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On Natural and Predictable Processes

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Abstract

We give short, nontechnical proofs of the equivalence of predictability and naturality of stochastic processes of integrable variation, as well as of the properties that any natural martingale of integrable variation is indistinguishable from zero, any local predictable martingale of locally integrable variation is constant a.s., any local martingale is predictable if and only if it is continuous a.s. These statements are well known, but their proofs are very complicated since they depend on deep results from the advanced theory of stochastic integration and local martingales. Our proofs use only basic concepts of stochastic calculus.

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1 Introduction

There are some important results which should be attributed to the very beginning of the stochastic analysis. Such results include, e.g., the Doléans theorem on equivalence of the notions of natural and predictable increasing processes, theorems about natural and predictable martingales of integrable variation, the theorem on continuity of predictable local martingales. Besides the original proof (Doléans-Dade, 1967) of the Doléans theorem, there are some other proofs of the theorem and some of its parts, for example, in Kallenberg (1997), Medveguyev (2007), Protter (2004), and O'Cinneide and Protter (2001). These statements are well known, but their proofs are very complicated since they depend on deep results from the advanced theory of stochastic integration and local martingales. Our proofs use only basic concepts of stochastic calculus.

2 Background

Let there be given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = \{\mathcal{F}_t \colon \mathcal{F}_t \subseteq \mathcal{F}, t \geq 0\}$ satisfying the usual hypotheses. We will deal with real-valued stochastic processes defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process $X = \{X_t, t \geq 0\}$ is right-regular if almost all its trajectories are right-continuous and have left limits. Protter's book (2004) will be used as a source for definitions of most notions. Recall the notion of a stochastic process of integrable variation important for the paper. Let $X = \{X_t, t \geq 0\}$ be a right-regular, $\mathbb{F}$-adapted stochastic process. Define the *variation process* $V = \{V_t, t \geq 0\}$ as

$$V_t = \sup \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|$$

where the supremum is taken over all partitions $0 = t_0 < \cdots < t_n = t$ of segment $[0, t]$ and for all $n \in \mathbb{N} = \{1, 2, \ldots\}$. The stochastic process X is called *a process of integrable variation* if $\mathbb{E}V_t < \infty$ for each $t \geq 0$. One can verify, as in the proof of the Jordan theorem on decomposition of functions of bounded variation, that V and $V - X = \{V_t - X_t, t \geq 0\}$ are right-regular, increasing $\mathbb{F}$-submartingales. It follows that any stochastic process X of integrable variation is the difference $X = V - (V - X)$ of two right-regular, increasing $\mathbb{F}$-submartingales.

A stochastic process $A = \{A_t, t \geq 0\}$ of integrable variation with $A_0 = 0$ a.s. is called *natural* if the equality

$$\mathbb{E} \int_0^t Z_s \, dA_s = \mathbb{E} \int_0^t Z_{s-} \, dA_s \tag{1}$$

holds for each $t > 0$ and for each bounded, right-regular $\mathbb{F}$-martingale $Z = \{Z_t, t \geq 0\}$.

The notion of a natural process was introduced by Meyer (1963a, b) for increasing processes. A more general definition given above can be found, for example, in Protter (2004).

Let $A = \{A_t, t \geq 0\}$ be a predictable process of integrable variation. Since A is adapted to the filtration $\mathbb{F}$, the stochastic processes $V = \{V_t, t \geq 0\}, A_- = \{A_{t-}, t \geq 0\}, V_- = \{V_{t-}, t \geq 0\}$ are also $\mathbb{F}$-adapted. Stochastic processes V_- and A_- being left-continuous are predictable. It follows from the equality $|A_t - A_{t-}| = V_t - V_{t-}$ that $V = V_{t-} + (V_t - V_{t-})$ is a predictable process and hence A is the difference $A = V - (V - A)$ of two right-regular, predictable, increasing $\mathbb{F}$-submurtingales.

3 Main Results

This section contains the proofs of the statements mentioned in the Introduction.

THEOREM 1. *If $A = \{A_t, t \geq 0\}$ is a process of integrable variation with $A_0 = 0$ a.s. and an $\mathbb{F}$-martingale, then the equality*

$$\mathbb{E}\int_0^t X_s \, dA_s = 0 \tag{2}$$

holds for each $t > 0$ and for each bounded predictable process $X = \{X_t, t \geq 0\}$.

PROOF. Let $X = \{X_t, t \geq 0\}$ be a left-continuous, $\mathbb{F}$-adapted stochastic process bounded by a constant $c > 0$. Fix $t > 0$ and partition the segment $[0, t]$ by points $t_{n,k} = k2^{-n}t, k = 0, 1, \dots, 2^n$. Define the stochastic process $X^{(n)} = \{X_s^{(n)}, s \in [0, t]\}$ as

$$X_0^{(n)} = X_0, X_s^{(n)} = X_{t_{n,k-1}}, \text{ for } s \in (t_{n,k-1}, t_{n,k}], k = 1, \dots, 2^n.$$

Note that $|X_s^{(n)}| \leq c$ for all $s \in [0, t]$, and $\lim_{n\to\infty} X_s^{(n)} = X_s$ for all $s \in [0, t]$ by left-continuity of X. By the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \int_0^t X_s^{(n)} \, dA_s = \int_0^t X_s \, dA_s \text{ a.s.}, \lim_{n\to\infty} \mathbb{E}\int_0^t X_s^{(n)} \, dA_s = \mathbb{E}\int_0^t X_s \, dA_s.$$

The mathematical expectation under the limit sign is zero. Indeed,

$$\mathbb{E}\int_0^t X_s^{(n)} \, dA_s = \sum_{k=1}^{2^n} \mathbb{E}(X_{t_{n,k-1}} \mathbb{E}(A_{t_{n,k}} - A_{t_{n,k-1}} \mid \mathcal{F}_{t_{n,k-1}})) = 0$$

since all summands are zeros. Now one can apply a standard monotone class argument and to prove equality (2). The theorem is proved.

THEOREM 2. *If $A = \{A_t, t \geq 0\}$ is a natural process of integrable variation and an $\mathbb{F}$-martingale, then A is indistinguishable from zero.*

PROOF. Equality (1) holds for any bounded, right-regular $\mathbb{F}$-martingale $Z = \{Z_t, t \geq 0\}$. The bounded stochastic process $Z_- = \{Z_{t-}, t \geq 0\}$ is predictable since it is left-continuous and $\mathbb{F}$-adapted. By Theorem 1 the value on the right in (1) is zero, and hence

$$\mathbb{E}\int_0^t Z_s \, dA_s = 0 \text{ for each } t > 0. \tag{3}$$

If A is a monotone process then (Yex, 1995, p. 169) the equality

$$\mathbb{E}(Z_t A_t) = \mathbb{E}\int_0^t Z_s \, dA_s \tag{4}$$

holds. In fact the proof can be applied to this more general case. It follows from (3) and (4) that $\mathbb{E}(Z_t A_t) = 0$. With the help of a standard argument (Yex, 1995, p. 177–178) one can prove that the stochastic process A is indistinguishable from zero. The theorem is proved.

Recall that a right-regular, $\mathbb{F}$-adapted stochastic process $A = \{A_t, t \geq 0\}$ is called *a local martingale* if there is an increasing sequence $\{\tau_n\}_{n \geq 1}$ of stopping times such that $\lim_{n\to\infty} \tau_n = \infty$ a.s., and for each $n \in \mathbb{N}$ the stopped stochastic process $A^{(\tau_n)} - A_0 = \{A_{t\wedge\tau_n} - A_0, t \geq 0\}$ is a uniformly integrable $\mathbb{F}$-martingale. A right-regular, $\mathbb{F}$-adapted stochastic process A is called *a process of locally integrable variation* if there exists an increasing sequence $\{\tau_n\}_{n\geq 1}$ of stopping times such that $\lim_{n\to\infty} \tau_n = \infty$ a.s. and for each $n \in \mathbb{N}$ the stopped stochastic process $A^{(\tau_n)} - A_0$ is a process of integrable variation.

THEOREM 3. *If a predictable process $A = \{A_t, t \geq 0\}$ is a local martingale and a process of locally integrable variation, then A is constant, $A = A_0$ a.s.*

PROOF. It saffices to prove the theorem for any predictable $\mathbb{F}$-martingale A of integrable variation with $A_0 = 0$ a.s. If so, then $B = \{B_t, t \geq 0\}, B_t = (A_t - A_{t-})/(1 + |A_t - A_{t-}|^2)$, is a bounded, predictable process. It is known (Yex, 1995, p. 147) that almost each trajectory of any right-regular stochastic process has a finite or countable set of discontinuity points, say $t_n = t_n(\omega), n \in \mathbb{N}$. By Theorem 1 we have

$$0 = \mathbb{E}\int_0^t B_s\, dA_s = \mathbb{E}\sum_{n:\, t_n \leq t} \frac{|A_{t_n} - A_{t_n-}|^2}{1 + |A_{t_n} - A_{t_n-}|^2}.$$

It follows that there is an event Ω_t such that $\mathbb{P}\{\Omega_t\} = 1$ and for each $\omega \in \Omega_t$ the function $A_s(\omega), s \in [0, t]$, is continuous. Put $t = m \in \mathbb{N}$ and $\Omega' = \cap_{m=1}^{\infty} \Omega_m$. Note that $\mathbb{P}\{\Omega'\} = 1$ and for each $\omega \in \Omega'$ the trajectory $A_t(\omega), t \geq 0$, is continuous. For each $n \in \mathbb{N}$ the function $\tau_n = \inf\{t \geq 0 \colon |A_t| > n\}$ is an $\mathbb{F}$-stopping time. Note that $\tau_n \uparrow \infty$ a.s. as $n \to \infty$, and $|A_t^{(\tau_n)}| = |A_{t\wedge\tau_n}| \leq n$ a.s. The stopped stochastic process $A^{(\tau_n)} = \{A_{t\wedge\tau_n}, t \geq 0\}$ is a predictable process and also it is an $\mathbb{F}$-martingale with $A_0^{(\tau_n)} = A_0 = 0$ a.s. For any points $0 = s_0 < \cdots < s_r = t$ the inequalities

$$\sum_{k=1}^{r} |A_{s_k}^{(\tau_n)} - A_{s_{k-1}}^{(\tau_n)}| = \sum_{k=1}^{r} |A_{s_k\wedge\tau_n} - A_{s_{k-1}\wedge\tau_n}| \leq \sum_{k=1}^{r} |A_{s_k} - A_{s_{k-1}}| \leq V_t$$

hold and hence $A^{(\tau_n)}$ is a process of integrable variation. By equality (4) and Theorem 1 with $Z = A^{(\tau_n)}$ we get $\mathbb{E}A^2_{t\wedge\tau_n} = 0$ and $A_{t\wedge\tau_n} = 0$ a.s. It follows that the stochastic process A is indistinguishable from zero. The theorem is proved.

Recall that a function $\tau\colon \Omega \to \overline{\mathbb{R}}_+ = [0,\infty]$ is called *a predictable stopping time* if there is an increasing sequence $\{\tau_n\}_{n\geq 1}$ of $\mathbb{F}$-stopping times such that $\lim_{n\to\infty} \tau_n = \tau$ a.s., and on the set $\{\tau > 0\}$ the inequality $\tau_n < \tau$ holds for any $n \in \mathbb{N}$. The sequence $\{\tau_n\}_{n\geq 1}$ is called *the announcing sequence* for τ.

Let τ be an $\mathbb{F}$-stopping time. One can define (Protter, 2004, p. 5, 105) two σ-algebras $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau-}$ associated with τ. They are connected (Medveguyev, 2007, p. 185–186, Protter, 2004, p. 105) with the relation $\mathcal{F}_{\tau-} \subseteq \mathcal{F}_\tau$, and the stopping time τ is measurable with respect to $\mathcal{F}_{\tau-}$. If τ is a predictable stopping time and $\{\tau_n\}_{n\geq 1}$ is an announcing sequence for τ then $\mathcal{F}_{\tau-} = \sigma(\cup_{n=1}^\infty \mathcal{F}_{\tau_n})$.

Let $A = \{A_t, t \geq 0\}$ be a predictable process. There are several proofs that the functions

$$\tau_{\alpha,\beta} = \inf\{t \geq \beta\colon |A_t - A_\beta| \geq \alpha\}, \alpha, \beta \in \mathbb{Q}_0, \tag{5}$$

where $\mathbb{Q}_0$ is the set of strictly positive rational numbers, are predictable stopping times. We are interested in the proofs that do not use the theory of stochastic integration. A proof of such a kind can be found in (Rogers and Williams, 2000, p. 340). A proof of the next theorem based on the theory of predictable projections one can read in (Medveguyev, 2007, p.192–206). We give a direct proof.

THEOREM 4. *A local martingale $A = \{A_t, t \geq 0\}$ is predictable if and only if it is continuous a.s.*

PROOF. Any continuous a.s. local martingale is predictable by the very definition of the notion of predictability. Now suppose that A is a predictable local martingale. One may assume that $A_0 = 0$. Let $\{\tau_n\}_{n\geq 1}$ be an increasing sequence of stopping times such that $\lim_{n\to\infty} \tau_n = \infty$ a.s. and for each $n \in \mathbb{N}$ the stopped stochastic process $A^{(\tau_n)} = \{A_{t\wedge\tau_n}, t \geq 0\}$ is a uniformly integrable $\mathbb{F}$-martingale. It inherits the properties of right-regularity and predictability from A. Suppose that all martingales $A^{(\tau_n)}, n \in \mathbb{N}$, are continuous a.s. Then all trajectories $A_t^{(\tau_n)}(\omega), t \geq 0, n \in \mathbb{N}$, are continuous for all ω from a set $\Omega' \in \mathcal{F}, \mathbb{P}\{\Omega'\} = 1$. One can assume that $\tau_n(\omega) \uparrow \infty$ as $n \to \infty$ for each $\omega \in \Omega'$. Since $\lim_{n\to\infty} A_t^{(\tau_n)}(\omega) = A_t(\omega)$ for any $\omega \in \Omega'$ and $t \geq 0$, the trajectory $A_t(\omega), t \geq 0$, is continuous. We need to prove that any predictable, uniformly integrable $\mathbb{F}$-martingale $A = \{A_t, t \geq 0\}$ is continuous a.s. Without loss of generality one can assume that all trajectories of A are right-regular (that is, right-continuous with left limits) and $A_0 = 0$. By a traditional convention we put $A_{0-} = A_0$. Let τ and $\{\tau_n\}_{n\geq 1}$

be a predictable stopping time and an announcing sequence for τ. Since A is a uniformly integrable $\mathbb{F}$-martingale, the function A_τ is well defined and is integrable. This function is well defined (Yex, 1995, p. 136) for any stopping time τ, not necessary for predictable one. It is known (Medveguyev, 2007, p. 186) that the function A_τ is measurable with respect to $\mathcal{F}_{\tau-}$. By the other known theorem (Yex, 1995, p. 136) the equality $A_{\tau_n} = \mathbb{E}(A_\tau | \mathcal{F}_{\tau_n})$ a.s. holds. With the help of the Doob theorem (Medveguyev, 2007, p. 41, or Yex, 1995, p. 112) we obtain

$$A_{\tau-} = \lim_{n\to\infty} A_{\tau_n} = \lim_{n\to\infty} \mathbb{E}(A_\tau | \mathcal{F}_{\tau_n}) = \mathbb{E}(A_\tau | \mathcal{F}_{\tau-}) = A_\tau \text{ a.s.}$$

It follows that the set $\Omega_\tau = \{A_\tau = A_{\tau-}\}$ is an event of the unit probability, and for each $\omega \in \Omega_\tau$ the value $s = \tau(\omega)$, if it is finite, can not be a discontinuity point of the trajectory $A_t(\omega), t \geq 0$.

Note that any discontinuity point of any trajectory $A_t(\omega), t \geq 0$, is a value of a predictable stopping time of type (5). Indeed, if $s > 0$ is a discontinuity point, then it easy to see that for some rational $\alpha > 0$ and rational $0 < \beta < s$, one has $s = \inf\{t > \beta \colon |A_t - A_\beta| \geq \alpha\}$, and hence s is a value of the function (5) with α and β.

It was mentioned above that the sets $\Omega_{\tau_{\alpha,\beta}} = \{A_{\tau_{\alpha,\beta}} = A_{\tau_{\alpha,\beta}-}\}$ and $\Omega_0 = \cap_{\alpha,\beta\in\mathbb{Q}_0} \Omega_{\tau_{\alpha,\beta}}$ are events of the unit probability. It was has been proved above that for each $\omega \in \Omega_0$ the trajectory $A_t(\omega), t \geq 0$, is continuous. The theorem is proved.

Theorem 5. *If $A = \{A_t, t \geq 0\}$ is a predictable process of integrable variation with $A_0 = 0$ a.s., then A is a natural process.*

Proof. It was mentioned above that the stochastic process A can be represented as the difference $A = V - (V - A)$ of two increasing predictable processes with $V_0 = 0$ and $A_0 = 0$ a.s. The stochastic process A is natural if V and $V - A$ are natural. So one can assume that A is an increasing predictable process with $A_0 = 0$ a.s. We need to verify the equality (1) for any bounded, right-regular $\mathbb{F}$-martingale $Z = \{Z_t, t \geq 0\}$. One can rewrite the equality (1) as

$$\mathbb{E}\int_0^t (Z_s - Z_{s-})\, dA_s = \mathbb{E}\left(\sum_{n:t_n\leq t} (Z_{t_n} - Z_{t_n-})(A_{t_n} - A_{t_n-})\right) \tag{6}$$

where $t_n = t_n(\omega), n \in \mathbb{N}$, are discontinuity points of A. With the help of stochastic process A one can construct functions (5). By the well-known theorem (Rogers and Williams, 2000, p. 340) all these functions are predictable stopping times. The functions (5) can be written as a sequence $\{\tau_n\}_{n\geq 1}$. It was proved in Theorem 4 that any discontinuity point of A is a value of a function $\tau_n, n \in \mathbb{N}$, for some n. On the other hand, if $A_{\tau_n} - A_{\tau_n-} > 0$ for some $n \in \mathbb{N}$ then τ_n is a discontinuity point of A. It follows that

$$\sum_{n:t_n\leq t} (Z_{t_n} - Z_{t_n-})(A_{t_n} - A_{t_n-}) = \sum_{n=1}^{\infty} \mathbb{1}_{\{\tau_n\leq t\}}(Z_{\tau_n} - Z_{\tau_n-})(A_{\tau_n} - A_{\tau_n-}).$$

Each of the two series converges absolutely. Indeed, there is a number $c > 0$ such that $|Z_t| \leq c$ for all $t \geq 0$, and hence

$$\sum_{n=1}^{\infty} \mathbb{1}_{\{\tau_n\leq t\}}|(Z_{\tau_n} - Z_{\tau_n-})(A_{\tau_n} - A_{\tau_n-})| \leq 2c \sum_{n:\tau_n\leq t} (A_{\tau_n} - A_{\tau_n-}) \leq 2cA_t.$$

Equality (6) can be written in the following way

$$\mathbb{E}\int_0^t (Z_s - Z_{s-})\, dA_s = \sum_{n=1}^{\infty} \mathbb{E}\left(\mathbb{1}_{\{\tau_n\leq t\}}(Z_{\tau_n} - Z_{\tau_n-})(A_{\tau_n} - A_{\tau_n-})\right).$$

Let us prove that all terms of the last series are equal to zero. To this end we note that the function $\mathbb{1}_{\{\tau_n\leq t\}}(A_{\tau_n} - A_{\tau_n-})$ is integrable and $\mathcal{F}_{\tau_n-}$-measurable. Let $\{\tau_{n,m}\}_{m\geq 1}$ be an announcing sequence for τ_n. Since Z is bounded, right-regular $\mathbb{F}$-martingale, the well-defined functions $Z_{\tau_{n,m}}$ and Z_{τ_n} are integrable. By the theorem (Yex, 1995, p. 136) the equality $Z_{\tau_{n,m}} = \mathbb{E}(Z_{\tau_n}|\mathcal{F}_{\tau_{n,m}})$ a.s. holds. By the Doob theorem (Medveguyev, 2007, p. 41, or Yex, 1995, p. 112) we get

$$Z_{\tau_n-} = \lim_{m\to\infty} Z_{\tau_{n,m}} = \lim_{m\to\infty} \mathbb{E}(Z_{\tau_n}|\mathcal{F}_{\tau_{n,m}}) = \mathbb{E}(Z_{\tau_n}|\mathcal{F}_{\tau_n-}) \text{ a.s.}$$

It follows that

$$\mathbb{E}\left(\mathbb{1}_{\{\tau_n\leq t\}}(Z_{\tau_n} - Z_{\tau_n-})(A_{\tau_n} - A_{\tau_n-})\right) = \mathbb{E}\left(\mathbb{1}_{\{\tau_n\leq t\}}(A_{\tau_n} - A_{\tau_n-})\mathbb{E}(Z_{\tau_n} - Z_{\tau_n-}|\mathcal{F}_{\tau_n-})\right) = 0.$$

The desired equality (1) and the theorem are proved.

Theorem 6. *Any natural process $A = \{A_t, t \geq 0\}$ of integrable variation is predictable.*

Proof. The stochastic process A can be written as the difference $A = V - (V - A)$ of two increasing, right-regular $\mathbb{F}$-submartingales V and $V - A$ with $V_0 = A_0 = 0$ a.s. There is a proof (Beiglböck et al., 2012) of the

Doob-Meyer decomposition which states, in particular, that every positive, right-regular $\mathbb{F}$-submartingale can be written as the sum of a right-regular $\mathbb{F}$-martingale and an increasing predictable process. It is important for us that the proof does not use the theory of stochastic integration. Thus, V and $V-A$ can be written as sums $V = M_1 + B_1$ and $M_2 + B_2$, where M_1 and M_2 are right-regular $\mathbb{F}$-martingales, and B_1 and B_2 are increasing predictable processes. It follows that A can be written as the sum $A = (M_1 - M_2) + (B_1 - B_2)$ of the right-regular $\mathbb{F}$-martingale $M_1 - M_2$ and the predictable process $B_1 - B_2$ of integrable variation. By Theorem 5 the predictable process $B_1 - B_2$ is natural, and hence $M_1 - M_2$ is a natural $\mathbb{F}$-martingale of integrable variation. By Theorem 2 the natural martingale $M_1 - M_2$ is indistinguishable from zero, and hence A and $B_1 - B_2$ are indistinguishable. It follows that A is predictable. The theorem is proved.

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