order to make interpolations arguments at further stages of the proof.

We shall denote by $\mathbb{BL}_{\mathcal{G}_{\rho_0}}$ the set of all 1-Lipschitz functions on $\big(\mathcal{B}(\mathcal{G}_{\rho_0}), ||\cdot||_{\mathcal{G}_{\rho_0}}\big)$ that are bounded by 1. We shall also consider an arbitrary countable subclass $\tilde{\mathcal{G}}_{\rho_0} \subset \mathcal{G}_{\rho_0}$ that is dense in $\mathcal{G}_{\rho_0}$ for the pointwise convergence (this is allowed since $\mathcal{G}_{\rho_0}$ fulfills $(HG1)$). For $\psi \in \mathcal{B}(\mathcal{G}_{\rho_0})$, $p \geq 1$ and $\mathbf{g} = (g_1, \ldots, g_p) \in \mathcal{G}_{\rho_0}^p$ we shall denote by $\psi(\mathbf{g})$ the vector $\big(\psi(g_1), \ldots, \psi(g_p)\big)$. We shall also write, for $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p$, the map

$$\phi_{\boldsymbol{\theta},\mathbf{g}} : \ \psi \to \exp\big(\boldsymbol{i} < \boldsymbol{\theta}, \psi(\mathbf{g}) > \big). \tag{2.6}$$

Here $< \cdot, \cdot >$ stands for the canonical scalar product. Also, for $\tau, \delta > 0$, we shall define

$$\tilde{\phi}_{\delta,\tau} : \ \psi \to \mathbb{1}_{A_{\delta,\tau}}(\psi), \ \text{with} \tag{2.7}$$

$$A_{\delta,\tau} := \Big\{\psi \in \mathcal{B}(\mathcal{G}_{\rho_0}), \sup_{(g,g')\in\mathcal{G}_{\rho_0}^2,\, ||g-g'||_{\mathbf{P}_0,2}\leq\delta} \mid \psi(g') - \psi(g) \mid\geq \tau\Big\}. \tag{2.8}$$

Finally, for $u > 0$ and $g \in \tilde{\mathcal{G}}_{\rho_0}$ we shall denote by $\overline{\phi}_{u,g}$ the map

$$\overline{\phi}_{u,g} : \ \psi \to \big(u \mid \psi(g) \mid \big) \wedge 2. \tag{2.9}$$

We shall denote by E_H the space of all real Borel functions on H. The next proposition is an adaptation of already known results on uniform convergence in law for families of processes. Its proof is postponed to the Appendix.

Proposition 1. *Let $(\psi_{n,h})_{n\geq 1,\, h\in\mathcal{H}_n}$ be a collection of maps from $\mathcal{G}_{\rho_0} \times H$ to $\mathbb{R}$ such that $\big\{\psi_{n,h}(\cdot, z),\ z \in H, n \geq 1, h \in \mathcal{H}_n\big\} \subset \mathcal{B}(\mathcal{G}_{\rho_0})$, $\big\{\psi_{n,h}(g, \cdot),\ g \in \mathcal{G}_{\rho_0}, n \geq 1, h \in \mathcal{H}_n\big\} \subset E_H$, and such that :*

1. *For each $n \geq 1$, $h \in \mathcal{H}_n$, $p \geq 1$, $\mathbf{g} \in \tilde{\mathcal{G}}_{\rho_0}^p$ and $\boldsymbol{\theta} \in \mathbb{Q}^p$, we have :*

$$\lim_{n\to\infty} \sup_{h\in\mathcal{H}_n} \Big| \int_H \phi_{\boldsymbol{\theta},\mathbf{g}}\big(\psi_{n,h}(\cdot, z)\big) d\mu(z) - \mathbb{E}\Big(\phi_{\boldsymbol{\theta},\mathbf{g}}\big(\mathcal{W}_{\mathcal{G}_{\rho_0}}\big)\Big)\Big| = 0; \tag{2.10}$$

2. *For each $\tau \in \mathbb{Q}^+$, we have :*

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \sup_{h\in\mathcal{H}_n} \int_H \tilde{\phi}_{\delta,\tau}\big(\psi_{n,h}(\cdot, z)\big) d\mu(z) = 0; \tag{2.11}$$

3. *For each $g \in \tilde{\mathcal{G}}_{\rho_0}$, we have :*

$$\lim_{u\to 0} \overline{\lim_{n\to\infty}} \sup_{h\in\mathcal{H}_n} \int_H \overline{\phi}_{u,g}\big(\psi_{n,h}(\cdot, z)\big) d\mu(z) = 0. \tag{2.12}$$

Then the following equivalent assertions are true :

– *For any closed set F of* $\big(\mathcal{B}(\mathcal{G}_{\rho_0}), ||\cdot||_{\mathcal{G}_{\rho_0}}\big)$, *we have :*

$$\overline{\lim_{n\to\infty}} \sup_{h\in\mathcal{H}_n} \mu^*\Bigg(\{z\in H,\ \psi_{n,h}(\cdot,z)\in F\}\Bigg) \le \mathbb{P}\big(\mathcal{W}_{\mathcal{G}_{\rho_0}}\in F\big); \qquad (2.13)$$

– *For any* $\phi\in\mathbb{BL}_{\mathcal{G}_{\rho_0}}$, *we have :*

$$\lim_{n\to\infty} \sup_{h\in\mathcal{H}_n} \Big| \int_H^* \phi\big(\psi_{n,h}(\cdot,z)\big)d\mu(z) - \mathbb{E}\Big(\phi\big(\mathcal{W}_{\mathcal{G}_{\rho_0}}\big)\Big)\Big| = 0. \qquad (2.14)$$

Here $\int_H^*$ *denotes the outer integral on H with respect to μ.*

Remark. The reader might notice that assumption (2.10) plays the role of a finite dimensional convergence condition, while (2.11) plays the role of an asymptotic equicontinuity condition. The role of (the unexpected) condition (2.12) is to enable an extrapolation of (2.10) to non rational vectors. Moreover the claimed equivalence between (2.13) and (2.14), can be proved, e.g., by enriching the arguments of Van der Vaart and Wellner (1996, Theorem 1.3.4, p.18) with a uniformity in $h\in\mathcal{H}_n$.

2.2. Step 2 : Poissonization and Variance Argument. In this subsection, we will give a rigorous statement for the heuristic (1.2), and hence justify a *variance argument*, by making use of an abstract poissonization technique due to Giné et al. (2003). Note that we can suppose without loss of generality that $\mathbb{P}\big(Z_1\in H\big) < 1/2$, since a simple union argument would imply Theorem 2 in its full generality. We first introduce, for $n\ge 1,\ h>0,\ g\in\mathcal{G}_{\rho_0},\ z\in H$, a poissonized version of $\Delta_n(g,h,z)$, namely :

$$\Delta\Pi_n(g,h,z) := \frac{\sum_{i=1}^{\eta_n}\Big[g_{h,z}(Z_i) - \mathbb{E}\big(g_{h,z}(Z_i)\big)\Big]}{\sqrt{nh^d f(z)}}, \qquad (2.15)$$

where η_n is a Poisson random variable with expectation n, independent of $(Z_i)_{i\ge 1}$. For measurability concerns, we shall write $\mathcal{T}$ for the σ-algebra of $\mathcal{B}(\mathcal{G}_{\rho_0})$ spanned by the collection of maps $\big\{\psi\to\psi(g),\ g\in\mathcal{G}_{\rho_0}\big\}$. The next proposition is a suitable generalization of Varron (2011, Proposition 3.2). We shall write the "diagonal" sets

$$A(h) := \Big\{(z,z')\in H^2,\ \{z+hJ\}\cap\{z'+hJ\}\neq\emptyset\Big\},$$

which play the same role as the sets $D(h)$ in the introduction.

Proposition 2. *There exists $h_0 > 0$ such that, for each $h \in (0, h_0)$, $n \in \mathbb{N}^*$, and for each measurable function ϕ from $\big(\mathcal{B}(\mathcal{G}_{\rho_0}), \mathcal{T}\big)$ to $\mathbb{C}$, we have :*

$$\mathbb{E}\Bigg(\Bigg(\int_H \Big[\phi\big(\Delta_n(\cdot,h,z)\big) - \mathbb{E}\Big(\phi\big(\Delta\Pi_n(\cdot,h,z)\big)\Big)\Big] d\mu(z)\Bigg)^2\Bigg)$$
$$\leq 2\int_{A(h)} \operatorname{Cov}\Big(\phi\big(\Delta\Pi_n(\cdot,h,z)\big), \phi\big(\Delta\Pi_n(\cdot,h,z')\big)\Big) d\mu(z)d\mu(z').$$

In particular, if ϕ is bounded by 1, then

$$\mathbb{E}\Bigg(\Bigg(\int_H \Big[\phi\big(\Delta_n(\cdot,h,z)\big) - \mathbb{E}\Big(\phi\big(\Delta\Pi_n(\cdot,h,z)\big)\Big)\Big] d\mu(z)\Bigg)^2\Bigg) \leq 2\delta(h).$$

PROOF. We choose h_0 such that $\mathbb{P}\big(Z_1 \in H + h_0 J\big) < 1/2$, with $H + h_0 J := \big\{x + h_0 y,\ x \in H,\ y \in J\big\}$. We then consider the semigroup $(D, +)$ generated by the signed measures $\delta_z - \mathbb{P}^{Z_1}$, $z \in \mathbb{R}^d$, with "+" denoting the usual sum of two finite signed measures. Given $\nu \in D$, $z \in H$, and $h > 0$, we shall write $\psi_{\nu,h,z} : g \to \int_{\mathbb{R}^d} g_{h,z} d\nu$ (as an element of $\mathcal{B}\big(\mathcal{G}_{\rho_0}\big)$). For measurability concerns, we endow D with the σ-algebra $\mathcal{D}$ spanned by the maps of the form $\nu \to \int_H \mathfrak{f}\big(\phi\big(\psi_{\nu,h,z}\big), z\big) d\mu(z)$, with ϕ measurable from $\big(\mathcal{B}(\mathcal{G}_{\rho_0}), \mathcal{T}\big)$ to $\mathbb{C}$ and $\mathfrak{f}$ bounded, Borel from $\mathbb{C} \times H$ to $\mathbb{R}$. By construction, $X_i := \delta_{Z_i} - \mathbb{P}^{Z_1}$, $i \geq 1$, defines a $\mathcal{D}$ measurable i.i.d. sequence.

Now, for fixed $h \in (0, h_0)$ and ϕ measurable from $\big(\mathcal{B}(\mathcal{G}_{\rho_0}), \mathcal{T}\big)$ to $\mathbb{C}$, we define

$$\mathfrak{H} :\ \nu \to \ \Bigg(\int_H \Big[\phi(\psi_{\nu,h,z}) - \mathbb{E}\Big(\phi(\Delta\Pi_n(\cdot,h,z))\Big)\Big] d\mu(z)\Bigg)^2,$$

as a measurable map from $(D, \mathcal{D})$ to $\mathbb{R}^+$. Consider the event $B := \big\{\nu \in \mathcal{D},\ \nu\big(H + h_0 J\big) + \mathbb{P}^{Z_1}(H + h_0 J) > 0\big\}$ and note that $\big\{X_1 \in B\big\} = \big\{Z_1 \in H + h_0 J\big\}$. Since $(HG2)$ implies $(g_{h,z}(\boldsymbol{z}))_{g \in \mathcal{G}_{\rho_0}} \equiv 0$ for $z \in H$ and $\boldsymbol{z} \notin H + h_0 J$, we have

$$\mathfrak{H}\Bigg(\sum_{i=1}^n X_i\Bigg) = \mathfrak{H}\Bigg(\sum_{i=1}^n \mathbb{1}_B(X_i) X_i\Bigg),$$

almost surely for each $n \in \mathbb{N}^*$. Hence, applying Giné (2003, Lemma 2.1) to X_i, $\mathfrak{H}$ and B defined as above, we have (since $\mathbb{P}(X_1 \in B) < 1/2$) :

$$\mathbb{E}\Bigg(\Bigg(\int_H \Big[\phi\big(\Delta_n(\cdot,h,z)\big) - \mathbb{E}\Big(\phi\big(\Delta\Pi_n(\cdot,h,z)\big)\Big)\Big] d\mu(z)\Bigg)^2\Bigg)$$
$$\leq 2\int_{H^2} \operatorname{Cov}\Big(\phi\big(\Delta\Pi_n(\cdot,h,z)\big), \phi\big(\Delta\Pi_n(\cdot,h,z')\big)\Big) d\mu(z)d\mu(z').$$

But, for fixed z, z' fulfilling $\{z+hJ\}\cap\{z'+hJ\} = \emptyset$, the usual properties of Poisson random measures imply $\Delta\Pi_n(\cdot, h, z) \perp\!\!\!\perp \Delta\Pi_n(\cdot, h, z')$, since two arbitrary functions $g_{h,z}, g'_{h,z'}$ (with $(g, g') \in \mathcal{G}^2_{\rho_0}$) have disjoint supports (by assumption $(HG2)$). Hence the last integral can be restricted to $A(h)$, which concludes the proof.

2.3. Step 3 : Expectation Argument for Poissonized Versions. Given a parameter $\rho \in (1, \rho_0)$ that shall be adjusted in the sequel, we define the subsequence $n_k := [\rho^k]$, $k \geq 1$. Clearly, Proposition 1 suggests to establish almost sure convergences of random quantities $\mu_n(\phi, h)$, for ϕ of the form $\phi_{\boldsymbol{\theta},\mathbf{g}}, \tilde{\phi}_{\delta,\tau}$ or $\overline{\phi}_{u,g}$. By Proposition 2, we already have a sharp control of their quadratic distances to the poissonized expectations $\int_H \mathbb{E}\big(\phi\big(\Delta\Pi_n(\cdot, h, z)\big)\big) d\mu(z)$. In this section, we shall provide upper bounds for quantities of the form

$$\overline{\lim_{k\to\infty}} \sup_{h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}} \int_H \Big| \mathbb{E}\Big(\phi(\Delta\Pi_{n_k}(\cdot, h, z)\Big) - a \Big| d\mu(z), \tag{2.16}$$

where a will be chosen depending upon the form of ϕ. Note that, by Fatou's lemma, it is sufficient to establish pointwise asymptotic results, as soon as limits do not depend on $z \in H_0$. This will be achieved through three separate propositions.

Proposition 3. *For each $p \geq 1$, $\boldsymbol{\theta} \in \mathbb{R}^p$ and $\mathbf{g} \in \mathcal{G}^p$, we have :*

$$\forall z \in H_0,\ \lim_{k\to\infty} \sup_{h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}} \Big| \mathbb{E}\Big(\phi_{\boldsymbol{\theta},\mathbf{g}}\Big(\Delta\Pi_{n_k}(\cdot, h, z)\Big)\Big) - \mathbb{E}\Big(\phi\big(\mathcal{W}_{\mathcal{G}_{\rho_0}}\big)\Big) \Big| = 0.$$

Proof. The proof is standard calculus. We omit it for sake of brevity.

Proposition 4. *For each $\tau > 0$, there exists a function r_τ, tending to zero at zero, and such that*

$$\forall z \in H_0, \forall \delta > 0,\ \overline{\lim_{k\to\infty}} \sup_{h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}} \mathbb{E}\Big(\tilde{\phi}_{\delta,\tau}\big(\Delta\Pi_{n_k}(\cdot, h, z)\big)\Big) \leq r_\tau(\delta).$$

Proof. For fixed $h > 0$ and $z \in H_0$, a direct consequence of Einmahl and Mason (1997, Proposition 3.1) is the following equality in law, as vectors indexed by $\mathcal{G}_{\rho_0}$:

$$\Delta\Pi_{n_k}(g, h, z) \stackrel{\mathcal{L}}{=} \frac{\sum_{i=1}^{\eta_{k,h,z}} g(Y_i^{(h,z)}) - \mathbb{E}\big(g(Y_i^{(h,z)})\big)}{\sqrt{n_k h^d f(z)}} + \pi_{k,h,z}\, \mathbb{E}\big(g(Y_i^{(h,z)})\big), \tag{2.17}$$

where $\eta_{k,h,z}$ is a Poisson random variable independent of $(Y_i^{(h,z)})_{i\geq 1}$, with expectation $n_k a(h,z)$ (recall (2.3)), $(Y_i^{(h,z)})_{i\geq 1}$ is i.i.d. having distribution

$$\mathbf{P}_{h,z} := \; A \to \mathbb{P}\Big(h^{-1}(Z_1 - z) \in A \Big| \; h^{-1}(Z_1 - z) \in J\Big), \tag{2.18}$$

and $\pi_{k,h,z}$ is defined as $(n_k h^d f(z))^{-1/2}(\eta_{k,h,z} - n_k a(h,z))$. Let us write

$$\begin{aligned} \mathcal{G}'_{\rho_0,\delta} &:= \big\{g - g',\; (g,g') \in \mathcal{G}^2_{\rho_0},\; \| \, g' - g \, \|_{\mathbf{P}_0,2} \leq \delta\big\}, \text{ and} \\ \boldsymbol{\Delta}(h,z,\delta) &:= \sup_{g \in \mathcal{G}'_{\rho_0,\delta}} \mathbb{E}\Big(\mid g\big(Y_i^{(h,z)}\big) \mid \Big). \end{aligned} \tag{2.19}$$

Equality (2.17) entails, for fixed $z \in H_0$, $k \geq 1$, and $h \in [h_{n_k}, \mathfrak{h}_{n_{k-1}}]$:

$$\begin{aligned} &\mathbb{E}\Big(\tilde{\phi}_{\delta,\tau}\big(\Delta\Pi_{n_k}(\cdot,h,z)\big)\Big) \\ \leq\; &\mathbb{P}\Big(\eta_{k,h,z} > 2n_k a(h,z)\Big) \\ &+ \max_{m \leq [2n_k a(h,z)]} \mathbb{P}\left(\frac{\sup_{g\in\mathcal{G}'_{\rho_0,\delta}} \Big| \sum_{i=1}^m \Big[g(Y_i^{(h,z)}) - \mathbb{E}\big(g(Y_i^{(h,z)})\big)\Big]\Big|}{\sqrt{n_k h^d f(z)}} > \frac{\tau}{2}\right) \\ &+\mathbb{P}\left(\mid \pi_{k,h,z} \mid \geq \frac{\tau}{2\boldsymbol{\Delta}(h,z,\delta)}\right) \\ =:\; &\mathbb{P}^{(1)}_{k,h,z} + \mathbb{P}^{(2)}_{k,h,z,\delta,\tau} + \mathbb{P}^{(3)}_{k,h,z,\delta}. \end{aligned}$$

The proof of Proposition 4 is achieved through the following three lemmas.

Lemma 1. *We have :*

$$\forall z \in H_0,\; \lim_{k\to\infty} \sup_{h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}} \mathbb{P}^{(1)}_{k,h,z} = 0.$$

PROOF. Apply the Chernoff bound and notice that

$$\forall z \in H_0,\; \lim_{k\to\infty} \inf_{h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}} n_k a(h,z) = \infty. \tag{2.20}$$

Lemma 2. *We have, for each $\tau > 0$:*

$$\forall z \in H_0,\; \forall \delta > 0,\; \overline{\lim_{k\to\infty}} \sup_{h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}} \mathbb{P}^{(3)}_{k,h,z,\delta} \leq \frac{2\delta}{\tau}.$$

Proof. Apply Markov's inequality to $| \pi_{k,h,z} |$ and notice that, by (2.23) below, we have, for each $\delta > 0$ and $z \in H_0$:

$$\overline{\lim_{k\to\infty}} \sup_{h_{n_k} \le h \le \mathfrak{h}_{n_{k-1}}} \mathbf{\Delta}(h, z, \delta) \le \delta.$$

For the next lemma, we shall use the notation

$$\mathcal{J}_{\mathcal{G}_{\rho_0}}(u) := \int_0^u \sup_{Q \text{ probab.}} \sqrt{\log N(\epsilon \,||\, G \,||_{Q,2}, \mathcal{G}_{\rho_0}, ||\cdot||_{Q,2})}d\epsilon, \quad \text{for } u > 0,$$

and $\mathcal{J}_{\mathcal{G}_{\rho_0}}(\infty) := \lim_{u\to\infty} \mathcal{J}_{\mathcal{G}_{\rho_0}}(u) < \infty$.

Lemma 3. *For some universal constant C_1, we have, for all $z \in H_0$, $\tau > 0$, $\delta > 0$*

$$\overline{\lim_{k\to\infty}} \sup_{h_{n_k} \le h \le \mathfrak{h}_{n_{k-1}}} \mathbb{P}^{(2)}_{k,h,z,\delta,\tau} \le \frac{C_1}{\tau} \mathcal{J}_{\mathcal{G}_{\rho_0}}\Big(\frac{\delta}{||\, G \,||_{\mathbf{P}_0,2}}\Big) \,||\, G \,||_{\mathbf{P}_0,2} \,. \tag{2.21}$$

Proof. Fix $\delta > 0$, $\tau > 0$, $z \in H_0$. The core of the proof essentially consists in adapting the usual Donsker and Glivenko-Cantelli arguments for classes satisfying (*Unif. entropy*), enriching them to a uniformity in h. Toward that aim, we heavily borrow arguments from Sheehy and Wellner (1992). Fix $k \ge 1$, and $h \in [h_{n_k}, \mathfrak{h}_{n_{k-1}}]$. We first apply the Montgomery-Smith inequality (Montgomery-Smith, 1993, Theorem 1) to obtain :

$$\begin{aligned}
\mathbb{P}^{(2)}_{k,h,z,\delta,\tau} &\le 3\,\mathbb{P}\Bigg(\sup_{g\in\mathcal{G}_{\rho_0,\delta}} \frac{\Big|\sum_{i=1}^{[2n_k a(h,z)]} \Big[g(Y_i^{(h,z)}) - \mathbb{E}\big(g(Y_i^{(h,z)})\big)\Big]\Big|}{\sqrt{n_k h^d f(z)}} > \frac{\tau}{20}\Bigg) \\
&\le \frac{60}{\tau}\mathbb{E}\Bigg(\sup_{g\in\mathcal{G}_{\rho_0,\delta}} \frac{\Big|\sum_{i=1}^{[2n_k a(h,z)]} \Big[g(Y_i^{(h,z)}) - \mathbb{E}\big(g(Y_i^{(h,z)})\big)\Big]\Big|}{\sqrt{n_k h^d f(z)}}\Bigg).
\end{aligned}$$

We then make use of the usual Rademacher symmetrization combined with subgaussianity of Rademacher processes (Van der Vaart and Wellner, 1996, p. 127-128) to obtain the following bound (C_1 denoting a universal constant) :

$$\mathbb{P}^{(2)}_{k,h,z,\delta,\tau} \le \frac{1}{\tau}\frac{C_1}{2}\sqrt{\frac{[2n_k a(h,z)]}{n_k h^d f(z)}}\mathbb{E}\Big(\mathcal{J}_{\mathcal{G}_{\rho_0}}\Big(\frac{\Theta}{\Gamma}\Big)\Gamma\Big),$$

$$\text{with } \Gamma^2 = \Gamma^2(n_k, h, z) \quad := \quad \frac{\sum_{i=1}^{[2n_k a(h,z)]} G^2\Big(Y_i^{(h,z)}\Big)}{[2n_k a(h,z)]},$$

$$\Theta^2 = \Theta^2(n_k, h, z) \quad := \quad \sup_{g \in \mathcal{G}'_{\rho_0,\delta}} \frac{\sum_{i=1}^{[2n_k a(h,z)]} g^2(Y_i^{(h,z)})}{[2n_k a(h,z)]}.$$

Recall that $G \geq 1$ by $(HG2)$. First note that, by standard probability calculus, if we prove that

$$\lim_{k\to\infty} \sup_{\substack{h_{n_k} \leq h \\ \leq \mathfrak{h}_{n_{k-1}}}} \mathbb{E}\left(\sup_{g \in \mathcal{G}'_{\rho_0,\delta}} \frac{\left| \sum_{i=1}^{[2n_k a(h,z)]} \Big[g^2(Y_i^{(h,z)}) - \mathbb{E}\big(g^2(Y_i^{(h,z)})\big)\Big] \right|}{[2n_k a(h,z)]} \right) = 0, \tag{2.22}$$

$$\overline{\lim_{k\to\infty}} \sup_{h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}} \sup_{g \in \mathcal{G}'_{\rho_0,\delta}} \mathbb{E}\Big(g^2(Y_1^{(h,z)})\Big) \leq \delta^2, \tag{2.23}$$

$$\forall \epsilon > 0, \ \overline{\lim_{k\to\infty}} \sup_{h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}} \mathbb{E}\Big(\Big| \Gamma^2(n_k, h, z) - || \, G \, ||^2_{\mathbf{P}_0,2} \Big| \Big) = 0, \tag{2.24}$$

then the following uniform convergence in probability holds

$$\forall \epsilon > 0, \ \lim_{k\to\infty} \sup_{\substack{h_{n_k} \leq h \\ \leq \mathfrak{h}_{n_{k-1}}}} \mathbb{P}\left(\mathcal{J}_{\mathcal{G}_{\rho_0}}\Big(\frac{\Theta(n_k,h,z)}{\Gamma(n_k,h,z)} \Big) \geq \mathcal{J}_{\mathcal{G}_{\rho_0}}\Big(\frac{\delta}{|| \, G \, ||_{\mathbf{P}_0,2}} \Big) + \epsilon \right) = 0.$$

This would imply, by the bounded convergence theorem (see., e.g. Williams (1991, p. 130), since each of the involved random variables take their values in $[0, 2C_1 \lambda(J) \mathcal{J}_{\mathcal{G}_{\rho_0}}(\infty)]$:

$$\overline{\lim_{k\to\infty}} \sup_{\substack{h_{n_k} \leq h \\ \leq \mathfrak{h}_{n_{k-1}}}} \mathbb{E}\left(\left(\frac{C_1}{2} \sqrt{\frac{[2n_k a(h,z)]}{n_k h^d f(z)}} \mathcal{J}_{\mathcal{G}_{\rho_0}}\Big(\frac{\Theta}{\Gamma}\Big) \right)^2 \right) \leq \left(C_1 \mathcal{J}_{\mathcal{G}_{\rho_0}}\Big(\frac{\delta}{|| \, G \, ||_{\mathbf{P}_0,2}} \Big) \right)^2,$$

and hence would conclude the proof, by using the Cauchy-Schwartz inequality in combination with (2.24).

To prove (2.22), and also (2.23) along the way, we shall adapt the Glivenko-Cantelli arguments of Van der Vaart and Wellner (1996, p. 123). First, we notice that :

$$\lim_{K\to\infty} \overline{\lim_{k\to\infty}} \sup_{h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}} \mathbb{E}\Big(G^2(Y_1^{(h,z)}) \mathbb{1}_{\{G^2(Y_1^{(h,z)}) > K\}} \Big) = 0. \tag{2.25}$$

Indeed we have, for fixed $K \in \mathbb{N}$ and $h > 0$:

$$\begin{aligned}\mathbb{E}\Big(G^2 \mathbb{1}_{\{G^2(\cdot)>K\}}(Y_1^{(h,z)})\Big) &\le \frac{\mathbb{E}\Big(G_{h,z}^2(Z_1)\mathbb{1}_{\{G_{h,z}^2(Z_1)\ge K\}\cap\{h^{-1}(Z_1-z)\in J\}}\Big)}{a(h,z)} \\ &= \frac{\mathbb{E}\Big(G_{h,z}^2(Z_1)\mathbb{1}_{\{G_{h,z}^2(Z_1)\ge K\}}\Big)}{a(h,z)}, \text{ by (HG2)}, \\ &\underset{h\to 0}{\to} \frac{1}{\lambda(J)}\int_J G^2(u)\mathbb{1}_{\{G^2(u)\ge K\}}du, \text{ by (2.1, 2.2)}. \quad (2.26)\end{aligned}$$

Note that replacing $G^2\mathbb{1}_{\{G^2(\cdot)>K\}}$ by $g-g'$ with $\| g-g' \|_{\mathbf{P}_0,2} \le \delta$ also implies (2.23), the required uniformity in (g,g') being provided by $(Hf2)$.
Next, following the steps of Van der Vaart and Wellner (1996, p. 123), by Rademacher symmetrization, we have, for fixed $K \in \mathbb{N}$, $k \in \mathbb{N}^*$, $h \in [h_{n_k}, \mathfrak{h}_{n_{k-1}}]$ and $\epsilon > 0$:

$$\mathbb{E}\left(\sup_{g\in\mathcal{G}'_{\rho_0,\delta}} \frac{\left|\sum_{i=1}^{[2n_k a(h,z)]}\Big[g^2(Y_i^{(h,z)})\mathbb{1}_{\{G^2(Y_i^{(h,z)})\le K\}} - \mathbb{E}\big(g^2(Y_i^{(h,z)})\mathbb{1}_{\{G^2(Y_i^{(h,z)})\le K\}}\big)\Big]\right|}{[2n_k a(h,z)]}\right)$$

$$\le \sqrt{24K}\,\mathbb{E}\left(\frac{\sqrt{1+\log N(\epsilon,\mathcal{F}'_\delta,\|\cdot\|_{\overline{\mathbb{P}}_{k,h,z},1})}}{\sqrt{[2n_k a(h,z)]}}\right), \quad (2.27)$$

where $\mathcal{F}'_\delta := \{g^2,\ g \in \mathcal{G}'_{\rho_0,\delta}\}$ and $\overline{\mathbb{P}}_{k,h,z}$ is the (random) empirical measure on $\big(Y_1^{(h,z)},\ldots,Y_{[2n_k a(h,z)]}^{(h,z)}\big)$. Now, by usual comparisons of covering numbers (see, e.g., Van der Vaart and Wellner (1996, p. 128)) we have, for each $\epsilon > 0$,

$$\begin{aligned}N\Big(\epsilon,\mathcal{F}'_\delta,\|\cdot\|_{\overline{\mathbb{P}}_{k,h,z},1}\Big) &\le N\left(\frac{\epsilon}{\| 4G \|_{\overline{\mathbb{P}}_{k,h,z},2}},\mathcal{G}'_{\rho_0,\delta},\|\cdot\|_{\overline{\mathbb{P}}_{k,h,z},2}\right) \\ &= N^2\left(\frac{\epsilon\,\| G \|_{\overline{\mathbb{P}}_{k,h,z},2}}{8\Gamma^2(n_k,h,z)},\mathcal{G}_{\rho_0},\|\cdot\|_{\overline{\mathbb{P}}_{k,h,z},2}\right).\end{aligned}$$

Moreover, since the map $\mathfrak{f}:\ u \to \sup_{Q \text{ probab.}} \sqrt{\log N\Big(u\,\| G \|_{Q,2},\mathcal{G}_{\rho_0},\|\cdot\|_{Q,2}\Big)}$ is positive and decreasing, we have, almost surely :

$$\sqrt{\log N\left(\frac{\epsilon\,\| G \|_{\overline{\mathbb{P}}_{k,h,z},2}}{8\Gamma^2(n_k,h,z)},\mathcal{G}_{\rho_0},\|\cdot\|_{\overline{\mathbb{P}}_{k,h,z},2}\right)} \le \frac{8\Gamma^2(n_k,h,z)}{\epsilon}\mathcal{J}_{\mathcal{G}_{\rho_0}}(\infty). \quad (2.28)$$

Now, as $\mathbb{E}\Big(\Gamma^2(n_k,h,z)\Big) = \mathbb{E}\Big(G^2\big(Y_1^{(h,z)}\big)\Big)$, and by (2.20), (2.25) and (2.27), we readily obtain (2.22).

Now the proof of (2.24) is made by first noticing that the latter assertion is true if G is formally replaced by $G\mathbb{1}_{G<K}$, for arbitrary $K \in \mathbb{N}$: just use the Bienamym-Tchebytchev inequality and (2.26). Then $G\mathbb{1}_{G\geq K}$ is made negligible by Markov's inequality together with (2.26). We omit details.

Now a combination of the three above lemmas concludes the proof of Proposition 4.

Proposition 5. *We have, for fixed $z \in H_0$, $g \in \tilde{\mathcal{G}}_{\rho_0}$ and $u > 0$:*

$$\overline{\lim_{k\to\infty}} \sup_{h_{n_k}\leq h\leq \mathfrak{h}_{n_{k-1}}} \mathbb{E}\Big(\overline{\phi}_{u,g}\big(\Delta\Pi_{n_k}(\cdot, h_{k,\ell}, z)\big)\Big) \leq (30\sqrt{2}+1) \mid\mid G \mid\mid_{\mathbf{P}_0,2} u.$$

PROOF. Fix $g \in \tilde{\mathcal{G}}_{\rho_0}$, $u > 0$ and $z \in H_0$. Using again representation (2.17), we write, for fixed k and $h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}$:

$$\begin{aligned} &\mathbb{E}\Big(\overline{\phi}_{u,g}\big(\Delta\Pi_{n_k}(\cdot, h, z)\big)\Big) \\ &\quad \leq 2\mathbb{P}\Big(\eta_{k,h,z} > 2n_k a(h,z)\Big) + \max_{m\leq[2n_k a(h,z)]} \frac{\mathbb{E}\Big(u\Big|\sum_{i=1}^{m} g\big(Y_i^{(h,z)}\big) - \mathbb{E}\Big(g\big(Y_i^{(h,z)}\big)\Big)\Big|\Big)}{\sqrt{n_k h^d f(z)}} \\ &\qquad + u\mathbb{E}\Big(\mid \pi_{k,h,z} \mid\Big) \times \mathbb{E}\Big(\mid g(Y_i^{(h,z)}) \mid\Big). \end{aligned} \tag{2.29}$$

The first term tends to 0 uniformly in $h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}$ by the Chernoff bound. Next, integrating (in t) Theorem 1 in Montgomery-Smith (1993) we see that the second term in (2.29) is bounded by

$$30u\,\mathbb{E}\left(\frac{\Big|\sum_{i=1}^{[2n_k a(h,z)]}\Big[g\big(Y_i^{(h,z)}\big) - \mathbb{E}\big(g\big(Y_i^{(h,z)}\big)\big)\Big]\Big|}{\sqrt{n_k h^d f(z)}}\right) \leq 30u\sqrt{\frac{[2n_k a(h,z)]\mathbb{E}\Big(G^2\big(Y_1^{(h,z)}\big)\Big)}{n_k h^d f(z)}}.$$

Moreover, by (2.1) and (2.2), we have

$$\overline{\lim_{k\to\infty}} \sup_{h_{n_k}\leq h\leq \mathfrak{h}_{n_{k-1}}} \sqrt{\frac{[2n_k a(h,z)]\mathbb{E}\Big(G^2\big(Y_1^{(h,z)}\big)\Big)}{n_k h^d f(z)}} \leq \sqrt{2} \mid\mid G \mid\mid_{\mathbf{P}_0,2} .$$

It remains to bound the third term of (2.29). By Jensen's inequality, and recalling the law of $Y_1^{(h,z)}$, we have, for all $k \geq 1$ and $h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}$:

$$\mathbb{E}\Big(\mid \pi_{k,h,z} \mid\Big) \times \mathbb{E}\Big(\mid g(Y_1^{(h,z)}) \mid\Big) \leq \sqrt{\frac{n_k a(h,z)}{n_k h^d f(z)}}\sqrt{\frac{\mathbb{E}(G_{h,z}^2(Z_1))}{a(h,z)}} = \sqrt{\frac{\mathbb{E}(G_{h,z}^2(Z_1))}{h^d f(z)}},$$

which, by $(Hf2)$, tends to $\mid\mid G \mid\mid_{\mathbf{P}_0,2}$ uniformly in $h_{n_k} \leq h \leq \mathfrak{h}_{n_{k-1}}$.

2.4. Step 4 : Proof of a Weak Form of Theorem 1. Recall that $n_k := \left[\rho^k\right]$ and notice that, by assumptions (1.7) and (1.8), we have

$$\sum_{k\geq 1} \mathfrak{h}_{n_k}^\alpha < \infty. \tag{2.30}$$

Such a summability will immediately entail a weak form of Theorem 1. For $k \geq 1$, we introduce the following discretization of $[h_{n_k}, \mathfrak{h}_{n_{k-1}}]$:

$$h_{k,\ell} := \rho^\ell h_{n_k},\ \ell \in \{0, \ldots, R_k\},\ \text{with } R_k := \left\lceil \frac{\log(\mathfrak{h}_{n_{k-1}}/h_{n_k})}{\log(\rho)} \right\rceil.$$

Making use of Proposition 2 together with the union bound, we have, for fixed ϕ of the form $\phi_{\boldsymbol{\theta},\mathbf{g}}, \tilde{\phi}_{\delta,\tau}$ or $\overline{\phi}_{u,g}$, and for fixed $\epsilon > 0$, ultimately as $k \to \infty$:

$$\mathbb{P}\Big(\max_{0\leq \ell\leq R_k} \Big| \int_H \phi\Big(\Delta_{n_k}(\cdot, h_{k,\ell}, z)\Big) - \mathbb{E}\Big(\phi\big(\Delta\Pi_{n_k}(\cdot, h_{k,\ell}, z)\big)\Big) d\mu(z) \Big| > \epsilon \Big)$$
$$\leq \sum_{\ell=0}^{R_k} \frac{2\delta(h_{k,\ell})}{\epsilon^2} = O\Big(\sum_{\ell=0}^{R_k} h_{k,\ell}^\alpha\Big) = O\Big(h_{n_k}^\alpha \rho^{\alpha R_k}\Big) = O\Big(\mathfrak{h}_{n_{k-1}}^\alpha\Big),$$

which is summable by (2.30). A combination of the Borel-Cantelli lemma with the arguments of *Step 3* leads to the conclusion that $\Delta_{n_k}(\cdot, h, z)_{z\in H}$ almost surely satisfies the conditions of Proposition 1 (with the choice of $\mathcal{H}_{n_k} := \big\{h_{k,\ell},\ \ell \in \{0, \ldots, R_k\}\big\}$), from where, almost surely :

$$\sup_{\phi\in\mathbb{BL}_{\mathcal{G}_{\rho_0}}} \overline{\lim_{k\to\infty}} \max_{\ell\leq R_k} \Big| \int_H^* \phi\big(\Delta_{n_k}(\cdot, h_{k,\ell}, z)\big) d\mu(z) - \mathbb{E}\Big(\phi\big(\mathcal{W}_{\mathcal{G}_{\rho_0}}\big)\Big) \Big| = 0. \tag{2.31}$$

The next steps of the proof will suitably interpolate (2.31), between consecutive n_{k-1}, n_k and $h_{k,\ell}, h_{k,\ell+1}$, in order to recover the full version of Theorem 2.

2.5. Step 5 : Interpolation Between Consecutive n_k. As usual for i.i.d. sums in Banach spaces, the main tools to interpolate between consecutive n_{k-1} and n_k are maximal inequalities. In order to use a *variance argument*, we shall use a poissonization technique due to Varron (2011), which adapts well to those kind of inequalities (see (2.37) in the sequel).

First, a close look at the arguments used to prove (2.31) leads to the conclusion that we also have, for each $\delta > 0$, almost surely :

$$\sup_{\substack{\phi\in\\ \mathbb{BL}_{\mathcal{G}_{\rho_0}}}} \overline{\lim_{k\to\infty}} \max_{\ell\leq R_k} \Big| \int_H^* \phi\big(\Delta_{[\delta(n_k-n_{k-1})]}(\cdot, h_{k,\ell}, z)\big) d\mu(z) - \mathbb{E}\big(\phi\big(\mathcal{W}_{\mathcal{G}_{\rho_0}}\big)\big) \Big| = 0, \tag{2.32}$$

because $[\delta(n_k - n_{k-1})]h^d_{n_k} \sim \delta(1-\rho^{-1})n_k h^d_{n_k}$, as $k \to \infty$. Also, using the same techniques as those used to obtain (2.21), it can be shown that, for fixed $z \in H_0$:

$$\overline{\lim_{k\to\infty}} \max_{\ell \le R_k} \mathbb{E}\Big(\| \Delta_{[\delta(n_k-n_{k-1})]}(\cdot, h_{k,\ell}, z) \|_{\mathcal{G}_{\rho_0}} \Big) \le C_2 \mathcal{J}_{\mathcal{G}_{\rho_0}}(\infty) \| G \|_{\mathbf{P}_0,2}, \tag{2.33}$$

where C_2 is a universal constant.

From now on, we shall work on selecting $\rho > 1$ close enough to 1 to interpolate (2.31) between the n_k, and then between the $h_{k,\ell}$ in *Step 6*. To avoid cumbersome notations, we shall make a repeated use of the symbol $\lim_{\rho\downarrow 1}$ without explicitly writing the dependency of n_k and $h_{k,\ell}$ upon the parameter ρ. Setting $N_k := \{n_{k-1}+1, \ldots, n_k\}$, we shall first show that, almost surely :

$$\begin{aligned}&\lim_{\rho\downarrow 1} \sup_{\phi \in \mathbb{BL}_{\mathcal{G}_{\rho_0}}} \overline{\lim_{k\to\infty}} \max_{n \in N_k} \max_{\ell \le R_k} \\ &\Big| \int_H^* \phi\big(\overline{\Delta}_n(\cdot, h_{k,\ell}, z)\big) d\mu(z) - \mathbb{E}\big(\phi(\mathcal{W}_{\mathcal{G}_{\rho_0}})\big)\Big| = 0, \text{ with} \\ &\overline{\Delta}_n(g,h,z) := \frac{\sum_{i=1}^n \Big[g_{h,z}(Z_i) - \mathbb{E}\big(g_{h,z}(Z_i)\big)\Big]}{\sqrt{n_k h^d_{k,\ell} f(z)}},\end{aligned} \tag{2.34}$$

for $z \in H$, $k \ge 1$, $n \in N_k$, $\ell \le R_k$, and $g \in \mathcal{G}_{\rho_0}$, $h_{k,\ell} \le h \le \rho h_{k,\ell}$. First notice that

$$\forall \phi \in \mathbb{BL}_{\mathcal{G}_{\rho_0}},\ \forall\, \psi, \psi' \in \mathcal{B}(\mathcal{G}_{\rho_0}),\ |\ \phi(\psi) - \phi(\psi')\ | \le \| \psi - \psi' \|_{\mathcal{G}_{\rho_0}} \wedge 2.$$

Hence, by subadditivity of outer integrals, we are turned to prove that

$$\lim_{\rho\downarrow 1} \overline{\lim_{k\to\infty}} \max_{\substack{n \in N_k \\ \ell \le R_k}} \int_H \Big[\| \overline{\Delta}_n(\cdot, h_{k,\ell}, z) - \overline{\Delta}_{n_k}(\cdot, h_{k,\ell}, z) \|_{\mathcal{G}_{\rho_0}} \wedge 2 \Big] d\mu(z) = 0. \tag{2.35}$$

Now define, for $k \ge 1$ and $\ell \in \{0, \ldots, R_k\}$:

$$\mathfrak{m}_{k,\ell} := \int_H \mathbb{E}\bigg(\Big[\max_{n=1,\ldots,\overline{\eta}_k} \Big\| \sqrt{\frac{n}{n_k}} \Delta_n(\cdot, h_{k,\ell}, z) \Big\|_{\mathcal{G}_{\rho_0}} \wedge 2 \Big] \bigg) d\mu(z),$$

where $\overline{\eta}_k$ is an Poisson r.v., independent of $(Z_i)_{i\ge 1}$, with expectation $n_k - n_{k-1}$. For fixed $\epsilon > 0$ we start with the following bound :

$$\mathbb{P}\Big(\max_{\substack{n\in N_k\\ \ell\le R_k}}\Big|\int_H\Big[\,||\,\overline{\Delta}_n(\cdot,h_{k,\ell},z)-\overline{\Delta}_{n_k}(\cdot,h_{k,\ell},z)\,||_{\mathcal{G}_{\rho_0}}\wedge 2\Big]d\mu(z)-\mathfrak{m}_{k,\ell}\Big|>\epsilon\Big)$$
$$\le\sum_{\ell=0}^{R_k}\mathbb{P}\Big(\Big|\int_H\Big[\max_{n\le n_k-n_{k-1}}\,||\,\overline{\Delta}_n(\cdot,h_{k,\ell},z)\,||_{\mathcal{G}_{\rho_0}}\wedge 2\Big]d\mu(z)-\mathfrak{m}_{k,\ell}\Big|>\epsilon\Big)$$
$$=:\sum_{\ell=0}^{R_k}\mathbb{P}_{k,\ell}. \tag{2.36}$$

Each of these $\mathbb{P}_{k,\ell}$ shall be controlled by a Poissonization technique of Varron (2011, Proposition 2.1). To gain in concision we shall make free use of some definitions and notations introduced in Varron (2011, Section 2), namely : *truncating* functions, *zero irrelevant* functions, and the symbol $\Sigma_{i=p}^{\to p+q}$. We take $(D,\mathcal{D},+)$ and $h_0>0$ as in the proof of Proposition 2.4. Given $k\ge 1$, $\ell\le R_k$, $p\ge 1$ and $(\nu_1,\ldots,\nu_p)\in D^p$ define

$$\mathfrak{p}_{k,\ell}(\nu_1,\ldots,\nu_p):=\Big\{z\to\Big[\max_{j=1,\ldots,p}\frac{||\,\psi_{\nu_j,h_{k,\ell},z}\,||_{\mathcal{G}_{\rho_0}}}{\sqrt{n_k h_{k,\ell}^d f(z)}}\wedge 2\Big]\Big\},$$

where z varies in H. Clearly, $\mathfrak{p}_{k,\ell}$ is *truncating*, as a map on $\bigcup_{p\ge1}D^p$. Moreover, defining $B:=\big\{\nu\in D,\nu(H+\mathfrak{h}_{n_{k-1}}J)+\mathbb{P}^{Z_1}\big(H+\mathfrak{h}_{n_{k-1}}J\big)>0\big\}$, we have $\mathbb{P}\big(X_i\in B\big)<1/2$ as soon as $\mathfrak{h}_{n_{k-1}}\le h_0$ (see the proof of Proposition 2.4), and also $\mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\to n}\mathbb{1}_B(X_i)X_i\Big)=\mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\to n}X_i\Big)$, almost surely for each $n\in\mathbb{N}$. Hence, applying Varron (2011, Proposition 2.1), we get, for k large enough :

$$\mathbb{E}\Big(\Big(\int_H\Big[\max_{n=1,\ldots,n_k-n_{k-1}}\Big|\Big|\overline{\Delta}_n(\cdot,h_{k,\ell},z)\Big|\Big|_{\mathcal{G}_{\rho_0}}\wedge 2\Big]d\mu(z)-\mathfrak{m}_{k,\ell}\Big)^2\Big)$$
$$=\mathbb{E}\Big(\Big(\int_H\mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\to n_k-n_{k-1}}\mathbb{1}_B(X_i)X_i\Big)(z)d\mu(z)-\mathfrak{m}_{k,\ell}\Big)^2\Big)$$
$$\le 2\int_{H^2}\mathrm{Cov}\Big(\mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\to\overline{\eta}_k}X_i\Big)(z),\mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\to\overline{\eta}_k}X_i\Big)(z')\Big)d\mu(z)d\mu(z')$$
$$=2\int_{H^2}\mathrm{Cov}\Big(\mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\to\overline{\eta}_k}\mathbb{1}_{B_{z,k,l}}(X_i)X_i\Big)(z),\mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\to\overline{\eta}_k}\mathbb{1}_{B_{z',k,l}}(X_i)X_i\Big)(z')\Big)$$
$$d\mu(z)d\mu(z'), \tag{2.37}$$

with $B_{z,k,l}:=\big\{\nu\in\mathcal{D},\ \nu(z+h_{k,\ell}J)+\mathbb{P}^{Z_1}(z+h_{k,\ell}J)>0\big\},\ z\in H.$

Now, if z and z' satisfy $\{z + h_{k,\ell}J\} \cap \{z' + h_{k,\ell}J\} = \emptyset$, then $B_{z,k,l} \cap B_{z',k,l} = \emptyset$. Since $\mathfrak{p}_{k,\ell}$ is obviously *zero irrelevant*, we can use Varron (2011, Proposition 2.2) to deduce that

$$\{z + h_{k,\ell}J\} \cap \{z' + h_{k,\ell}J\} = \emptyset \;\Rightarrow\; \mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\rightarrow\overline{\eta}_k} X_i\Big)(z) \perp\!\!\!\perp \mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\rightarrow\overline{\eta}_k} X_i\Big)(z').$$

This implies that (2.37) is bounded by

$$\begin{aligned} &2\int_{A(h_{k,\ell})} \mathrm{Cov}\Big(\mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\rightarrow\overline{\eta}_k} X_i\Big)(z), \mathfrak{p}_{k,\ell}\Big(\sum_{i=1}^{\rightarrow\overline{\eta}_k} X_i\Big)(z')\Big) d\mu(z)d\mu(z') \\ &\quad \le 8\delta(h_{k,\ell}). \end{aligned}$$

Note that the above inequality holds uniformly in ℓ, for all k large enough, since $\mathfrak{h}_{n_k} \to 0$, from where

$$\sum_{\ell=0}^{R_k} \mathbb{P}_{k,\ell} = O\Big(\sum_{\ell=0}^{R_k} \rho^{\alpha\ell} h_{n_k}^{\alpha}\Big) = O(\mathfrak{h}_{n_{k-1}}^{\alpha}).$$

Invoking the Borel-Cantelli lemma, the proof of (2.35) boils down to the following lemma.

Lemma 4. *We have* $\lim_{\rho\downarrow 1} \overline{\lim_{k\to\infty}} \max_{\ell=0,\dots,R_k} \mathfrak{m}_{k,\ell} = 0$.

PROOF. For fixed $\delta > 0$, $z \in H_0$, $k \ge 1$, $\ell \le R_k$, we split

$$\begin{aligned} &\mathbb{E}\Big(\Big[\max_{n=1,\dots,\overline{\eta}_k} \Big|\Big|\sqrt{\frac{n}{n_k}}\Delta_n(\cdot, h_{k,\ell}, z)\Big|\Big|_{\mathcal{G}_{\rho_0}} \wedge 2\Big]\Big) \\ &\le 2\mathbb{P}\Big(\overline{\eta}_k > (1+\delta)(n_k - n_{k-1})\Big) \\ &\quad + \mathbb{E}\Big(\max_{n=1,\dots,[(1+\delta)(n_k-n_{k-1})]} \Big|\Big|\sqrt{\frac{n}{n_k}}\Delta_n(\cdot, h_{k,\ell}, z)\Big|\Big|_{\mathcal{G}_{\rho_0}}\Big) \\ &\le 2\mathbb{P}\Big(\overline{\eta}_k > (1+\delta)(n_k - n_{k-1})\Big) \\ &\quad + 270\mathbb{E}\Big(\Big|\Big|\sqrt{\frac{[(1+\delta)(n_k - n_{k-1})]}{n_k}}\Delta_{[(1+\delta)(n_k-n_{k-1})]}(\cdot, h_{k,\ell}, z)\Big|\Big|_{\mathcal{G}_{\rho_0}}\Big), \end{aligned}$$

where the last term is obtained by integrating in t the Montgomery-Smith inequality (Montgomery-Smith, 1993, Theorem 1 and Corollary 4). For, say, the choice of $\delta = \rho^{-1}$, the first term obviously tends to 0 as $k \to \infty$, while the second is controlled by (2.33). Hence, by (2.32) together with $(n_k - n_{k-1}) \sim (1-\rho^{-1})n_k$, we have :

$$\overline{\lim_{k\to\infty}} \max_{\ell=0,\dots,R_k} \mathbb{E}\Big(\Big[\max_{n=1,\dots,\overline{\eta}_k} \Big|\Big|\sqrt{\frac{n}{n_k}}\Delta_n(\cdot,h_{k,\ell},z)\Big|\Big|_{\mathcal{G}_{\rho_0}} \wedge 2\Big]\Big) \leq \sqrt{\big(1+\frac{1}{\rho}\big)\big(1-\frac{1}{\rho}\big)}C_2\mathcal{J}_{\mathcal{G}_{\rho_0}}(\infty) \mid\mid G \mid\mid_{\mathbf{P}_0,2},$$

which proves Lemma 4 by Fatou's lemma.

2.6. Step 6 : Interpolating Between Consecutive $h_{k,\ell}$. To conclude the proof of Theorem 2, we shall interpolate between the $h_{k,\ell}$, by using the continuity assumption $(HG3)$. First fix $\rho \in (1,\rho_0)$ and notice that, almost surely, for any $k \geq 1$, $n \in N_k$, $\ell \leq R_k$, $z \in H_0$, $h \in [h_{k,\ell}, \rho h_{k,\ell}]$ and $\phi \in \mathbb{BL}_{\mathcal{G}}$:

$$\mid \phi(\overline{\Delta}_n(\cdot,h,z)) - \phi(\overline{\Delta}_n(\cdot,h_{k,\ell},z)) \mid \leq \mid\mid \psi_{\tilde\rho}(\overline{\Delta}_n(\cdot,h_{k,\ell},z)) - \overline{\Delta}_n(\cdot,h_{k,\ell},z) \mid\mid_{\mathcal{G}} \wedge 2,$$
$$\text{with } \tilde\rho := \frac{h}{h_{k,\ell}} \in [1,\rho], \text{ and } \psi_{\tilde\rho} : g \to \psi\Big(g\big(\tilde\rho^{-1}\cdot\big)\Big).$$

Hence, writing

$$\mathcal{G}'_\rho := \left\{g(\tilde\rho^{-1}\cdot) - g(\cdot),\ g\in\mathcal{G},\ \tilde\rho\in[1,\rho]\right\},$$

we have :

$$\begin{aligned}
&\sup_{\phi\in\mathbb{BL}_{\mathcal{G}}} \overline{\lim_{k\to\infty}} \max_{\ell=0,\dots,R_k} \max_{n\in N_k} \sup_{h_{k,\ell}\leq h\leq \rho h_{k,\ell}} \\
&\qquad \Big| \int_H^* \phi(\overline{\Delta}_n(\cdot,h,z))d\mu(z) - \int_H^* \phi(\overline{\Delta}_n(\cdot,h_{k,\ell},z))d\mu(z)\Big| \\
&\leq \overline{\lim_{k\to\infty}} \max_{\ell=0,\dots,R_k} \max_{n\in N_k} \sup_{1\leq\tilde\rho\leq\rho} \int_H \Big[\Big|\Big|\psi_{\tilde\rho}(\overline{\Delta}_n(\cdot,h_{k,\ell},z)) - \overline{\Delta}_n(\cdot,h_{k,\ell},z)\Big|\Big|_{\mathcal{G}} \wedge 2\Big] d\mu(z) \\
&= \overline{\lim_{k\to\infty}} \max_{\ell=0,\dots,R_k} \max_{n\in N_k} \int_H \Big[\mid\mid \overline{\Delta}_n(\cdot,h_{k,\ell},z) \mid\mid_{\mathcal{G}'_\rho} \wedge 2\Big] d\mu(z) \\
&= \mathbb{E}\Big(\mid\mid \mathcal{W}_{\mathcal{G}'_\rho} \mid\mid_{\mathcal{G}'_\rho} \wedge 2\Big), \text{ by (2.34).}
\end{aligned}$$

Now assumption $(HG3)$ plays its role here, as it entails :

$$\lim_{\rho\downarrow 1} \mathbb{E}\Big(\mid\mid \mathcal{W}_{\mathcal{G}'_\rho} \mid\mid_{\mathcal{G}'_\rho} \wedge 2\Big) = 0, \tag{2.38}$$

which in turn implies (by the dominated convergence theorem under μ) :

$$\lim_{\rho\downarrow 1} \overline{\lim_{k\to\infty}} \sup_{h_{n_k}\leq h\leq \mathfrak{h}_{n_{k-1}}} \max_{n\in N_k} \Big| \int_H^* \phi(\overline{\Delta}_n(\cdot,h,z))d\mu(z) - \mathbb{E}\big(\phi(\mathcal{W}_{\mathcal{G}'_\rho})\big)\Big| = 0, \text{ a.s.}$$

The proof of Theorem 2 is then concluded by noticing that, for large enough k, and for each $n \in N_k, \ell \leq R_k, h \in [h_{k,\ell}, \rho h_{k,\ell}]$, we can write

$$\Delta_n(\cdot, h, z) = \Big(\frac{nh^d}{n_k h_{k,\ell}^d}\Big)^{1/2} \overline{\Delta}_n(\cdot, h, z), \text{ with } \rho^{-1/2} \leq \Big(\frac{nh^d}{n_k h_{k,\ell}^d}\Big)^{1/2} \leq \rho^{d/2},$$

and again adjust ρ sufficiently close to 1. We omit details.

3 Proof of Theorem 3

It is well known that, in empirical processes theory, conditions (*Unif. entropy*) and (*Bracketing*) differ in their roles only for proving asymptotic equicontinuity. As a natural consequence, the proof of Theorem 3 is achieved the same way as in the preceding section, but with one change that we will describe immediately : of course, Proposition 1 remains true if we substitute (2.11) by the following assumption :

For each $\tau > 0$, there exists a finite partition $\mathcal{G}_1, \ldots, \mathcal{G}_p$ of $\mathcal{G}$ such that :

$$\begin{aligned} &\forall q \in \{1, \ldots, p\}, \ \overline{\lim_{k\to\infty}} \sup_{h \in \mathcal{H}_n} \textstyle\int_H \overset{\approx}{\phi}_{q,\tau}(\psi_{n,h}(\cdot, z)) \leq \tau, \text{ with} \\ &\overset{\approx}{\phi}_{q,\tau} : \ \psi \to 1\!\!1_{[\tau, +\infty]}\Big(\sup_{(g,g') \in \mathcal{G}_q^2} \Big|\psi(g) - \psi(g')\Big|\Big). \end{aligned} \tag{3.1}$$

Then the rest of the proof remains exactly the same as in the proof of Theorem 2, until the step where Lemma 3 has to be replaced by the following lemma.

Lemma 5. *For any $\tau > 0$, there exists a finite partition $\mathcal{G}_1, \ldots, \mathcal{G}_p$ of $\mathcal{G}_{\rho_0}$ such that, for each $q \leq p$:*

$$\overline{\lim_{k\to\infty}} \max_{\ell=0,\ldots,R_k} \int_H \mathbb{E}\Big(\overset{\approx}{\phi}_{q,\tau}\big(\Delta\Pi_{n_k}(\cdot, h_{k,\ell}, z)\big)\Big) d\mu(z) \leq \tau. \tag{3.2}$$

Proof. We shall write, for a class of functions $\mathcal{F}$, a measure Q, and $\delta > 0$:

$$\mathcal{J}_{[]}\big(\delta, \mathcal{F}, ||\cdot||_{Q,2}\big) := \int_0^\delta \sqrt{\log N_{[]}\big(\epsilon, \mathcal{F}, ||\cdot||_{Q,2}\big)} d\epsilon. \tag{3.3}$$

Now fix $\tau > 0$. By assumption $(Hf1')$, there exists $1 \leq L < \infty$, a finite union of open balls O, and $h_0 > 0$ such that :

$$\mu(H_0 \cap O) \geq \mu(H) - \tau/2, \tag{3.4}$$

$$\frac{1}{L} \leq f(z) \leq L \text{ for all } z \in \big(O + h_0 J\big) \cap H_0. \tag{3.5}$$

Recall that $\mathbf{P}_{h,z}$ is defined by (2.18). As O is a finite union of open balls, a combination of (3.5) and $(HG2)$ entails $L^{-2}\mathbf{P}_0 \le \mathbf{P}_{h,z} \le L^2\mathbf{P}_0$, for $h \le h_0$ small enough, and for each $z \in H_0 \cap O$. This implies

$$\forall \delta > 0,\ N_{[]}\big(\delta, \mathcal{G}'_q, ||\cdot||_{\mathbf{P}_{h_{k,\ell},z},2}\big) \le N_{[]}\Big(\frac{\delta}{L^2}, \mathcal{G}'_q, ||\cdot||_{\mathbf{P}_0,2}\Big),$$
$$\text{with } \mathcal{G}'_q := \big\{g - g',\ (g,g') \in \mathcal{G}_q^2\big\}. \tag{3.6}$$

Now consider a parameter $\delta > 0$ that will be adjusted in the sequel. We split $\mathcal{G}_{\rho_0}$ into a finite partition $\mathcal{G}_1, \ldots, \mathcal{G}_p$, for which each $\mathcal{G}_q$ is included in a bracket $[\mathfrak{g}^q, \overline{\mathfrak{g}}^q]$ with $||\ \mathfrak{g}^q - \overline{\mathfrak{g}}^q\ ||_{\mathbf{P}_0,2} < \delta$. In order to apply the dominated convergence theorem, we will investigate the following limit, for fixed $z \in H_0 \cap O$ and $q \le p$:

$$\overline{\lim_{k\to\infty}} \max_{\ell=0,\ldots,R_k} \mathbb{E}\Big(\sup_{g\in\mathcal{G}'_q} \Big|\Delta\Pi_{n_k}(g, h_{k,\ell}, z)\Big|\Big). \tag{3.7}$$

To bound that limit, we first fix $k \ge 1$, $\ell \le R_k$ and proceed as at the beginning of the proof of Proposition 4, until we need to bound, for fixed $z \in H_0$:

$$\mathbb{P}^{[2]}_{k,\ell,z} := \mathbb{P}\left(\frac{\sup_{g\in\mathcal{G}'_q}\Big|\sum_{i=1}^{[2n_k a(h_{k,\ell},z)]}\Big[g(Y_i^{(h_{k,\ell},z)}) - \mathbb{E}\big(g(Y_i^{(h_{k,\ell},z)})\big)\Big]\Big|}{\sqrt{n_k h^d_{k,\ell} f(z)}} > \frac{\tau}{20}\right).$$

We then apply Van der Vaart (1998, Lemma 19.34) with envelope $2G$ combined with Markov's inequality to obtain (for a universal constant C_3)

$$\mathbb{P}^{[2]}_{k,\ell,z} \le \frac{C_3}{\tau}\sqrt{\frac{[2n_k a(h_{k,\ell},z)]}{n_k h^d_{k,\ell} f(z)}}\Big[\mathcal{J}_{[]}(\delta, \mathcal{G}'_q, ||\cdot||_{\mathbf{P}_{h_{k,\ell},z},2})$$
$$+\frac{1}{\mathfrak{a}(\delta, h_{k,\ell}, z)}\mathbb{E}\Big(G^2\big(Y_1^{(h_{k,\ell},z)}\big)\mathbb{1}_{\big\{G\big(Y_1^{(h_{k,\ell},z)}\big)\ge \mathfrak{a}(\delta,h_{k,\ell},z)\sqrt{[2n_k a(h_{k,\ell},z)]}\big\}}\Big)\Big],$$

where

$$\mathfrak{a}(\delta, h, z) := \frac{\delta}{\sqrt{\log N_{[]}(\delta, \mathcal{G}'_q, ||\cdot||_{\mathbf{P}_{h,z},2})}}. \tag{3.8}$$

But, as soon as $\mathfrak{h}_{n_{k-1}}$ is small enough we have, by (3.6), for each $\ell \le R_k$:

$$\mathcal{J}_{[]}\Big(\delta, \mathcal{G}'_q, ||\cdot||_{\mathbf{P}_{h_{k,\ell},z},2}\Big) \le L^2\mathcal{J}_{[]}\Big(\frac{\delta}{L^2}, \mathcal{G}'_q, ||\cdot||_{\mathbf{P}_0,2}\Big), \text{ and}$$
$$\mathfrak{a}(\delta, h_{k,\ell}, z) \ge \frac{\delta}{\sqrt{\log N_{[]}\big(\delta, \mathcal{G}'_q, ||\cdot||_{L^{-2}\mathbf{P}_0,2}\big)}}. \tag{3.9}$$

Hence

$$\underline{\lim}_{k\to\infty} \min_{\ell=0,\dots,R_k} \mathfrak{a}(\delta, h_{k,\ell}, z) \geq \frac{\delta}{\sqrt{\log N_{[]}(\delta, \mathcal{G}'_q, ||\cdot||_{L^{-2}\mathbf{P}_0,2})}} > 0, \text{ thence, by (2.26),}$$

$$\overline{\lim}_{k\to\infty} \max_{\ell=0,\dots,R_k} \mathbb{E}\Big(G^2\big(Y_1^{(h_{k,\ell},z)}\big)\mathbb{1}_{\big\{G\big(Y_1^{(h_{k,\ell},z)}\big)\geq \mathfrak{a}(\delta,h_{k,\ell},z)\sqrt{[2n_k a(h_{k,\ell},z)]}\big\}}\Big) = 0.$$

By (2.2) and (3.9) we conclude that, for each $z \in H_0 \cap O$:

$$\overline{\lim}_{k\to\infty} \max_{\ell\leq R_k} \mathbb{P}^{[2]}_{k,\ell,z} \leq \frac{C_3}{\tau}\sqrt{2}L^2 \mathcal{J}_{[]}\Big(\frac{\delta}{L^2}, \mathcal{G}'_q, ||\cdot||_{\mathbf{P}_0,2}\Big) \leq \frac{4C_3L^2}{\tau}\mathcal{J}_{[]}\Big(\frac{\delta}{L^2}, \mathcal{G}, ||\cdot||_{\mathbf{P}_0,2}\Big),$$

where the last bound is a consequence of $N_{[]}(\epsilon, \mathcal{G}'_q, ||\cdot||_{\mathbf{P}_0,2}) \leq N_{[]}(\frac{\epsilon}{2}, \mathcal{G}, ||\cdot||_{\mathbf{P}_0,2})^2$ for each $\epsilon > 0$. As neither L, O, h_0, C_0 or τ does depend upon δ, we can now choose $\delta > 0$ so that the last bound is less than $\tau/2$. This, combined with (3.4) and Fatou's lemma, concludes the proof.

The remainder of the proof of Theorem 3 follows the same steps as in the proof of Theorem 2. We omit details.

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Appendix : Some Minor Proofs

Proof of Corollary 1. Assume that the conditions of Theorem 2 are satisfied. Fix $n \geq 1$, $a \in \mathbb{Q} \cap]1/2d, 1/d]$, and $b \in \mathbb{Q} \cap]1/\alpha, 2/\alpha[$. Then set $h_n(a) := n^{-a}$, and define $\mathfrak{h}_n(b) := \log(n)^{-b}$ whenever $\log(n)^{-b} > h_n(1/2d)$ and $\mathfrak{h}_n(b) := h_n(1/2d)$ otherwise. Applying Theorem 2, repeatedly for $a \in \mathbb{Q} \cap]1/2d, 1/d]$ and $b \in \mathbb{Q} \cap]1/\alpha, 2/\alpha[$, to the sequences $(h_n(a), \mathfrak{h}_n(b))_{n \geq 1}$, we have, with probability one :

"For each $a \in \mathbb{Q} \cap]1/2d, 1/d]$ and $b \in \mathbb{Q} \cap]1/\alpha, 2/\alpha[$, and for each closed set F of $\big(\mathcal{B}(\mathcal{G}), || \cdot ||_{\mathcal{G}}\big)$, we have

$$\overline{\lim_{n\to\infty}} \sup_{h_n \leq h \leq \mathfrak{h}_n} \mu^* \bigg(\{z \in H,\ \Delta_n(\cdot, h, z) \in F\} \bigg) \leq \mathbb{P}\big(\mathcal{W}_{\mathcal{G}} \in F\big)". \tag{4.1}$$

Combine that result with the following consequence of (1.10) and (1.11)

$$\mathbb{P}\bigg(\exists a \in \mathbb{Q} \cap]1/2d, 1/d], \exists b \in \mathbb{Q} \cap]1/\alpha, 2/\alpha[, \exists n_0 \geq 1, \forall n \geq n_0,\ h_n^* \in [h_n(a), \mathfrak{h}_n(b)]\bigg) = 1,$$

to conclude the proof.

Proof of Proposition 1. We consider the $\psi_{n,h}(\cdot, z)$ as random elements of $\mathcal{B}(\mathcal{G}_{\rho_0})$, with underlying probability space $\big(H, \mathcal{B}, \mu\big)$ ($\mathcal{B}$ being the Borel σ-algebra). By $(HG2), (HG3)$ and the dominated convergence theorem (with envelope G), the class $\tilde{\mathcal{G}}_{\rho_0}$ is dense in the $|| \cdot ||_{\mathbf{P}_0,2}$-totally bounded set $\mathcal{G}_{\rho_0}$. According to (2.11), for any $\tau > 0$, we can find a finite collection $(\mathcal{G}_j, g_j)_{j=1,\ldots,p}$, with $(\mathcal{G}_j)_{j=1,\ldots,p}$ being a partition of $\mathcal{G}_{\rho_0}$ and $g_j \in \tilde{\mathcal{G}}_{\rho_0} \cap \mathcal{G}_j$, such that

$$\overline{\lim_{n\to\infty}} \sup_{h \in \mathcal{H}_n} \mu\Bigg(\bigg\{ z \in H,\ \Big|\Big| \psi_{n,h}(\cdot, z) - \sum_{j=1}^{p} \psi_{n,h}(g_j, z) \mathbb{1}_{\mathcal{G}_j}(\cdot) \Big|\Big|_{\mathcal{G}_{\rho_0}} \geq \tau \bigg\} \Bigg) \leq \tau.$$

Hence, the proof amounts to show that, for each $p \geq 1$ and $\boldsymbol{g} \in \tilde{\mathcal{G}}_{\rho_0}^p$,

$$\lim_{n\to\infty} \sup_{h \in \mathcal{H}_n} \sup_{\mathcal{L} \in \mathbb{BL}(\mathbb{R}^p)} \Big| \int_H \mathcal{L}\big(\psi_{n,h}(\boldsymbol{g}, z)\big) d\mu(z) - \mathbb{E}\Big(\mathcal{L}\big(\mathcal{W}_{\mathcal{G}_{\rho_0}}(\boldsymbol{g})\big)\Big) \Big| = 0, \tag{4.2}$$

where $\mathbb{BL}(\mathbb{R}^p)$ denotes the set of all 1-Lipschitz functions from $\mathbb{R}^p$ to $\mathbb{R}$ (with Euclidian norms) that are bounded by 1. To prove (4.2), we first notice that, by regularity of $x \to e^{ix}$, we have, for each $n \geq 1, h \in \mathcal{H}_n$ and $\boldsymbol{\theta}, \boldsymbol{\theta'} \in \mathbb{R}^p$:

$$\Big| \int_H \phi_{\boldsymbol{\theta}, \mathbf{g}}\big(\psi(\cdot, h, z)\big) d\mu(z) - \int_H \phi_{\boldsymbol{\theta'}, \mathbf{g}}\big(\psi(\cdot, h, z)\big) d\mu(z) \Big|$$

$$\leq p \max_{j=1,\dots,p} \int_H \overline{\phi}_{|\boldsymbol{\theta}-\boldsymbol{\theta}'|_p, g_j}\big(\psi_{n,h}(\cdot, z)\big) d\mu(z),$$

where $| \boldsymbol{u} |_p := \max\big\{ | u_1 |, \dots, | u_p | \big\}$ for $\boldsymbol{u} = (u_1, \dots, u_p) \in \mathbb{R}^p$. Hence, as (2.12) provides a suitable control of that uniform continuity modulus, uniformly in $h \in \mathcal{H}_n$ as $n \to \infty$, the validity of assertion (2.10) can be extended to each $\boldsymbol{\theta} \in \mathbb{R}^p,\ p \in \mathbb{N}^*$.
The next lemma (for which we were not able to find an appropriate statement in the existing literature) provides the missing link between uniform convergences of characteristic functions and uniform convergence of probability measures with respect to the bounded-Lipschitz distance. This will conclude the proof of Proposition 1.

Lemma 6. *Let $(P_{n,h})_{n\geq 1,\, h\in\mathcal{H}_n}$ be a sequence of collections of probability measures on $\mathbb{R}^p$ and P another probability measure on $\mathbb{R}^p$. Write $\phi_{P_{n,h}}$ and ϕ_P for the corresponding characteristic functions. Assume that, for each $\boldsymbol{\theta} \in \mathbb{R}^p$, we have*

$$\lim_{n\to\infty} \sup_{h\in\mathcal{H}_n} | \phi_{P_{n,h}}(\boldsymbol{\theta}) - \phi_P(\boldsymbol{\theta}) | = 0, \tag{4.3}$$

with $| \cdot |$ denoting the modulus on $\mathbb{C}$. Then we have :

$$\lim_{n\to\infty} \sup_{h\in\mathcal{H}_n} \sup_{\mathcal{L}\in\mathbb{BL}(\mathbb{R}^p)} \Big| \int_{\mathbb{R}^p} \mathcal{L} dP_{n,h} - \int_{\mathbb{R}^p} \mathcal{L} dP \Big| = 0.$$

PROOF. For fixed $n \geq 1,\ h \in \mathcal{H}_n$, we have (see, e.g., (Dudley, 1999, p. 325))

$$\forall\epsilon > 0,\ \mathbf{P}_{n,h}\Big(\big\{\boldsymbol{u} \in \mathbb{R}^p,\ | \boldsymbol{u} |_p > 1/\epsilon\big\}\Big) \leq \frac{7p}{\epsilon} \max_{j=1,\dots,p} \int_0^\epsilon \big(1 - Re\big(\phi_{n,h}(t\boldsymbol{e}_j)\big)\big) dt,$$

$\boldsymbol{e}_j$ being the canonical j^{th} vector of $\mathbb{R}^p$. Hence, by continuity of ϕ_P, we have:

$$\lim_{M\to\infty} \overline{\lim_{n\to\infty}} \sup_{h\in\mathcal{H}_n} P_{n,h}\Big(\big\{\boldsymbol{u} \in \mathbb{R}^p,\ | \boldsymbol{u} |_p > M\big\}\Big) = 0. \tag{4.4}$$

Denote by $\mathbb{L}$ the (vector) algebra spanned by the maps $\boldsymbol{u} \to \exp\big(\boldsymbol{i} < \boldsymbol{\theta}, \boldsymbol{u} >\big)$, $\boldsymbol{\theta} \in \mathbb{R}^p$. The Stone-Weierstrass approximation theorem together with the Arzela-Ascoli theorem imply, for fixed $\tau > 0$, the existence of a finite collection $(\mathcal{L}_1, \dots, \mathcal{L}_R) \in \mathbb{L}^R$ such that

$$\sup_{\mathcal{L}\in\mathbb{BL}(\mathbb{R}^p)} \inf_{r=1,\dots,R} \sup_{\boldsymbol{u}\in[-M,M]^p} | \mathcal{L}(\boldsymbol{u}) - \mathcal{L}_r(\boldsymbol{u}) | \leq \tau. \tag{4.5}$$

Since, by (4.3),

$$\overline{\lim_{n\to\infty}} \sup_{h\in\mathcal{H}_n} \max_{r=1,\dots,R} \left| \int_{\mathbb{R}^p} \mathcal{L}_r dP_{n,h} - \int_{\mathbb{R}^p} \mathcal{L}_r dP \right| = 0, \tag{4.6}$$

the proof is straightforwardly concluded with the triangle inequality.

Davit Varron
Laboratoire de Mathématiques de
Besançon, UMR CNRS 6623
Université de Franche-Comté
16, route de Gray, 25000
Besançon, France
E-mail: dvarron@univ-fcomte.fr

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