Sankhyā : The Indian Journal of Statistics
2016, Volume 78-A, Part 1, pp. 105-123

A Version of Komlós Theorem for Additive Set Functions

Gianluca Cassese
Università Milano Bicocca, Milano, Italy

Abstract

We provide a version of the celebrated theorem of Komlós in which, rather than random quantities, a sequence of finitely additive measures is considered. We obtain a form of the subsequence principle and some applications.

AMS (2000) *subject classification.* 60F05, 60B10, 46B42.
Keywords and phrases. Komlós Lemma, Subsequence principle, Strong law of large numbers, Weak compactness

1 Introduction

In 1967 (Komlós, 1967) proved the following subsequence principle: a norm bounded sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $L^1(P)$, with P a probability law, admits a subsequence $\langle g_n \rangle_{n\in\mathbb{N}}$ and $g \in L^1(P)$ such that, for any further subsequence $\langle h_n \rangle_{n\in\mathbb{N}}$,

$$P\left(\lim_{N\to\infty} \frac{h_1 + h_2 + \ldots + h_N}{N} = g\right) = 1 \tag{1.1}$$

The proof uses a truncation technique, weak compactness in $L^2(P)$ and martingale convergence. In this paper we prove a form of this result in which the random quantities f_n are replaced by additive set functions and countable additivity is not assumed.

The original work of Komlós has originated a number of subsequent contributions extending its validity in several directions. Chatterji (1970) replaced L^1 with L^p for $0 < p < 2$; Schwartz (1986) gave two different proofs still using truncation and weak compactness; Berkes (1990) showed that the subsequence may be selected so that each permutation of its elements still satisfies (1.1). Other proofs of this same result (or some extension of it) were subsequently given also by Balder (1990) and Trautner (1990). von Weizsäcker (2004) explored the possibility of dropping the boundedness property while Lennard (1993) showed that this property is necessary for a convex subset of L^1 to have each sequence satisfying the subsequence principle. Other papers considered cases in which the functions f_n take their values in some vector space other than $\mathbb{R}$. These include Balder (1989) and Guessous (1997). Balder and Hess (1996) considered multifunctions with values in Banach spaces with the Radon Nikodym property. Day and Lennard (2010) and Jiménez Fernández et al. (2011) proved equivalence with the Fatou property. Eventually, Halevy (1979) considered the case in which the probability measure P is replaced by an independent strategy, a finitely additive set function of a special type introduced by Dubins and Savage (2014).

Even disregarding the obvious interest in the strong law of large numbers, it is often very useful in applied problems to extract from a given sequence an a.s. converging subsequence. Komlós theorem implies that this may be done upon replacing the original sequence with one formed by convex combinations of elements of arbitrarily large index. In this somehow different formulation, Komlós theorem has been exploited extensively, e.g. by Burkholder (1973) to give a simple proof of Kingman ergodic theorem, or by Cvitanić and Karatzas (2001) for application to statistics.

However, consider replacing each element f_n in the original sequence with the indicator $\mathbf{1}_B(f_n)$ of the event $f_n \in B$ and to construct the resulting empirical distribution:

$$F_N(B) = \frac{\sum_{n=1}^N \mathbf{1}_B(f_n)}{N} \qquad B \in \mathscr{B} \tag{1.2}$$

In order to apply Komlós to the time honored problem of the convergence of the empirical distribution, one should be able to select a subsequence so as to obtain convergence for all B in $\mathscr{B}$. But this may hardly be possible if $\mathscr{B}$ is not countably generated, a situation quite common in the theory of stochastic processes when $\mathscr{B}$ is the Borel σ algebra of some non separable metric space. This difficulty is emphasized if one requires a more sophisticated notion of convergence than setwise convergence.

In a highly influential paper, Blackwell and Dubins (1962) modeled the evolution of probability in response to some observable phenomenon as a sequence of regular posterior probabilities:

$$F_n(B) = P(B|f_1, \ldots, f_n) \qquad B \in \mathscr{B} \tag{1.3}$$

The merging of opinions obtains whenever the posteriors originated by two countably additive probability measures converge to 0 in total variation for all histories $f_1, f_2, \ldots$ save possibly on a set of measure zero. In this formulation we are confronted with set functions taking values in a vector space of measurable functions, a setting in which the subsequence principle in its original formulation appears even more troublesome.

The problem just considered provides a good case in point of the advantage or the need of working with finite additivity. On the one hand one may wish to define each F_n in (1.3) on a larger class than $\mathscr{B}$, e.g. the class of all subsets of the underlying sample space X. Classical conditional expectation may then be extended fairly easily to this larger class (although not in a unique way), e.g. via (Cassese, 2008, Theorem 1). On the other hand, one may consider this same problem in more general cases than with X a complete, separable metric space so that the existence of a regular conditional probability may not be guaranteed. In either case, by virtue of the lifting theorem, posterior probability may be defined as a vector valued, additive set function but the countable additivity property has to be abandoned.

The main result of this paper establishes that if a sequence of (finitely additive) probabilities is transformed by taking convex combinations and restrictions, then norm convergence obtains. The proof, although rather different from those given in the cited references, retains from the original work of Komlós, the idea of achieving weak compactness via truncations. In Section 2 we prove the basic version of our result, valid for general Banach lattices with order continuous norm (but not assuming the Radon Nikodym property), as the key argument in our proof just uses lattice properties. With such degree of generality this result, perhaps of its own interest, appears to be rather weak. In Section 3 we specialize to the space $ba(\mathscr{A})$ of scalar valued, additive, bounded set functions for which the convergence statement is significantly stronger. We provide some applications, such as a finitely additive version of the strong law of large numbers, Corollary 4, and explore the implications of assuming independence, Corollary 2 and of dropping norm boundedness, Theorem 3. In Theorem 4 we prove a version valid for a special space $ba_0(\mathscr{A}, X)$ of set functions taking values in some vector lattice X.

We refer throughout to a given, non empty set Ω and an algebra $\mathscr{A}$ of its subsets. $\mathscr{S}(\mathscr{A})$ and $ba(\mathscr{A})$ designate the families of simple, $\mathscr{A}$ measurable functions $f : \Omega \to \mathbb{R}$ and, as in Dunford and Schwartz (1988), the family of real valued, additive set functions on $\mathscr{A}$ which are bounded with respect to the total variation norm,$\|\lambda\| = \lambda(\Omega)$, respectively. When $\mu \in ba(\mathscr{A})$ and $B \in \mathscr{A}$ we define $\mu_B \in ba(\mathscr{A})$ implicitly by letting $\mu_B(A) = \mu(B \cap A)$ for each $A \in \mathscr{A}$. $\mathbb{P}(\mathscr{A})$ denotes the family of finitely additive probabilities on $\mathscr{A}$. Countable additivity is never assumed, unless otherwise explicitly stated. If

$\lambda \in ba(\mathscr{A})_+$, we say that a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of functions on Ω λ-converges to 0 when

$$\lim_n \lambda^*(|f_n| > \eta) = 0 \qquad \eta > 0 \tag{1.4}$$

where, as usual, λ^* denotes the outer measure

$$\lambda^*(B) = \inf_{\{A\in\mathscr{A}: B\subset A\}} \lambda(A) \qquad B \subset \Omega \tag{1.5}$$

Likewise the expressions λ-Cauchy or λ-bounded refer to the Cauchy property or to boundedness formulated relatively to the topology of λ-convergence.

2 Banach Space Preliminaries

The proof of the main theorem is based on the two technical results proved in this section in which X is taken to be a real Banach space. If $K \subset X$ we write $\overline{K}$ and $\overline{K}^w$ to denote its closure n the strong and in the weak topology, respectively, and $\operatorname{co}(K)$ for its convex hull. The symbols $\overline{\operatorname{co}}(K)$ and $\overline{\operatorname{co}}^w(K)$ will indicate the closed-convex hulls of K in the corresponding topology. If $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in X the symbol $\Gamma(x_1, x_2, \ldots)$ will be used for the collection of those sequences $\langle y_n \rangle_{n\in\mathbb{N}}$ in X such that $y_n \in \operatorname{co}(x_n, x_{n+1}, \ldots)$ for each $n \in \mathbb{N}$.

We denote a given but arbitrary Banach limit on ℓ^∞ generically by LIM. We extend this notion to X as follows: if $\langle x_n \rangle_{n\in\mathbb{N}}$ is a norm bounded sequence in X, then there exists a unique $x^{**} \in X^{**}$ satisfying

$$x^{**}(x^*) = \operatorname*{LIM}_n (x^* x_n) \qquad x^* \in X^* \tag{2.1}$$

and we write $\operatorname{LIM}_n x_n = x^{**}$. In the following we identify an element of X with its isomorphic image under the natural homomorphism $\kappa : X \to X^{**}$ so that, when appropriate, we write $\operatorname{LIM}_n x_n \in X$. Two properties follow easily from (2.1): (i) $\| \operatorname{LIM}_n x_n \| \le \operatorname{LIM}_n \|x_n\|$ and (ii) x_n converges weakly to x if and only if $\operatorname{LIM}_n x'_n = x$ for all subsequences $\langle x'_n \rangle_{n\in\mathbb{N}}$.[1] A third one, less obvious, is the following implication of Krein - Šmulian theorem:

Lemma 1. *If K is a relatively weakly compact subset of a Banach space X, then*

$$\operatorname*{LIM}_n x_n \in \bigcap_i \overline{\operatorname{co}}(x_i, x_{i+1}, \ldots) \quad \textit{for every sequence} \quad x_1, x_2, \ldots \in K \tag{2.2}$$

[1] Or, in yet other terms, if and only if all Banach limits on the original sequence coincide with x.

Proof. Pick a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in K, write $K_i = \overline{\text{co}}(x_i, x_{i+1}, \ldots)$ and observe that

$$\inf_{x\in K_i} x^*(x) \le (\operatorname*{LIM}_n x_n)(x^*) \le \sup_{x\in K_i} x^*(x) \qquad x^* \in X^*$$

by the properties of the Banach limit and (2.1). The set K_i is convex and, by assumption and theorem (Dunford and Schwartz, 1988, V.6.4), weakly compact. It follows from a theorem of Šmulian (Dunford and Schwartz, 1988, p. 464) that there exists $y_i \in K_i$ such that $x^*(y_i) = (\text{LIM}_n x_n)(x^*)$ for each $x^* \in X^*$. In other words, $\text{LIM}_n x_n \in \bigcap_i K_i$.

Banach limits will be important in what follows but are used in other parts of the theory of finitely additive set functions, e.g. to show the existence of densities. Let us mention that this tool was also used by Ramakrishnan (1981) in the setting of finitely additive Markov chains.

A partially ordered, normed vector space X is said to possess property $(\boldsymbol{P})$ when every increasing, norm bounded net $\langle x_\alpha\rangle_{\alpha\in\mathfrak{A}}$ in X admits a least upper bound $x \in X$ and $\lim_\alpha \|x_\alpha - x\| = 0$. Clearly, a normed vector lattice possessing property $(\boldsymbol{P})$ is a complete lattice and its norm is order continuous, i.e. if $\langle x_\alpha\rangle_{\alpha\in\mathfrak{A}}$ is an increasing net in X and if $x = \sup_\alpha x_\alpha \in X$ then $\lim_\alpha \|x - x_\alpha\| = 0$. Examples of Banach lattices with property $(\boldsymbol{P})$ are the classical Lebesgue spaces L^p as well as $ba(\mathscr{A})$.

We recall that a Banach lattice X is a vector lattice endowed with a norm such that $x, y \in X$ and $|x| \le |y|$ imply $\|x\| \le \|y\|$ and X is norm complete. A Banach lattice with order continuous norm is complete as a lattice.[2]

Lemma 2. *Let X be a Banach lattice possessing property $(\boldsymbol{P})$ and denote by $ba_0(\mathscr{A}, X)$ the space of all finitely additive set functions $F : \mathscr{A} \to X$ endowed with the norm*

$$\|F\|_{ba_0(\mathscr{A},X)} = \sup_{\pi\in\Pi(\mathscr{A})} \left\| \sum_{A\in\pi} |F(A)| \right\|_X \tag{2.3}$$

Then $ba_0(\mathscr{A}, X)$ is a Banach lattice with property $(\boldsymbol{P})$.

Proof. First of all it is clear that (2.3) defines a norm. Given our exclusive focus on $ba_0(\mathscr{A}, X)$ in this proof, we shall use the symbol $\|F\|$ in place of $\|F\|_{ba_0(\mathscr{A},X)}$. Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in $ba_0(\mathscr{A}, X)$. By (2.3) the sequence $\langle F_n(A)\rangle_{n\in\mathbb{N}}$ is Cauchy in X and converges thus in norm

[2] The proof of this claim is contained in that of (Aliprantis and Burkinshaw, 1985, 12.9).

to some limit $F(A)$, for each $A \in \mathscr{A}$. The set function implicitly defined $F : \mathscr{A} \to X$ is additive. Moreover,

$$\left\|\sum_{A\in\pi}|(F-F_n)(A)|\right\|_X \leq \sup_{r>n}\left\|\sum_{A\in\pi}|(F_r-F_n)(A)|\right\|_X \leq \sup_{r>n}\|F_r-F_n\|$$

so that $\lim_n \|F-F_n\| = 0$ and $\|F\| \leq \limsup_n \|F_n\|$. $ba_0(\mathscr{A}, X)$ is thus a Banach space. We can introduce a partial order by saying that $F \geq G$ whenever $F(A) \geq G(A)$ for all $A \in \mathscr{A}$. Let $\langle F_\alpha\rangle_{\alpha\in\mathfrak{A}}$ be a norm bounded, increasing net in $ba_0(\mathscr{A}, X)_+$. Fix $A \in \mathscr{A}$. Then $\langle F_\alpha(A)\rangle_{\alpha\in\mathfrak{A}}$, an increasing, norm bounded net in X, converges in norm to some $F(A) \in X_+$ by the property $(\boldsymbol{P})$. Again F is additive, $F \geq F_\alpha$ for all $\alpha \in \mathfrak{A}$ and

$$\left\|\sum_{A\in\pi}F(A)\right\|_X = \lim_\alpha\left\|\sum_{A\in\pi}F_\alpha(A)\right\|_X \leq \sup_\alpha\|F_\alpha\|$$

In addition,

$$\left\|\sum_{A\in\pi}|(F-F_\alpha)(A)|\right\|_X = \left\|F(\Omega)-F_\alpha(\Omega)\right\|_X$$

so that $\lim_\alpha \|F - F_\alpha\| = 0$ and $ba_0(\mathscr{A}, X)$ possesses property $(\boldsymbol{P})$ and its norm, as a consequence, is order continuous. It remains to show that it is a lattice and that the norm is a lattice norm, i.e. that $|F| = \sup\{F, -F\}$ exists in $ba_0(\mathscr{A}, X)$ and that $\|F\| = \||F|\|$.

Denote by $\Pi(\mathscr{A})$ the collection of all partitions of Ω into finitely many elements of $\mathscr{A}$. If $F \in ba_0(\mathscr{A}, X)$ and $\pi \in \Pi(\mathscr{A})$ define the subadditive set function $F_\pi : \mathscr{A} \to X_+$ by letting

$$F_\pi(A) = \sum_{E\in\pi}|F(A\cap E)| \qquad A \in \mathscr{A} \tag{2.4}$$

Observe that $F_\pi \in ba_0(\mathscr{A}_\pi, X)$, where $\mathscr{A}_\pi \subset \mathscr{A}$ denotes the sub algebra generated by the partition $\pi \in \Pi(\mathscr{A})$. The net $\langle F_\pi\rangle_{\pi\in\Pi(\mathscr{A})}$ is increasing and norm bounded so that it converges in norm to some $F_* : \mathscr{A} \to X_+$ which is additive in restriction to $\mathscr{A}_\pi$ for all $\pi \in \Pi(\mathscr{A})$, i.e. $F_* \in ba_0(\mathscr{A}, X)$. Moreover, $F_* \geq \{F, -F\}$. Any $G \in ba_0(\mathscr{A}, X)$ with $G \geq \{F, -F\}$ is also such that $G(A) = \sum_{E\in\pi} G(A\cap E) \geq \sum_{E\in\pi} |F(A\cap E)| = F_\pi(A)$ and so $G \geq F_*$. This proves that $|F| = F_*$ and thus that $ba_0(\mathscr{A}, X)$ is a vector lattice. To see that its norm is a lattice norm observe that $\|F_*\| = \|F_*(\Omega)\|_X = \lim_\pi \|F_\pi(\Omega)\|_X = \|F\|$.

The norm introduced on $ba_0(\mathscr{A}, X)$ differs from the variation and semivariation norms usually considered for vector measures. It seems to be appropriate for the somehow unusual case in which the set functions take value in a vector space endowed with a lattice structure. A nice consequence of the lattice property is the relative ease of the weak compactness condition, compared to general spaces of vector measures, see Brooks (1972) and Brooks and Dinculeanu (1974), and the nice interplay between norm and order which is crucial to our approach.

The following lattice inequalities will be useful:

$$(x+y)\wedge z \leq (x\wedge z)+(y\wedge z) \qquad x,y,x\in X_+ \tag{2.5a}$$

$$|x\wedge z - y\wedge z| \leq |x-y| \qquad x,y,z\in X \tag{2.5b}$$

Theorem 1. *Let X be a Banach lattice with order continuous norm and $\langle x_n\rangle_{n\in\mathbb{N}}$ a sequence in X_+. Fix $z\in X_+$. There exist three sequences in X_+, (i) $\langle y_n\rangle_{n\in\mathbb{N}}$ in $\Gamma(x_1,x_2,\ldots)$, (ii) $\langle \zeta_n\rangle_{n\in\mathbb{N}}$ with $\zeta_n\leq y_n$ for $n=1,2,\ldots$ and (iii) $\langle \xi_n\rangle_{n\in\mathbb{N}}$ increasing, such that*

$$\lim_n \left\|\zeta_n\wedge 2^k z-\xi_k\right\| = 0 \quad \textit{and} \quad y_n\wedge 2^k z \xrightarrow{\textit{weakly}} \xi_k \qquad \textit{for all} \quad k\in\mathbb{N} \tag{2.6}$$

This theorem proves that any positive sequence may be suitably transformed to obtain some form of convergence via convexification and truncation. The important fact is that *the same* convex sequence, $\langle y_n\rangle_{n\in\mathbb{N}}$, possesses the weak convergence property *for any* truncation adopted. This delicate property is obtained exploiting the order structure of Banach lattices with order continuous norm. If we replace the family of convex sequences by those sequences which are dominated by an element of such family, then we obtain norm convergence.

PROOF. Fix the following families:

$$\begin{aligned} &\mathscr{C}(n) = \left\{u\in X_+ : u\leq u' \text{ for some } u'\in \mathrm{co}(x_n,x_{n+1},\ldots)\right\} \\ \text{and} \quad &\mathscr{C} = \bigcap_n \overline{\mathscr{C}(n)} \end{aligned} \tag{2.7}$$

and notice that $x\in\mathscr{C}$ implies $\|x\|\leq \limsup_n \|x_n\|$ and, by (2.5b), $x\wedge u\in\mathscr{C}$ for all $u\in X$.

In a Banach lattice all sets admitting a lower as well as an upper bound are relatively weakly compact, (Aliprantis and Burkinshaw, 1985, Theorem 12.9). Thus, by Lemma 1, for every sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $\Gamma(x_1,x_2,\ldots)$

$$\operatorname*{LIM}_n \left(u_n\wedge 2^k z\right) \in \bigcap_n \overline{\mathrm{co}}\left(u_n\wedge 2^k z, u_{n+1}\wedge 2^k z,\ldots\right)\subset \mathscr{C} \tag{2.8}$$

Let Ξ designate the family of all sequences $\tilde{y}=\langle y_n\rangle_{n\in\mathbb{N}}$ in $\mathscr{C}$ with $y_{n-1}\leq y_n\leq 2^n z$. Let Ξ be partially ordered by the product order and let $\Xi_0=\{\tilde{y}^\alpha$:

$\alpha \in \mathfrak{A}\}$ be a chain in Ξ. Then, $\{y_n^\alpha : \alpha \in \mathfrak{A}\}$ is a chain in $\mathscr{C}$ admitting $2^n z$ as an upper bound in X. Since X is a complete lattice, $y_n = \sup_\alpha y_n^\alpha$ exists in X and, by order continuity of the norm, in $\mathscr{C}$. Of course, $y_{n-1} \leq y_n \leq 2^n z$ so that $\tilde{y} = \langle y_n \rangle_{n\in\mathbb{N}}$ is an upper bound for Ξ_0. By Zorn's lemma, Ξ admits a maximal element which we denote by $\tilde{\xi} = \langle \xi_n \rangle_{n\in\mathbb{N}}$.

If $j > 0$ and $\xi_n^j = \xi_{n+j} \wedge 2^n z$, then $\langle \xi_n^j \rangle_{n\in\mathbb{N}}$ is an element of Ξ dominating $\tilde{\xi}$. Thus,

$$\xi_n \wedge 2^k z = \xi_k \qquad n \geq k \tag{2.9}$$

By the inclusion $\xi_k \in \mathscr{C}$, there exist two sequences $\langle \zeta_k \rangle_{k\in\mathbb{N}}$ and $\langle y_k \rangle_{k\in\mathbb{N}}$ such that $0 \leq \zeta_k \leq y_k \in \mathrm{co}(x_k, x_{k+1}, \ldots)$ and $\|\xi_k - \zeta_k\| < 2^{-k}$. It follows from (2.5b) and (2.9) that $|\xi_k - \zeta_n \wedge 2^k z| = |\xi_n \wedge 2^k z - \zeta_n \wedge 2^k z| \leq |\xi_n - \zeta_n|$ and so

$$\xi_k = \lim_n \zeta_n \wedge 2^k z \leq \mathop{\mathrm{LIM}}_n \left(y_n \wedge 2^k z\right) \equiv \hat{y}_k \tag{2.10}$$

Thus, by (2.8), $\langle \hat{y}_k \rangle_{k\in\mathbb{N}}$ is yet another sequence in Ξ dominating $\langle \xi_n \rangle_{n\in\mathbb{N}}$ so that $\hat{y}_k = \xi_k$. Since (2.10) holds for any subsequence, we obtain that $y_n \wedge 2^k z$ converges to ξ_k weakly for every $k \geq 0$.

We stress that this result does not assume norm boundedness of the original sequence $\langle x_n \rangle_{n\in\mathbb{N}}$.

3 Komlós Theorem for Additive Set Functions

In this section we apply Theorem 1 to $X = ba(\mathscr{A})$. The following is the main result of the paper.

Theorem 2. *Let $\langle F_n \rangle_{n\in\mathbb{N}}$ be a norm bounded sequence in $ba(\mathscr{A})_+$, $\delta > 0$ and $\lambda \in \mathbb{P}(\mathscr{A})$. There exist (a) $\xi \in ba(\lambda)_+$ with*

$$\|\xi\| \geq \sup_k \limsup_n \left\|F_n \wedge 2^k \lambda\right\| - \delta \tag{3.1}$$

(b) $\langle G_n \rangle_{n\in\mathbb{N}}$ in $\Gamma(F_1, F_2, \ldots)$ and (c) $\langle A_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that, letting $\bar{G}_n = G_{n,A_n}$,

$$\lim_n \left\|\bar{G}_n - \xi\right\| = 0 \quad \textit{and} \quad \sum_n \lambda(A_n^c) < \infty \tag{3.2}$$

Moreover, the following are equivalent: (i) $\xi = 0$ is the only choice that satisfies (3.2) for some λ, $\langle G_n \rangle_{n\in\mathbb{N}}$ and $\langle A_n \rangle_{n\in\mathbb{N}}$ as above; (ii) the sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ is asymptotically orthogonal, i.e.

$$\lim_n \left\|F_n \wedge F_j\right\| = 0 \qquad \textit{for} \quad j = 1, 2, \ldots \tag{3.3}$$

Proof. Let $\eta = \lim_k \limsup_n \|F_n \wedge 2^k\lambda\| - \delta$. Passing to a subsequence, we assume with no loss of generality that $\lim_k \liminf_n \||F_n| \wedge 2^k\lambda\| > \eta$.

Since $ba(\mathscr{A})$ is a Banach lattice with order continuous norm, we can invoke Theorem 1 with F_n, G_n and H_n in place of x_n, y_n and ζ_n respectively. Then $G_n \wedge 2^k\lambda$ and $H_n \wedge 2^k\lambda$ converge weakly to ξ_k but $G_n \wedge 2^k\lambda \geq H_n \wedge 2^k\lambda$. This implies that $G_n \wedge 2^k\lambda - H_n \wedge 2^k\lambda$ converges to 0 in norm and therefore that

$$\lim_n \left\|G_n \wedge 2^k\lambda - \xi_k\right\| = 0 \qquad k \in \mathbb{N} \tag{3.4}$$

As $\langle \xi_k \rangle_{k\in\mathbb{N}}$ is increasing and norm bounded, property $(\boldsymbol{P})$ implies that it converges in norm to some $\xi \ll \lambda$. Upon passing to a subsequence we deduce

$$\lim_n \left\|G_n \wedge 2^n\lambda - \xi\right\| = 0 \tag{3.5}$$

and in turn

$$\|\xi\| = \lim_n \left\|G_n \wedge 2^n\lambda\right\| \geq \lim_k \liminf_n \left\|F_n \wedge 2^k\lambda\right\| > \eta$$

Moreover, selecting a further subsequence if necessary, we can assume that the sequence $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ of convex weights associated with $\langle G_n \rangle_{n\in\mathbb{N}}$ via $G_n = \sum_i \alpha_{n,i} F_i$ is disjoint, i.e. $\alpha_{n,i}\alpha_{m,i} = 0$ when $n \neq m$.

Choose $A_n \in \mathscr{A}$ such that $G_n(A_n) + 2^n\lambda(A_n^c) \leq \|G_n \wedge 2^n\lambda\| + 2^{-n}$ and observe that

$$\sum_{j\geq n} \lambda(A_j^c) \leq \sum_{j\geq n} 2^{-j}\left[\|G_j \wedge 2^j\lambda\| + 2^{-j}\right] \leq 2^{-n}\Big(1 + \sup_i \|F_i\|\Big) \tag{3.6}$$

Let $\bar{G}_n = G_{n,A_n}$ and $\hat{G}_n = \bar{G}_n + 2^n\lambda_{A_n^c}$. It follows that $\hat{G}_n \geq G_n \wedge 2^n\lambda$ and therefore

$$\|\hat{G}_n - G_n \wedge 2^n\lambda\| = \|\hat{G}_n\| - \|G_n \wedge 2^n\lambda\| = G_n(A_n) + 2^n\lambda(A_n^c) - \|G_n \wedge 2^n\lambda\| \leq 2^{-n}$$

$\hat{G}_n$ converges thus in norm to ξ and we conclude that

$$\|\bar{G}_n - \xi\| \leq |\hat{G}_n - \xi|(A_n) + \xi(A_n^c) \leq \|\hat{G}_n - \xi\| + \xi(A_n^c)$$

Thus, (3.2) follows easily from (3.6) and absolute continuity.

Suppose that (i) holds. Then it must be that $\limsup_n \|F_n \wedge \mu\| = 0$ for each $\mu \in ba(\mathscr{A})_+$, including F_j for $j = 1, 2, \ldots$. Conversely, assume (ii) fix $\lambda \in \mathbb{P}(\mathscr{A})$ and let $H = \sum_n 2^{-n} F_n$. By induction it is easily established the decomposition $\lambda = \sum_{j=0}^{\infty} \lambda_j^{\perp}$ where $F_0 \equiv \lambda_0^{\perp} \perp H$ while $F_j \gg \lambda_j^{\perp} \perp$

$\{F_0, \ldots, F_{j-1}\}$ for $j \geq 1$. Observe that, by orthogonality, $\|\lambda\| = \sum_j \|\lambda_j^\perp\|$. Moreover, for fixed $k, \varepsilon, j > 0$ there exists $t > 1$ such that, by (2.5a),

$$\left\| F_n \wedge 2^k \Big(\textstyle\sum_{0\leq i\leq j} \lambda_i^\perp \Big) \right\| \leq \varepsilon + \left\| F_n \wedge t \textstyle\sum_{1\leq i\leq j} F_i \right\| \leq \varepsilon + t \textstyle\sum_{1\leq i\leq j} \|F_n \wedge F_i\| \tag{3.7}$$

We conclude that

$$\limsup_n \left\| G_n \wedge 2^k \lambda \right\| = \lim_j \limsup_n \left\| G_n \wedge 2^k \sum_{i>j} \lambda_i^\perp \right\| \leq 2^k \lim_j \sum_{i>j} \left\| \lambda_i^\perp \right\| = 0$$

and thus that $\|\xi\| = \lim_k \left\|\xi \wedge 2^k \lambda\right\| = \lim_k \lim_n \left\|G_n \wedge 2^k \lambda\right\| = 0$.

It is possible to drop the assumption that the sequence $F_1, F_2, \ldots$ is positive although at the cost of loosing some information on the limit ξ.

Corollary 1. *Let $\langle F_n \rangle_{n\in\mathbb{N}}$ be a norm bounded sequence in $ba(\mathscr{A})$ and $\lambda \in \mathbb{P}(\mathscr{A})$. There exist (a) $\xi \in ba(\lambda)$, (b) $\langle G_n \rangle_{n\in\mathbb{N}}$ in $\Gamma(F_1, F_2, \ldots)$ and (c) $\langle A_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that, letting $\bar{G}_n = G_{n,A_n}$,*

$$\lim_n \left\| \bar{G}_n - \xi \right\| = 0 \quad \textit{and} \quad \sum_n \lambda(A_n^c) < \infty \tag{3.8}$$

Proof. From Theorem 2 we conclude that $\lim_n \|H_{n,B_n} - \chi\| = 0$ where $\langle H_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\Gamma(F_1^+, F_2^+, \ldots)$, $\langle B_n \rangle_{n\in\mathbb{N}}$ a sequence in $\mathscr{A}$ with $\sum_n \lambda(B_n^c) < \infty$ and $\chi \in ba(\lambda)_+$. Let $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ be the disjoint sequence of convex weights associated with H_n via

$$H_n = \sum_i \alpha_{n,i} F_i^+ \qquad n \in \mathbb{N}$$

Write $\bar{F}_n = \sum_i \alpha_{n,i} F_i^-$ and apply Theorem 2 to $\langle \bar{F}_n \rangle_{n\in\mathbb{N}}$. We obtain a disjoint sequence $\langle \beta_k \rangle_{k\in\mathbb{N}}$ of convex weights, a sequence $\langle C_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{A}$ and $\zeta \in ba(\lambda)_+$ such that, letting $K_j = \sum_n \beta_{j,n} \bar{F}_n$,

$$\lim_j \|K_{j,C_j} - \zeta\| = 0 \quad \text{and} \quad \sum_j \lambda(C_j^c) < \infty$$

Let $\gamma_{j,i} = \sum_n \beta_{j,n} \alpha_{n,i}$, $G_j = \sum_i \gamma_{j,i} F_i$ and $A_j = C_j \cap \bigcap_{\{n:\beta_{j,n}>0\}} B_n \in \mathscr{A}$. Observe that $G_j \in \mathrm{co}(F_j, F_{j+1}, \ldots)$. Moreover, since $\beta_{j,n}\beta_{j',n} = 0$ when $j \neq j'$,

$$\sum_j \lambda(A_j^c) = \sum_j \lambda(C_j^c) + \sum_j \sum_{\{n:\beta_{j,n}>0\}} \lambda(B_n^c) \leq \sum_j \lambda(C_j^c) + \sum_n \lambda(B_n^c) < \infty$$

But then $G_j = \sum_n \beta_{j,n} H_n - K_j$ so that, letting $\xi = \chi - \zeta$,

$$\|G_{j,A_j} - \xi\| \le |\chi(A_j^c)| + |\zeta(A_j^c)| + \sum_n \beta_{j,n} \|H_{n,A_j} - \chi_{A_j}\| + \|K_{j,A_j} - \zeta_{A_j}\| \longrightarrow 0$$

A few comments are in order.

(1). Assume that the original sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ is relatively weakly compact. It is then uniformly absolutely continuous with respect to some λ (see [Brooks and Dinculeanu (1974),Theorems 2.3 and 4.1]) so that $\lim_k \sup_n \|G_n - G_n \wedge 2^k \lambda\| = 0$ and the sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ converges thus strongly to ξ. In this special case, Theorem 2 is nothing more than the classical result of Banach and Saks [Dunford and Schwartz (1988) III.3.14]. In the general case, however, the sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ need not converge to ξ, not even in restriction to a single, fixed set. If one assumes that $\mathscr{A}$ is a σ algebra and λ is countably additive, then it becomes possible to replace the sets $A_1, A_2, \ldots$ with $A_k^* = \bigcap_{n>k} A_n$ and conclude that for each ε there is $A \in \mathscr{A}$ such that $\lambda(A^c) < \varepsilon$ while G_n converges in norm to ξ in restriction to A. This improvement on the statement of Theorem 2 may also be obtained upon introducing a form of the independence property valid in the finitely additive context. The first step in this direction was made long ago by Purves and Sudderth (1976) relatively to strategies, a notion due to Dubins and Savage (2014) and too lengthy to explain here. Briefly put, an independent strategy is a finitely additive probability λ defined over the algebra of clopen sets of the product space $X^{\mathbb{N}}$, with X an arbitrary non void set, and satisfying

$$\lambda(H_1 \times H_2 \times \ldots H_N \times \ldots) = \gamma_1(H_1)\gamma_2(H_2)\ldots\gamma_N(H_N)\ldots \tag{3.9}$$

where $\langle H_n \rangle_{n \in \mathbb{N}}$ is a sequence of subsets of X and $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ a sequence of finitely additive probabilities defined on all subsets of X. This notion has found a number of applications in replicating finitely additive theorems on the convergence of random quantities. Given that our interest focuses instead on additive functions, we think that the following is a reasonable adaptation of that same notion to our setting.

Say that a sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ in $ba(\mathscr{A})$ is independent relatively to $\lambda \in ba(\sigma\mathscr{A})$ if $\lambda \gg F_n$ for $n = 1, 2, \ldots$ and there exists a sequence $\langle \mathscr{A}_n \rangle_{n \in \mathbb{N}}$ of subalgebras of $\mathscr{A}$ with the property that (i) $\inf_{h \in \mathscr{S}(\mathscr{A}_n)} \|F_n - \lambda_h\| = 0$ and (ii) for any countable partition $\{N_1, N_2, \ldots\}$ of $\mathbb{N}$ into finite subsets

$$\lambda\left(\bigcap_i B_i\right) = \prod_{i=1}^{\infty} \lambda(B_i) \qquad B_i \in \bigvee_{n \in N_i} \mathscr{A}_n, \; i = 1, 2, \ldots \tag{3.10}$$

We state the following corollary only for the case of positive set functions.

Corollary 2. *Let* $\langle F_n\rangle_{n\in\mathbb{N}}$ *be a bounded sequence in* $ba(\sigma\mathscr{A})_+$ *which is independent relatively to* $\lambda\in\mathbb{P}(\sigma\mathscr{A})$ *and let* $\delta>0$*. There exist* $\xi\in ba(\mathscr{A})_+$ *satisfying* $\|\xi\|\geq\sup_k\limsup_n\left\|F_n\wedge 2^k\lambda\right\|-\delta$*, and a sequence* $\langle G_n\rangle_{n\in\mathbb{N}}$ *in* $\Gamma(F_1,F_2,\ldots)$ *such that, for each* $\varepsilon>0$*, there is* $A_\varepsilon\in\mathscr{A}$ *such that*

$$\xi(A_\varepsilon^c)<\varepsilon \quad \text{and} \quad \lim_n|G_n-\xi|(A_\varepsilon)=0 \tag{3.11}$$

Proof. Choose $f_n\in\mathscr{S}(\mathscr{A}_n)$ such that $\|F_n-\lambda_{f_n}\|<2^{-n-1}$. Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be the sequence in (3.2) with $G_n=\sum_i\alpha_{n,i}F_i$ and write $g_n=\sum_i\alpha_{n,i}f_i$. Choose the sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ to be disjoint. Observe that $\|G_n-\lambda_{g_n}\|\leq 2^{-n-1}$. For each $A\in\mathscr{A}$

$$\begin{aligned} G_n(A)+2^n\lambda(A^c) &\geq -2^{-n-1}+\lambda_{g_n}(A)+2^n\lambda(A^c)\\ &\geq -2^{-n-1}+\lambda(g_n\wedge 2^n)\\ &= -2^{-n-1}+\lambda_{g_n}\big(g_n\leq 2^n\big)+2^n\lambda\big(g_n>2^n\big)\\ &\geq -2^{-n}+G_n\big(g_n\leq 2^n\big)+2^n\lambda\big(g_n>2^n\big) \end{aligned}$$

so that $G_n(g_n\leq 2^n)+2^n\lambda(g_n>2^n)\leq 2^{-n}+\|G_n\wedge 2^n\lambda\|$. Thus we can replace A_n with $\{g_n\leq 2^n\}$ in Theorem 2 and obtain

$$\sum_{n>N}\lambda\left(g_n>2^n\right)\leq 2^{-N}\left(1+\sup_n\|F_n\|\right)$$

Given that the sets $N_n=\{i:\alpha_i^n\neq 0\}$ are disjoint and that $g_n\in\mathscr{S}\big(\bigvee_{i\in N_n}\mathscr{A}_i\big)$ we conclude, following the classical proof of the Borel-Cantelli lemma,

$$\lambda\left(\bigcap_{n>N}\{g_n\leq 2^n\}\right)=\prod_{n>N}\lambda(g_n\leq 2^n)\geq 1-\sum_{n>N}\lambda\left(g_n>2^n\right)\geq 1-2^{-N}\Big(1+\sup_n\|F_n\|\Big)$$

i.e. $\lim_N\lambda\left(\bigcap_{n>N}\{g_n\leq 2^n\}\right)=1$ and, by absolute continuity, $\lim_N\xi\left(\bigcup_{n>N}\{g_n>2^n\}\right)=0$. One can then fix $A_\varepsilon=\bigcap_{n>N}\{g_n\leq 2^n\}\in\mathscr{A}$ with N sufficiently large. The claim follows from (3.2).

(2). Notice the special case in which $F_n=\int f_n d\lambda$ with $\langle f_n\rangle_{n\in\mathbb{N}}$ a bounded sequence in $L^1(\lambda)$ so that G_n takes the form $G_n=\int g_n d\lambda$ for some $g_n\in\mathrm{co}(f_n,f_{n+1},\ldots)$. Then, with the notation of Corollary 1,

$$\begin{aligned} \lambda^*(|g_n-g_m|>c) &\leq \lambda^*(|g_n-g_m|\mathbf{1}_{A_n\cap A_m}>c)+\lambda(A_n^c\cup A_m^c)\\ &\leq c^{-1}\int_{A_n\cap A_m}|g_n-g_m|d\lambda+\lambda(A_n^c\cup A_m^c)\\ &\leq c^{-1}(\|\bar G_{n,A_m}-\xi\|+\|\bar G_{m,A_n}-\xi\|)+\lambda(A_n^c\cup A_m^c) \end{aligned}$$

so that the sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ is λ-Cauchy, a claim proved in [Cassese (2013), Theorem 6.3] for the case $f_n \geq 0$. If, in addition, λ is countably additive, then by completeness we obtain that g_n λ-converges to some limit $h \in L^1(\lambda)$ or even converges a.s., upon passing to a subsequence. This is the form in which the subsequence principle attributed to Komlós is often stated in applications.

(3). In a possible interpretation of Theorem 2, one may take F_n to be an expression of the disagreement $|F_n^1 - F_n^2|$ between two different opinions. Theorem 2 suggests that either disagreement is progressively smoothed out, in accordance with condition (3.3), or that it converges to some final divergence of opinions. It would be interesting to see if this result may be useful to get more insight in the classical problem of merging of opinions described in the well known paper of Blackwell and Dubins (1962).

We close this section with two generalizations of Theorem 2. In the first we drop the norm boundedness condition; in the second one we consider vector valued set function.

Theorem 3. *Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence in $ba(\mathscr{A})_+$ and $\lambda \in \mathbb{P}(\mathscr{A})$. There exist $\xi : \mathscr{A} \to \mathbb{R}_+ \cup \{\infty\}$ finitely additive and a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\Gamma(F_1, F_2, \ldots)$ such that*

$$\lim_n \left|G_n \wedge 2^n\lambda - \xi\right|(A) = 0 \qquad \text{for each } A \in \mathscr{A} \text{ with } \xi(A) < \infty \tag{3.12}$$

Proof. Given that Theorem 1 does not require norm boundedness, the sequence $\langle \xi_k\rangle_{k\in\mathbb{N}}$ is obtained exactly as in the proof of Theorem 2. Define the extended real valued, finitely additive set function

$$\xi(A) = \lim_k \xi_k(A) \qquad A \in \mathscr{A} \tag{3.13}$$

and notice that (3.4) holds so that the sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ may be chosen in such a way that $\|G_n \wedge 2^n\lambda - \xi_n\| < 2^{-n}$. Thus, if $A \in \mathscr{A}$ and $\xi(A) < \infty$,

$$\begin{aligned}\lim_n \left|\xi - G_n \wedge 2^n\lambda\right|(A) &= \lim_n |\xi - \xi_n|(A) = \lim_n \sup_{\pi\in\Pi(\mathscr{A})} \sum_{B\in\pi} |(\xi - \xi_n)(A\cap B)| \\ &= \lim_n (\xi - \xi_n)(A) = 0\end{aligned}$$

Spaces such as L^1 or *ba* have the special property that a sequence of positive elements converges weakly to 0 if and only if it converges in norm too. For this special spaces we obtain the following:

Theorem 4. *Let $(W, \mathscr{B}, P)$ be a classical probability space – i.e. W non empty, Σ a σ algebra of subsets of W and P σ additive – and set $X =$*

$L^1(W,\mathscr{B},P)$. *Let* $\langle F_n\rangle_{n\in\mathbb{N}}$*, a norm bonded sequence in* $ba_0(\mathscr{A},X)_+$*, and* $\lambda\in\mathbb{P}(\mathscr{A})$ *be such that the set*

$$\mathcal{R}=\text{co}\left\{\sup_{A\in\pi}F_n(A)/\lambda(A):n\in\mathbb{N},\ \pi\in\Pi(\mathscr{A})\right\}\tag{3.14}$$

is P*-bounded. There exist* $\xi\in ba_0(\mathscr{A},X)_+$ *and* $\langle G_n\rangle_{n\in\mathbb{N}}$ *in* $\Gamma(F_1,F_2,\ldots)$ *such that*

$$\lim_n\left|G_n-\xi\right|(\Omega)=0\qquad P-a.s.\tag{3.15}$$

Proof. Given that X is a Banach lattice possessing property $(\boldsymbol{P})$ and that weak and strong convergence to 0 are equivalent properties for positive sequences in X, we deduce from Theorem 1 that there exists $\xi\in ba_0(\mathscr{A},X)_+$ and a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\Gamma(F_1,F_2,\ldots)$ such that

$$\lim_n\left\|G_n\wedge 2^n\lambda-\xi\right\|=0\tag{3.16}$$

Observe that for each $A\in\mathscr{A}$,

$$(G_n\wedge 2^n\lambda)(A)=\lim_\pi\sum_{A'\in\pi}G_n(A\cap A')\wedge 2^n\lambda(A\cap A')$$

and, since the net on the right hand side is decreasing with π, there exists $\pi_n=\{A_n^1,\ldots,A_n^{I_n}\}\in\Pi(\mathscr{A})$ such that

$$\begin{aligned}(G_n\wedge 2^n\lambda)(A)&\le\sum_{i=1}^{I_n}G_n(A\cap A_n^i)\wedge 2^n\lambda(A\cap A_n^i)\\&\le\sum_{i=1}^{I_n}G_n(A\cap A_n^i)\mathbf{1}_{B_n^i}+2^n\lambda(A\cap A_n^i)\mathbf{1}_{B_n^{ic}}\equiv\bar{G}_n(A)\end{aligned}$$

(the last equality being a definition of $\bar{G}_n\in ba_0(\mathscr{A},X)$) where

$$B_n^i=\left\{G_n(A_n^i)\le 2^n\lambda(A_n^i)\right\}$$

On the other hand,

$$\begin{aligned}\|\bar{G}_n-G_n\wedge 2^n\lambda\|&=\|\bar{G}_n(\Omega)\|_X-\|(G_n\wedge 2^n\lambda)(\Omega)\|_X\\&=\int\sum_{i=1}^{I_n}G_n(A_n^i)\wedge 2^n\lambda(A_n^i)dP-\lim_\pi\int\sum_{A\in\pi}G_n(A)\wedge 2^n\lambda(A)dP\end{aligned}$$

so that π_n may be so chosen that $\|\bar{G}_n - G_n \wedge 2^n\lambda\| \le 2^{-n}$. We conclude, $\lim_n \|\bar{G}_n - \xi\| = 0$. Observe that, with the above notation,

$$B_n = \bigcap_{i=1}^{I_N} B_n^i = \left\{ \sup_{A\in\pi_n} G_n(A)/\lambda(A) \le 2^n \right\}$$

Given that $\sup_{A\in\pi_n} G_n(A)/\lambda(A) \in \mathcal{R}$ then, by assumption, $\lim_n P(B_n) = 1$. Passing to a subsequence (still indexed by n) we obtain that $\sum_n P(B_n^c) < \infty$ and therefore that for each $k > 0$ there exists $N_k > N_{k-1}$ such that

$$P\left(\bigcap_{n>N_k} B_n\right) > 1 - \varepsilon \tag{3.17}$$

Let $H_k = \bigcap_{n>N_k} B_n$.

$$\begin{aligned}\int_{H_k} |G_n - \xi|(\Omega)dP &= \lim_\pi \int_{H_k} \sum_{A\in\pi} |\bar{G}_n(A_n) - \xi(A_n)|dP \\ &\le \|\bar{G}_n - G_n\| + \|G_n \wedge 2^n\lambda - \xi\|\end{aligned}$$

We obtain a subsequence such that $|G_n - \xi|(\Omega)$ converges to 0 P a.s. on H_k for each k. Given that the sequence $\langle H_k\rangle_{k\in\mathbb{N}}$ is increasing, we conclude that the sequence converges pointwise on $H = \bigcup_k H_k$ and that $P(H) = 1$.

4 Further Applications

A first, simple application of Theorem 2 is the following:

Corollary 3. *Let $\langle\mu_n\rangle_{n\in\mathbb{N}}$ be a norm bounded sequence in $ba(\mathscr{A})$ and let $K_n \subset K_{n+1} \cap L^1(\mu_n)$ for $n = 1, 2, \ldots$. Write $K = \bigcup_n K_n$ and assume that*

$$\sup_k \limsup_n \big\| |\mu_n| \wedge |\mu_k| \big\| > 0 \qquad \text{and} \qquad \limsup_n \|f\|_{L^1(\mu_n)} < \infty \qquad f \in K \tag{4.1}$$

Then there is $\mu \in \mathbb{P}(\mathscr{A})$ such that $K \subset L^1(\mu)$.

Proof. The statement remains unchanged if we replace μ_n with $|\mu_n|$ so that we can assume with no loss of generality that $\mu_n \ge 0$ for all $n \in \mathbb{N}$. Theorem 2 delivers the existence of $\bar{\mu} \in ba(\mathscr{A})_+$ such that $\|\bar{\mu}\| > 0$ and $\lim_n \|\bar{\mu} - m_n \wedge 2^n\lambda\| = 0$ for some $\lambda \in ba(\mathscr{A})_+$ and $\langle m_n\rangle_{n\in\mathbb{N}}$ in $\Gamma(\mu_1, \mu_2, \ldots)$. Write $\mu = \bar{\mu}/\|\bar{\mu}\| \in \mathbb{P}(\mathscr{A})$. Let $f \in K$ and for each $n \in \mathbb{N}$ sufficiently large, let $\langle f_j^n \rangle_{j\in\mathbb{N}}$ be a sequence of $\mathscr{A}$ simple functions such that $m_n^*(|f - f_j^n| > 2^{-j}) \le 2^{-j}$. Then, $\mu^*(|f - f_n^n| > 2^{-n}) \le [2^{-n} + \|\bar{\mu} - m_n \wedge 2^n\lambda\|]/\|\bar{\mu}\|$, which proves that f is μ measurable. Moreover,

$$\begin{aligned} \|\bar{\mu}\| \int |f| d\mu &= \lim_k \int (|f| \wedge k) d\bar{\mu} = \lim_k \lim_n \int (|f| \wedge k) d(m_n \wedge 2^n \lambda) \\ &\le \limsup_n \int |f| dm_n < \infty \end{aligned}$$

We also obtain the following form of the strong law, with

$$\mathbb{P}_*(\lambda) = \left\{\mu \in \mathbb{P}(\mathscr{A}) : \mu \ll \lambda \text{ and } \mu(A) = 0 \text{ if and only if } \lambda(A) = 0\right\} \tag{4.2}$$

Corollary 4 (Komlós). *Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of measurable functions such that*

$$\lim_{c\to\infty} \sup_{h\in \mathrm{co}(|f_1|,|f_2|,\ldots)} \lambda^*(h > c) = 0 \tag{4.3}$$

There exists a sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ in $\Gamma(f_1, f_2, \ldots)$ and $\mu \in \mathbb{P}_*(\lambda)$ such that, for any subsequence $\langle g_n^\alpha \rangle_{n\in\mathbb{N}}$, the partial sums

$$S_k^\alpha = \frac{g_1^\alpha + \ldots + g_k^\alpha}{k} \qquad k = 1, 2, \ldots \tag{4.4}$$

form a Cauchy sequence in $L^1(\mu)$.

Proof. By [Cassese (2013)Theorem 6.1] it is possible to find $\nu \in \mathbb{P}_*(\lambda)$ such that the set $\mathrm{co}(|f_1|, |f_2|, \ldots)$ is bounded in $L^1(\nu)$. We can then apply Corollary 1 and the remarks that follow and obtain a sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ in $\Gamma(f_1, f_2, \ldots)$ which is ν-Cauchy. By that same reference it is possible to find $\mu \in \mathbb{P}_*(\nu) \subset \mathbb{P}_*(\lambda)$ and a subsequence (still denoted by $\langle g_n \rangle_{n\in\mathbb{N}}$ for simplicity) such that $\mathrm{co}(|f_1|, |f_2|, \ldots)$ is bounded in $L^1(\mu)$ and that

$$\sup_p \int \sum_{n=n_r}^{n_r+p} |g_{n+1} - g_n| d\mu \le 2^{-r} \qquad r \in \mathbb{N} \tag{4.5}$$

for some suitably chosen sequence $\langle n_r \rangle_{r\in\mathbb{N}}$. Let $k > n_r$ and $S_k = k^{-1} \sum_{n=1}^k g_n$. Then,

$$k \int |S_k - g_{n_r}| \, d\mu \le \int \sum_{n=1}^{n_r} |g_n - g_{n_r}| d\mu + \int \sum_{n=n_r+1}^{k} |g_n - g_{n_r}| \, d\mu$$

$$\leq \int \sum_{n=1}^{n_r} |g_n - g_{n_r}| d\mu + \int \sum_{n=n_r+1}^{k} \sum_{i=n_r+1}^{n} |g_i - g_{i-1}| \, d\mu$$
$$\leq 2n_r \sup_n \|f_n\|_{L^1(\mu)} + 2^{-r} k$$

so that

$$\sup_{p,q} \|S_{k+p} - S_{k+q}\|_{L^1(\mu)} \leq 4(n_r/k) \sup_n \|f_n\|_{L^1(\mu)} + 2^{-(r-1)}$$

and the sequence $\langle S_k \rangle_{k\in\mathbb{N}}$ is Cauchy in $L^1(\mu)$. It is clear that (4.5), on which our preceding conclusion rests, holds for the sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ if it holds for all of its subsequences.

The comparison of Corollary 4 with the original result of Komlós illustrates the difficulties inherent in finite additivity. Not only is the original property of a.s. convergence replaced here by the Cauchy criterion, but also a change of measure is necessary to prove the claim. Given that λ and μ have the same null sets, these limitations are irrelevant in the case of a countably additive measure. It should be noted, however, that the change of measure technique is useful here to relax more familiar integrability conditions which are traditionally employed in the proof of the strong law.

In closing, one should mention that a finitely additive version of the strong law has been proved long ago, by Chen (1976) and later by Halevy (1979) who also proved a version of Komlós theorem. Other important papers that follow a similar approach to finitely additive limit theorems are those of Karandikar (1982) and of Ramakrishnan (1981). The setting adopted in these and related papers is however that of independent strategies mentioned above and is therefore radically different from ours. The connection between this approach and the one proposed in this work surely deserves further study.

References

ALIPRANTIS, C.D. and BURKINSHAW, O. (1985). *Positive Operators.* Academic Press, Orlando.

BALDER, E. J. (1989). Infinite-Dimensional Extension of a Theorem of Komlós. *Probab. Th. Rel. Fields,* **81**, 185–188.

BALDER, E. J. (1990). New Sequential Compactness Results for Spaces of Scalarly Integrable Functions. *J. Math. Anal. Appl.,* **151**, 1–16.

BALDER, E. J. and HESS, C. (1996). Two generalizations of Komlós' theorem with lower closure-type applications. *J. Convex Anal.,* **3**, 25–44.

BERKES, I. (1990). An extension of the Komlós subsequence theorem. *Acta Math. Hung.,* **55**, 103–110.

BLACKWELL, D. and DUBINS, L.E. (1962). Merging of opinions with increasing information. *Ann. Math. Stat.* **33**, 882–886.

BROOKS, J. K. (1972). Weak compactness in the space of vector measures. *Bull. Amer. Math. Soc.,* **78**, 284–287.

BROOKS, J.K. and DINCULEANU, N. (1974). Strong additivity, absolute continuity and compactness in spaces of measures. *J. Math. Anal. Appl.,* **45**, 156–175.

BURKHOLDER, D. L. (1973). Discussion on prof. Kingman's paper. *Ann. Probab.,* **1**, 900–902.

CASSESE, G. (2013). Convergence in measure under finite additivity. *Shankyā A,* **75**, 171–193.

CASSESE, G. (2008). Finitely additive supermartingales. *J. Theo. Probab.,* **21**, 586–603.

CHATTERJI, S. D. (1970). A general strong law. *Inventiones Math.,* **9**, 235–245.

CHEN, R. (1976). Some finitely additive versions of the strong law of large numbers. *Israel J. Math.,* **24**, 244–259.

CHEN, R. (1977). On Almost Sure Convergence in a Finitely Additive Setting, Z. *Wahrsch. Ver. Geb.,* **37**, 341–365.

CVITANIĆ, J. and KARATZAS, I. (2001). Generalized neyman-pearson lemma via convex duality. *Bernoulli,* **7**, 79–97.

DAY, J. B. and LENNARD, C. (2010). Convex Komlós sets in banach function spaces. *J. Math. Anal. Appl.,* **367**, 129–136.

DIESTEL, J. and UHL JR, J. J. (1977). *Vector measures, mathematical surveys, 15.* Providence.

DOLÉANS-DADE, C. (1968). Existence du Processus Croissant Naturel Associé à un Potentiel de la Classe (D). *Z. Wahrsch. verw. Geb.,* **9**, 309–314.

DUBINS, L. E. and SAVAGE, L. J. (2014). *How to gamble if you must. Inequalities for stochastic processes.* Dover, New York.

DUNFORD, N. and SCHWARTZ, J. T. (1988). *Linear operators. General theory.* Wiley, New York.

GUESSOUS, M. (1997). An elementary proof of Komlós-Révész theorem in hilbert space. *J. Convex Anal.,* **4**, 321–332.

HALEVY, A. (1979). On an analogous of Komlós's theorem for strategies. *Ann. Probab.,* **7**, 1073–1077.

JIMÉNEZ FERNÁNDEZ, E., JUAN, M. A. and SÁNCHEZ PÉREZ, E. A. (2011). A Komlós theorem for abstract banach lattices of measurable functions. *J. Math. Anal. Appl.,* **383**, 130–136.

KARANDIKAR, R.L. (1982). A general principle for limits theorems in finitely additive probability. *Trans. Amer. Math. Soc.,* **273**, 541–550.

KOMLÓS, J. (1967). A generalization of a problem of steinhaus. *Acta Math. Hung.,* **18**, 217–229.

LENNARD, C. (1993). A converse to a theorem of Komlós for convex subsets of L_1. *Pacific J. Math.,* **159**, 75–85.

PURVES, R.A. and SUDDERTH, W.D. (1976). Some finitely additive probability. *Ann. Probab.,* **4**, 259–276.

RAMAKRISHNAN, S. (1981). Finitely additive markov chains. *Trans. Amer. Math. Soc.,* **265**, 247–272.

SCHWARTZ, M. (1986). New proofs of a theorem of Komlós. *Acta Math. Hung.,* **47**, 181–185.

TRAUTNER, R. (1990). A new proof of the Komlós-Révész theorem. *Probab. Th. Rel. Fields,* **84**, 281–287.

VON WEIZSÄCKER, H. (2004). Can one drop L^1-Boundedness in Komlós's subsequence theorem? *Amer. Math. Month.,* **111**, 900–903.

Gianluca Cassese
Department of Economics
Statistics and Management
Università Milano Bicocca
via Bicocca degli Arcimboldi 8,
20126 Milano, Italy
E-mail: gianluca.cassese@unimib.it

Paper received: 28 November 2014.