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On Regularities of Mass Phenomena

Victor I. Ivanenko, Valery A. Labkovsky
Kyiv Polytechnical Institute, Kyiv, Ukraine

Abstract

This paper presents a result that provides a positive answer to the question of existence of regularities of the so called *random in a broad sense* mass phenomena (Kolmogorov, 1986). The theorem of existence of statistical regularities of mass phenomena in the form of closed in weak-$*$ topology families of finitely-additive probability distributions, and their significance to decision theory, constitute the content of this paper.

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1 Introduction

The interest in studying the properties of *mass phenomena* (MP) is not new. In Jarvik (1951) and Borel (1956) the authors pointed out the difficulties arising in the process of modeling of the social MP. The problem of revealing the regularities of MP becomes more and more important, especially in relation to the instability of financial markets and other economic objects, that makes forecasting in this area very unreliable (Taleb, 2001; Mandelbrot and Hudson, 2006; Munier, 2012).

Intuitively it is clear that MP can be deterministic (DMP), i.e., such that we can predict their behavior, as well as random (RMP), i.e., such that we can not predict their behavior. In turn, random mass phenomena can be of two types. In particular, in Borel (1956) one reads:"Some contemporary theoreticians ... think that probability could be defined as frequency for a very large number of trials. If for a very big number of trials this frequency does not tend to a limit, but fluctuates more or less between different limits, one needs to affirm that probability p does not remain constant and changes in the process of trials. This concerns, for example, human mortality rate in the course of centuries, since the progress of medicine and hygiene leads to the increase of life duration."

Khinchin (1961) raises the issue of which RMP posses the properties that are necessary for application of the probability theory to their description.

In Kolmogorov (1986) we find the following remark: "Speaking of randomness in the ordinary sense of this word, we mean those phenomena in which we do not find regularities allowing us to predict their behavior. Generally speaking, there are no reasons to assume that random in this sense phenomena are subject to some probabilistic laws. Hence, it is necessary to distinguish between randomness in this *broad sense* and *stochastic* randomness (which is the subject of probability theory)".

However, what do the words "do not find regularities allowing us to predict their behavior" mean? It is unlikely that these words imply that such regularities do not exist at all. More likely these words point to the problem of finding statistical regularities of *random in a broad sense* mass phenomena (RBSMP), that is regularities of asymptotic behavior of different average values that characterize these phenomena. These values can be frequencies of hitting in given subsets, arithmetic averages of some functionals, and so on.

Recall that RMP are called *statistically stable* or *stochastic* if with the increase of the number of "trials" all those different averages tend to limits (and if some other conditions are verified as well, see details in Kolmogorov (1986)). Thus probability theory assumes the existence of statistical regularities of stochastic random mass phenomena. At the same time the question of existence of regularities of non-stochastic random mass phenomena has remained open.[1]

Analogously, one can consider statistical regularities, i.e. properties of asymptotic behavior, of deterministic mass phenomena. Therefore in what follows we consider in this perspective RMP as well as DMP, and regard both of them simply as MP.

The theorem of existence of statistical regularities of mass phenomena in the form of closed in weak-$*$ topology sets of finitely-additive probability distributions, and their significance to decision theory, constitute the content of this paper.

Families of probability distributions appear in literature more and more often, and in different contexts. They were considered in game theory (Shapley, 1955) in order to study non-additive set functions. In the so called subjective decision theory these families appear as consequence of

[1]The term "nonstochastic" appeared in Vyugin (1985) in the context of Kolmogorov's complexity, meaning "more complex than stochastic".

the axioms of rational choice (Ivanenko and Labkovskii, 1986a, b; Gilboa and Schmeidler, 1989; Chateauneuf, 1991; Cassadesus-Masonel, Klibanoff and Ozdenoren, 2000), where, similarly to robust statistics (Huber, 1981), they were sometimes interpreted as families of a priori distributions. Families of probabilities appear in many other frameworks such as imprecise, interval, non-additive, upper and lower probabilities (Wiley and Fine, 1982; Walley, 1991; Kuznetsov, 1991; Marinacci, 1999). Statistical instability has been related to the concept of soft sets and the so called soft probabilities in Molodtsov (2013). Finally, in Fierens et al. (2009), for the finite sequence of random values with the so called chaotic probability, a solution of the statistical inference problem in the form of a family of probability distributions was proposed. The theorem of existence of statistical regularities of mass phenomena can serve as a complement for many such approaches to the problem of randomness in a broad sense. Namely, that in order to describe statistically unstable random phenomena one can continue using traditional probability theory methods, though somewhat extended.

The paper is organized as follows. In Section 2 we introduce all necessary definitions (statistically equivalent sequences and sampling nets, regularity and statistical regularity) and formulate the theorem of existence of statistical regularities. In particular, it turns out that any sampling net has a statistical regularity - a non-empty set of limit points of the corresponding net of frequency distributions, and vice versa, to any regularity - a closed in weak-$*$ topology set of finitely-additive probability distributions - one can put into correspondence a sampling net whereof the statistical regularity coincides with the original regularity. If some specific conditions (Definition 5) are satisfied, then the regularity reduces to one probability measure - the random mass phenomenon is stochastic. In Section 3 we discuss the relation of statistical regularities to decision theory, namely to the so called preference representation theorems. Section 4 contains the proof of the theorem of existence of statistical regularities. Section 5 contains some concluding remarks that we would like to emphasize.

2 Theorem of Existence of Statistical Regularities

An ordinary sequence is the simplest example of a mass phenomenon.

Definition 1. *Let X be an arbitrary set. Two sequences $\overline{x}^{(1)}$ and $\overline{x}^{(2)}$ of elements of the set X are called statistically equivalent (S-equivalent) if and* *only if for any natural number m and any bounded mapping $\gamma \in (X \to \mathbb{R}^m)$ the set of limit points of the sequence*

$$\left\{y_n^{(k)}; n \in \mathbb{N}\right\}, y_n^{(k)} = \frac{1}{n}\sum_{i=1}^{n} \gamma\left(x_i^{(k)}\right)$$

does not depend on $k \in \{1, 2\}$.

The class of S-equivalence of the sequence $\overline{x} \in X^{\mathbb{N}}$ will be denoted as $S(\overline{x})$. Our first goal is to find the invariant of the relation of S-equivalence. Let us introduce several notions.

Let M be a Banach space of bounded real functions, defined on the set X, M^* be the dual space of the space M, and τ - is a weak-$*$ topology in M^*. Let, further, $PF(X)$ be the subspace of the topological space (M^*, τ) defined by the formula

$$PF(X) = \{p \in M^* : p(f) \geq 0 \text{ if } f \geq 0, p(\mathbf{1}_X) = 1\},$$

where $\mathbf{1}_A(\cdot)$ is the characteristic function of the set A. In what follows, instead of $p(\mathbf{1}_A)$ we shall often write $p(A)$, identifying, by the same token, the elements of the set $PF(X)$ with the finitely additive and normed measures on 2^X. Obviously, $p(f)$ in this case is simply the integral $p(f) = \int f(x)p(dx)$, defined naturally due to boundedness of function f.

Associate to an arbitrary sequence $\overline{x} = \{x_n; n \in \mathbb{N}\} \in X^{\mathbb{N}}$ the sequence of measures from $PF(X)$ defined as

$$\left\{p_{\overline{x}}^{(n)}(\cdot); n \in \mathbb{N}\right\}, \ p_{\overline{x}}^{(n)}(A) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_A(x_i), \forall A \subseteq X. \tag{1}$$

Due to compactness of the space $PF(X)$ (as of a bounded closed set in (M^*, τ)), the sequence (1) of measures (the sequence of frequency distributions) will have a non-empty closed set of limit points, which we denote as $P_{\overline{x}}$ and call *the statistical regularity* of this sequence. Therefore introduce the following definition.

Definition 2. *Any non-empty closed subset of the space $PF(X)$ is called a regularity on X. Denote the set of all regularities on X as $\mathbb{P}(X)$ and associate to any sequence $\overline{x} \in X^{\mathbb{N}}$ its regularity $P_{\overline{x}}$. Finally, for $m \in \mathbb{N}$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in (X \to \mathbb{R}^m)$ and $P \in \mathbb{P}(X)$, the symbol $P(\gamma)$ denotes the set*

$$\left\{(r_1, r_2, \ldots, r_m) \in \mathbb{R}^m : \exists p \in P, r_i = p(\gamma_i), \forall i \in \overline{1, m}\right\},$$

and, in particular, $p(\gamma) = (p(\gamma_1), p(\gamma_2), \ldots, p(\gamma_m))$ for $p \in PF(X)$.

Consider the following proposition.

Proposition 1. *The mapping $\overline{x} \mapsto P_{\overline{x}}$ is the invariant of the relation of S-equivalence on $X^{\mathbb{N}}$.*

This statement will be proved below in a more general form. So far, however, let us agree to call the S-equivalence classes of sequences *the simplest mass phenomena*, and their statistical regularities - the regularities of the corresponding phenomena. Any sequence $\overline{x} \in X^{\mathbb{N}}$ is considered as a realization of a simplest mass phenomenon $S(\overline{x})$.

The connection of the notions introduced above with the probabilistic notions follows directly from the strong law of large numbers.

Proposition 2. *Let X be a finite set, μ - a probability distribution on X, and $\overline{\xi} = \{\xi_n; n \in \mathbb{N}\}$ - a sequence of independent (in the usual sense) random elements, taking values in X with distribution μ. Then with probability 1 the sequence $\overline{x}$ of the values of the sequence $\overline{\xi}$ will be a realization of the simplest random phenomenon $S(\overline{x})$ with statistical regularity $P_{\overline{x}} = \{\mu\}$, i.e. consisting of the single distribution μ.*

However, when the set X is infinite everything becomes considerably more difficult. In this case, the capabilities of sequences, generally speaking, are insufficient in order to guarantee that the frequencies of hitting in all measurable sets would tend to their probabilities simultaneously. Moreover, it is easy to see that the regularities of sequences, since they are concentrated only on a countable subset of the set X, constitute only a small part of the set of all regularities on X. This seems to reflect the fact that sequences constitute only a small part of all mass phenomena. A more general notion of *sampling net* is, as we shall see further, already sufficient for our goals.

Definition 3. *A* ***sampling net*** *(s.n.) in X is any net $\varphi = \{\varphi_\lambda, \lambda \in \Lambda, \succcurlyeq\}$ taking values in the sampling space*

$$X^\infty = \bigcup_{n=1}^{\infty} X^n, \quad X^n = \underbrace{X \times \cdots \times X}_{n}.$$

Moreover, if $\lambda \in \Lambda, \varphi_\lambda \in X^n$ then we denote $n = n_\lambda, \varphi_\lambda = (\varphi_{\lambda 1}, \varphi_{\lambda 2}, \ldots, \varphi_{\lambda n_\lambda})$ and associate to this λ the measure $p_\varphi^{(\lambda)} \in PF(X)$ defined as

$$p_\varphi^{(\lambda)}(A) = \frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} \mathbf{1}_A(\varphi_{\lambda i}), \quad A \subseteq X.$$

The set P_φ of limit points of the net $p_\varphi = \left\{p_\varphi^\lambda, \lambda \in \Lambda, \geq\right\}$ will be called ***the regularity*** *of the s.n. φ. The class of all s.n. in X will be denoted as $\Phi(X)$.*

Extend now the relation of S-equivalence onto the whole $\Phi(X)$.

Definition 4. *Sampling nets* $\varphi^{(k)} \in \Phi(X), k = 1, 2$ *are considered as* S-*equivalent if and only if for any* $m \in \mathbb{N}$ *and any bounded mapping* $\gamma \in (X \to \mathbb{R}^m)$ *the set of limit points of the net of averages*

$$\left\{ y_\lambda^{(k)}, \lambda \in \Lambda, \succcurlyeq \right\}, \quad y_\lambda^{(k)} = \frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} \gamma(\varphi_{\lambda i}^{(k)}) \tag{2}$$

does not depend on $k \in \{1, 2\}$.

We can now formulate the main theorem in the following way.

Theorem 1. (i) *For any s.n.* $\varphi \in \Phi(X)$, *any* $m \in \mathbb{N}$ *and any bounded mapping* $\gamma \in (X \to \mathbb{R}^m)$, *the set of limit points of the net (2) can be written as* $P_\varphi(\gamma)$.

(ii) *The mapping* $\varphi \mapsto P_\varphi$, *defined on* $\Phi(X)$, *is the invariant of the relation of* S-*equivalence.*

(iii) *This mapping is a mapping on the whole set* $\mathbb{P}(X)$ *(see Definition 2), i.e. the set* $\Phi(X)/S$ *of classes of* S-*equivalence and the set* $\mathbb{P}(X)$ *of regularities are put by this mapping into one-to-one correspondence.*

One can find an example of statistical regularity of some deterministic mass phenomenon in Zorych et al. (2000); Ivanenko (2010). With respect to random mass phenomena this theorem justifies the following definition.

Definition 5. *Any class of* S-*equivalence of sampling nets in* X *is called random mass phenomenon in* X. *The regularity* P_φ *is called the statistical regularity of the random mass phenomenon* $S(\varphi)$. *Any s.n.* $\varphi' \in S(\varphi)$ *is called a realization of the random mass phenomenon* $S(\varphi)$. *The random phenomenon, having statistical regularity* P, *is called* μ-*stochastic if and only if there exists a non-trivial* σ-*algebra* $\mathcal{A} \subseteq 2^X$, *on which* μ *is a* σ-*additive probability, and* $p(A) = \mu(A)$ *for all* $p \in P$, $A \in \mathcal{A}$.

3 Applications in Decision Theory

Considering decision problems, assume that we need to make a decision u from the set U of possible decisions, knowing that the result of making a decision depends on some uncontrolled parameter θ from the set Θ of possible values of this parameter and is described by the bounded real loss function $L : \Theta \times U \to \mathbb{R}$. If nothing is known about the behavior of the parameter θ, then we cannot, strictly speaking, exclude the scenario, where the value of θ is chosen in the worst possible for us way. In this case, the quality of decision u is evaluated by means of the loss function

$$L_1^*(u) = \sup_{\theta \in \Theta} L(\theta, u), \quad u \in U,$$

a so called "minmax" criterion.

If it is known that parameter θ is stochastic with the given distribution μ, then, trying to minimize the average losses, one makes use of the Bayes criterion

$$L_2^*(u) = \int L(\theta, u)\mu(d\theta), \quad u \in U.$$

Suppose now that parameter θ is random in a broad sense with the statistical regularity $P \in \mathbb{P}(\Theta)$. Let us show that in this case it is reasonable to chose the criterion in the form of

$$L_3^*(u) = \sup_{p \in P} \int L(\theta, u)p(d\theta) = \max_{p \in P} \int L(\theta, u)p(d\theta), \quad u \in U, \tag{3}$$

Indeed, let

$$r_1 < L_3^*(u) < r_2.$$

The following statement is straightforward.

Proposition 3. *Let $\{\varphi_\lambda, \lambda \in \Lambda, \succcurlyeq\}$ be a sampling net in Θ with the regularity P. Then for any $\lambda_1 \in \Lambda$ there is $\lambda \succcurlyeq \lambda_1$ such that*

$$\frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} L(\varphi_{\lambda i}, u) > r_1,$$

and, at the same time, there is such λ_2, that for all $\lambda \succcurlyeq \lambda_2$ there will be

$$\frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} L(\varphi_{\lambda i}, u) < r_2.$$

In other words, $L_3^*(u)$ is that natural border that separates the average losses, that can happen for a given u for an arbitrary "large" λ, from those average losses that are not "dangerous" to us, when λ is sufficiently "large".

It is easy to see that $L_3^*(u)$ becomes $L_1^*(u)$, when $P = PF(\Theta)$ (strictly nothing is known about θ, save the set Θ where it takes values), and that it becomes $L_2^*(u)$, when $P = \{\mu\}$ is a stochastic regularity and function $L(\cdot, u)$ is measurable relatively to the corresponding σ- algebra.

The inverse result appears as somewhat surprising. It turns out that if one subordinates a criterion choice rule to some conditions of consistency (known in decision theory as *axioms of rational choice*) with the triplet (Θ, U, L), then any rule, satisfying these conditions, leads to the criterion of the form (3), where P is some closed in weak-$*$ topology set of finitely-additive probability measures on Θ (cf. Ivanenko and Labkovskii 1986a, b; Gilboa and Schmeidler, 1989; Chateauneuf, 1991; Ivanenko and Munier, 2000; Cassadesus- Masonel et al., 2000; Mikhalevich, 2011).

Application of the above decision-theoretical formalism to financial theory (for example, to asset valuation) can lead to some useful results, involving sets of probability measures (Ivanenko and Munier, 2013).

One can suggest that regularity on Θ is, in a certain sense, the most general form of information about the behavior of θ. Indeed, there exists a relation between the notion of regularity, the above decision-theoretical formalism and information theory, in particular with information measurement (Ivanenko, 2010).

4 Proof of Theorem 1

Denote the set of limit points of an arbitrary net $g = \{g_\alpha, \alpha \in A, \succcurlyeq\}$ with values in X as $LIM(g)$ or $LIM\,\{g_\alpha, \alpha \in A, \succcurlyeq\}$. Denote the set of bounded mappings from X into $\mathbb{R}^m$ as M^m. In order to prove Theorem 1 we need to establish the three following facts:

(i) The relation

$$LIM\,\{y_\lambda, \lambda \in \Lambda, \succcurlyeq\} = P_\varphi(\gamma),$$

where

$$y_\lambda = \frac{1}{n_\lambda}\sum_{i=1}^{n_\lambda}\gamma(\varphi_{\lambda i}),$$

is true for all $m \in \mathbb{N}$, $\gamma \in M^m$, $\varphi \in \Phi(X)$.

(ii) If $P_1, P_2 \in \mathbb{P}(X), P_1 \neq P_2$, then there exist such $m \in \mathbb{N}$ and such $\gamma \in M^m$, that $P_1(\gamma) \neq P_2(\gamma)$.

(iii) For any regularity $P \in \mathbb{P}(X)$ there exist such s.n. $\varphi \in \Phi(X)$, that $P = P_\varphi$.

Begin with the proof of the proposition (i). Let $r \in LIM(y)$, where $y = \{y_\lambda, \lambda \in \Lambda, \succcurlyeq\}$. Then there exists a subnet of the net y converging to r, i.e. there exists (see Kelley (1957)) a directed set $(A, \succcurlyeq)$ and a function $f : A \to \Lambda$ such that the net $\overline{y} = y \circ f$ converges to r, and, in addition, for any $\lambda \in \Lambda$ there exists such $\alpha_1 \in A$ that $f(\alpha) \succcurlyeq \lambda$ for all $\alpha \succcurlyeq \alpha_1$.

Consider now the net of measures $\overline{p}_\varphi = p_\varphi \circ f$, where $p_\varphi = \left\{p_\varphi^{(\lambda)}, \lambda \in \Lambda, \succcurlyeq\right\}$. By virtue of compactness of the space $(PF(X), \tau)$ this net has at least one limit point. Denote it as p_0 and consider a subnet $\overline{\overline{p}}_\varphi$ of the net $\overline{p}_\varphi$, converging to p_0. Let it be $\overline{\overline{p}}_\varphi = \overline{p}_\varphi \circ g = p_\varphi \circ f \circ g, g : B \to A$. Then the net $\overline{\overline{y}} = y \circ f \circ g$, on the one hand, converges to r, and, on the other hand, $\overline{\overline{y}}_\beta = \overline{\overline{p}}_\varphi^{(\beta)}(\gamma), \beta \in B$, so that

$$r = \lim_\beta \overline{\overline{p}}_\varphi^{(\beta)}(\gamma) = p_0(\gamma) \in P_\varphi(\gamma).$$

By the same token, it is proved that $LIM(y) \subseteq P_\varphi(\gamma)$.

Conversely, if $p_0 \in P_\varphi, r = p_0(\gamma)$, then there exists a subnet $\widetilde{p}_\varphi = \left\{\widetilde{p}_\varphi^\alpha, \alpha \in A, \succcurlyeq\right\}$ of the net p_φ, converging to p_0. But in this case $\lim_\alpha \widetilde{p}_\varphi^{(\alpha)}(\gamma_i) = p_0(\gamma_i)$ for all $i \in \overline{1, m}$. It means that $\lim_\alpha \widetilde{p}_\varphi^{(\alpha)}(\gamma) = p_0(\gamma)$. And, since $\widetilde{p}_\varphi^{(\alpha)}(\gamma) = y_\lambda$ for $\lambda = f(\alpha)$, this proves (i).

In order to prove (ii) assume that there exists $p_1 \in P_1 \setminus P_2$. Since the set P_2 is closed, there exists a vicinity of the point p_1 that does not intersect P_2 and it means that there exist such $\epsilon > 0, \gamma_1, \gamma_2, \ldots, \gamma_m \in M$ that

$$\forall p_2 \in P_2, \exists i \in \overline{1, m}, |p_1(\gamma_i) - p_2(\gamma_i)| > \epsilon.$$

So that if $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, then $p_1(\gamma) \notin P_2(\gamma)$.

The complete proof of the statement (iii) can be found in Ivanenko and Labkovsky (1990b) and Ivanenko (2010).[2] Here we shall outline the main ideas of the proof. Let Q be the set of all such measures $q \in PF(X)$ that each one of them is concentrated on a finite set $X_q \subseteq X$, and in addition all numbers $q(x), x \in X_q$, are rational. One can show that the set Q is everywhere dense in $(PF(X), \tau)$.

Now, to an arbitrary regularity $P \in \mathbb{P}(X)$ we put into correspondence the directed set $(\Lambda, \succcurlyeq)$ such that

$$\Lambda = \mathbb{R}^+ \times M^\infty \times P, \;\; \mathbb{R}^+ =]0, \infty[, \;\; M^\infty = \bigcup_{m=1}^\infty M^m, M^m = \underbrace{M \times \cdots \times M}_{m}$$

and the relation $(\succcurlyeq)$ is given by the formula

[2] The original version of this statement can be found in Ivanenko and Labkovskii (1990a).

$$(\epsilon_1, \gamma_{11}, \gamma_{12}, \ldots, \gamma_{1n_1}, p_1) \succcurlyeq (\epsilon_2, \gamma_{21}, \gamma_{22}, \ldots, \gamma_{2n_2}, p_2) \Leftrightarrow \\ (\epsilon_1 \leq \epsilon_2, \{\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1n_1}\} \supseteq \{\gamma_{21}, \gamma_{22}, \ldots, \gamma_{2n_2}\}),$$

where no condition is imposed on p_1 and p_2.

Finally, to any $\lambda = (\epsilon, \gamma_1, \gamma_2, \ldots, \gamma_m, p) \in \Lambda$ we put into correspondence some

$$q_\lambda \in Q \bigcap \left\{ p^{'} \in PF(X) : \forall i \in \overline{1, m}, \left| p(\gamma_i) - p^{'}(\gamma_i) \right| < \epsilon \right\}.$$

It is proven further that with any $\lambda \in \Lambda$ one can associate simultaneously a sequence of points $x_1^{(\lambda)}, x_2^{(\lambda)}, \ldots, x_{n_\lambda}^{(\lambda)} \in X_q$ satisfying the condition

$$q_\lambda(A) = \frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} \mathbb{1}_A \left(x_i^{(\lambda)} \right), \quad \forall A \subseteq X.$$

It remains to chose $\varphi_\lambda = \left(x_1^{(\lambda)}, x_2^{(\lambda)}, \ldots, x_{n_\lambda}^{(\lambda)} \right)$ and we obtain a s.n. $\varphi : \lambda \mapsto \varphi_\lambda$ that has the regularity $P_\varphi = P$.

5 Concluding Remarks

One can conclude that the notions of a sampling net and its statistical regularity relate to each other analogously to the notions of random variable and its probability distribution. Besides, Theorem 1 shows that the statistical regularity of a mass phenomenon is, in the general case, a family of probability distributions, and in the specific case of a stochastic random mass phenomenon - a probability distribution (von Mises, 1928; Kolmogorov, 1963).

On the other hand, Theorem 1 extends the possibilities of classical probability theory to model random in a broad sense phenomena. It seems reasonable to suggest the notion of *extended probability space* (X, Ξ, P_X), where X is a set, Ξ is an algebra of subsets of X, P_X a regularity on X.

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Victor I. Ivanenko
Valery A. Labkovsky
Department of mathematical
modeling of economic systems,
Kyiv Polytechnical Institute,
57, Prospekt Peremogy,
Kyiv 03056, Ukraine
E-mail: victor.ivanenko.1@gmail.com

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