Sankhyā : The Indian Journal of Statistics
2016, Volume 78-B, Part 1, pp. 1-38

Variable Family Size Based Spatial Moving Correlations Model

Hensley H Mariathas and Brajendra C Sutradhar
Memorial University of Newfoundland, St. John's, NL, Canada

Abstract

It is well known that the autocorrelations among responses play a significant role in time series setup mainly for the purpose of forecasting. Similarly, in a spatial setup, spatial variation and correlations among responses collected from a large sequence of spatial locations are important parameters for any practical inferences. For example, variation in plant crop damages and correlations among neighboring plant crop damages are important parameters to understand before one can take suitable measure to prevent such damages in the future. In this setup, a group of neighboring plants or locations constitute a family, and the pairwise responses within a family of locations are likely to be correlated. Furthermore, the responses from neighboring families will also be correlated but they become uncorrelated when the locations are far apart. In this paper, we deal with modeling of spatial correlations for continuous data collected from non-linear sequence of locations and propose a pairwise linear mixed models-based moving or band correlation structure that reflects the correlations for within and between families. The proposed correlation structure is then exploited to develop the likelihood inferences for both variance and correlation parameters of the model. The regression parameters are also estimated. The correlation model and the inferences are illustrated using a monte carlo study for a simpler case with responses collected from a linear sequence of locations. The correlation mis-specification effects are also discussed.

AMS (2000) *subject classification.* Primary 62H11, 62H12, Secondary 62H20.
Keywords and phrases. Family of random effects, Generalized least square and likelihood estimation, Linear and non-linear spatial responses, Moving correlations band, Pair-wise spatial correlations, Regression function, Variable family size.

1 Introduction

Over the last two decades, analysis of spatial data has become an emerging area of research in many different fields, such as ecology, environmental science, epidemiology, geography, sociology, economics and forestry. The spatial data are realizations of random variables collected from a sequence of related geographical locations, where the responses collected from adjacent locations naturally become correlated. These correlations are referred to as *spatial correlations.* Note that apart from the influence of certain fixed covariates, a response at a given location, is usually influenced by some invisible, say random effects, associated to this and other adjacent locations. The variance of these random effects and their possible correlations may highly influence the variance and pair-wise correlations of the responses. In spatial setup, it is of main interest to examine the variance and correlations of the responses. It is also of interest to find the effects of the covariates after taking the spatial correlations of the responses into account. Note that the spatial correlations may not affect the consistent estimation of the regression effects in a continuous data setup but may effect when spatial responses are discrete such as counts taken from spatial locations. However, the modeling of correlations in a continuous data setup is nevertheless very important as it helps to understand the variation and correlations among spatial responses. For various modeling for correlations and analysis of continuous spatial data over the last two decades, we, for example, refer to Cressie (1993), Vecchia (1988, 1992), Jones and Vecchia (1993), Gaetan and Guyon (2010), Cressie and Johannesson (2008), Kang and Cressie (2011), and Kang et al. (2010).

With regard to modeling of the spatial correlation structure, Cressie (1991, p. 85–86) suggests for using various correlation structure such as exponential covariance function, Gaussian covariance function and a reciprocal covariance function. One of the common properties of these correlation structures is that the correlation decays as the distance between two objects increases. Similar correlation functions but by using the spectral approach, were exploited by Jones and Vecchia (1993) [see also Vecchia (1988, 1992) in order to define pairwise covariances]. More specifically, Jones and Vecchia (1993) use a modified Bessel function of the second kind in terms of the distance between two objects which also decays as distance increases. This type of covariance function, also known as Matèrn covariance function, has been exploited in some other studies (Berger et al. (2001, Section 1.1), Gelfand et al. (2003, Section 2)) using Bayesian framework for the analysis of spatially correlated data. But these studies do not appear to address the fact that in a spatial setup each of the pairwise responses may be influenced by the member locations of their own family as well as by certain member locations from the other family.

The family based spatial correlations are however reflected in some studies where higher order ARMA (autoregressive moving average) models such as ARMA(3,3) are lead for the construction of the pairwise correlations, see for example, Basu and Reinsel (1993a). However, one of the major difficulties with this ARMA model fitting is that the spatial objects are considered to be equi-spaced, which may not hold in many practical situations.

Note that developing a familial type correlation model for spatial data has many advantages. For example, by using a simple spatial random effects (SRE) model for the spatial random process involved in the observation model, Cressie and Johannesson (2008) [see also Kang and Cressie (2011, Eq. 6)] were able to use exact kriging approach, which in an empirical-Bayesian framework performs rapid kriging computation for required spatial predictions. Kang et al. (2010) have generalized the SRE model to the spatial-temporal setup. However, one of the limitations of this SRE model is that the model uses the same length of random effects vector (e.g. Kang and Cressie (2011, Eq. 6)), which is equivalent to use a balanced family size at all spatial location. To improve this SRE model, in this paper, we propose a pairwise unbalanced familial random effects model that eventually produces moving band type correlation structures for the spatial data. As far as the weights are concerned, similar to the previous studies, they will be assumed to be deterministic and known spatial functions. They are unbalanced and hence not orthogonal. The construction of the proposed unbalanced spatial random effects (USRE) model and the resulting correlation structures for the spatial responses are given in details in Sections 2.1 and 2.2. In Section 2.2, we also illustrate the computation of the variances and pairwise covariances based on two specialized unbalanced random effects correlation structure. The spatial correlation structure proposed in Section 2 are exploited in Section 3 to develop the Gaussian likelihood inferences for both regression and correlation parameters. We also discuss a hybrid moment-GLS technique for parameter estimation in simple cases. Note that because the model produces a moving band type patterned correlation matrix, its inversion for practically large data set is not a problem for the desired likelihood inferences. In Section 4, we carry out an intensive empirical study to illustrate the proposed correlation structure and the likelihood inferences. Further note that even though the spatial correlation model developed in Section 2 is quite general for non-linear sequence of responses (also may be referred to as two dimensional responses), in the simulation study for simplicity we have, however, applied the model for linear sequence of responses. This later case with one-dimensional spatial responses arises in spatial design where, for example, spatial objects such as plants are laid out in a long row and rows are independent. The correlation mis-specification effect is also examined empirically. The paper concludes in Section 5.

2 Familial Random Effects Based Spatial Model

Consider a region $\mathcal{S}$ containing S spatial locations, where the region $\mathcal{S}$, for example, may have circular, rectangular or linear form. When $\mathcal{S}$ is a line, that is, the sequence of locations form a line, they are known to be one dimensional locations. In other cases, they are known to be two dimensional locations. Now irrespective of the form of the region $\mathcal{S}$, we label the S locations in $\mathcal{S}$ as $\{1, \cdots, r, \cdots, s, \cdots, S\}$. Let y_r be the response at the r^{th} location, where this response may be influenced by a multi-dimensional fixed covariate vector $x_r = (x_{r1}, \cdots, x_{rp})'$ containing, for example, the epidemiological and environmental information from the r^{th} location. Also this response will be influenced by some invisible random effects from the neighboring locations those are within a pre-specified distance from the r^{th} location. We refer to these locations as the r^{th} family of size n_r and denote the family by f_r. To be specific, let d^* be the pre-specified distance determined by the experimenter, which should be large enough to assume that any two locations apart from each other by a distance more than d^* will be uncorrelated. Suppose that for any locations $k = 1, \cdots, S$, d^*_{rk} denote the Euclidian distance between the centers of the r^{th} and k^{th} locations. For convenience, we define an indicator variable δ_{rk} such that

$$\delta_{rk} = \begin{cases} 1 & \text{if } \ d^*_{rk} \leq d^* \ \text{ for } \ k = 1, \cdots, S \\ \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

implying that

$$\sum_{k \in f_r} \delta_{rk} = n_r. \tag{2.2}$$

Next, suppose that the individual random effects of all S locations in the whole region $\mathcal{S}$ are denoted by $\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_r, \cdots, \widetilde{\gamma}_S$; $\widetilde{\gamma}_r$ being the random effect corresponding to the r^{th} location. It is then clear that n_r random effects including $\widetilde{\gamma}_r$ corresponding to the n_r locations satisfying (2.1)–(2.2) will constitute a cluster of random effects for the r^{th} location. We label these random effects as

$$\gamma_r \quad = \quad (\gamma_{r1}, \cdots, \gamma_{rj_r} \cdots, \gamma_{rn_r})' \tag{2.3}$$

where without any loss of generality we can assume that $\gamma_{r1} = \widetilde{\gamma}_r$, and these n_r random effects are treated to be in family f_r. Thus, we do not distinguish between the family of locations and the family of random effects corresponding to the r^{th} location. More specifically, f_r will be used for both purpose, that is, to represent both family of n_r locations corresponding to the r^{th} location and the family of random effects collected from these n_r locations.

Note that the elements of the latent vector γ_r in (2.3) are likely to be correlated for all $r = 1, \cdots, S$, and this correlation structure will be obtained from the correlation structure of the original random effects $\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_r, \cdots, \widetilde{\gamma}_S$ corresponding to S locations of the region $\mathcal{S}$. Further note that even though there are many ways to consider the correlation structure for these random effects, in this paper, we assume a distance based truncated equi-correlation structure, such that marginally

$$\widetilde{\gamma}_r \sim N(0, \sigma_\gamma^2), \tag{2.4}$$

and pair-wise they follow the truncated equi-correlation structure, given by

$$\text{corr}(\widetilde{\gamma}_r, \widetilde{\gamma}_s) = \delta_{rs}\phi_{rs} = \begin{cases} 1 & \text{for } \; d^*_{rs} = 0 \\ \phi & \text{for } \; 0 < d^*_{rs} \leq d^* \\ 0 & \text{for } \; d^*_{rs} > d^*, \end{cases} \tag{2.5}$$

which generates a band correlation matrix with pairwise correlation ϕ within the band, where the band width is determined by the spatial distance (lag) d^* defined in (2.1). Next, we consider the responses to be continuous, and for all r ($r = 1, \cdots, S$), we propose that the responses y_r at the r^{th} location follow the unbalanced linear mixed model given by

$$y_r = x_r'\beta + w_r'\gamma_r + \epsilon_r, \tag{2.6}$$

where, β is the $p \times 1$ regression effects of the covariates x_r on y_r for all $r = 1, \cdots, S$; $w_r = (w_{r1}, \cdots, w_{rj_r}, \cdots, w_{rn_r})'$ be the known weight vector corresponding to $\gamma_r \in f_r$. Further, ϵ_r denote the model errors such that $\epsilon_r \overset{iid}{\sim} (0, \sigma_\epsilon^2)$ for all $r = 1, \cdots, s, \cdots, S$. Note that the mixed model in (2.6) is unbalanced because the r^{th} family of the random effects f_r have unequal sizes for $r = 1, \ldots, S$. Suppose that Γ_{rr} denotes the $n_r \times n_r$ variance covariance matrix of γ_r. Now because $E[\gamma_r] = 0$ by (2.5), one may then write the mean and variance of y_r as

$$\begin{aligned} E(Y_r) &= \mu_r = x_r'\beta, \\ \text{var}(Y_r) &= w_r'\text{var}(\gamma_r)\, w_r + \text{var}(\epsilon_r) \\ &= w_r'\Gamma_{rr}w_r + \sigma_\epsilon^2, \end{aligned} \tag{2.7}$$

where by (2.4)–(2.5), Γ_{rr} may be expressed as

$$\Gamma_{rr} = \sigma_\gamma^2 C_{n_r n_r}(\phi),$$

with $C_{n_r n_r}(\phi)$ as the $n_r \times n_r$ correlation matrix for the elements of the random effect vector γ_r which belongs to the family f_r. More specifically,

$$C_{n_r n_r}(\phi) = \phi 1_{n_r} 1'_{n_r} + (1-\phi) I_{n_r}, \tag{2.8}$$

by (2.5), where 1_{n_r} and I_{n_r} are the n_r dimensional unit vector and matrix, respectively.

Note that the aforementioned distance based equi-correlation structure for the member random effects of the same family is quite practical. However, apart from the correlation structure of the random effects in a family, the variance of the responses generated by the linear relationship (2.6) depends also on the corresponding weights $w_{r1}, \cdots, w_{rj_r}, \cdots, w_{rn_r}$ assigned to the random effects $\gamma_{r1}, \cdots, \gamma_{rj_r} \cdots, \gamma_{rn_r}$. These weights are, however, assumed to be known, and they are determined by the experimenter based on their experience or pilot studies. For example, for some experiments, it may be appropriate to assign equal weights to all random effects , namely

$$\begin{aligned} w_r &= (w_{r1}, \cdots, w_{rj_r}, \cdots, w_{rn_r})' \\ &= \frac{1}{\sqrt{n_r}} 1'_{n_r}, \end{aligned} \tag{2.9}$$

or, it may be appropriate to put distance based exponentially decaying weights such as

$$\begin{aligned} w_r &= (w_{r1}, \cdots, w_{rj_r}, \cdots, w_{rn_r})' \\ &= (c^0/m, c^1/m, \cdots, c^{j_r-1}/m, \cdots, c^{n_r-1}/m)', \end{aligned} \tag{2.10}$$

where c is a suitable constant fraction, and $m = [\sum_{j_r=1}^{n_r} \{c^{j_r-1}\}^2]^{\frac{1}{2}}$. Remark that the weights in (2.9) and (2.10) are chosen such that $w'_r w_r = 1$, indicating that when random effects are independent, that is, $\phi = 0$ in (2.5), the contribution from random effects toward the variance of the response (2.7) would be

$$w'_r \Gamma_{rr} w_r = w'_r [\sigma^2_\gamma I_{n_r}] w'_r = \sigma^2_\gamma,$$

same as the variance of a single random effect.

Next, because in the present spatial setup, the two neighboring families of random effects or locations will also likely to be correlated, in Section 2.1 below, we demonstrate how to construct two such correlated families, and subsequently obtain pair-wise correlations of the responses.

2.1. Construction of Two Correlated Families. The marginal properties of the response y_r collected from the r^{th} location for all $r = 1, \ldots, S$, are given by (2.7). Now for another location $s (s \neq r)$, the response y_s will satisfy the marginal mixed model (2.6) and hence y_s would be generated from

$$y_s = x_s'\beta + w_s'\gamma_s + \epsilon_s, \quad r \neq s, \tag{2.11}$$

where $\gamma_s = (\gamma_{s1}, \cdots, \gamma_{sj_s} \cdots, \gamma_{sn_s})'$ is the cluster of random effects belonging to f_s, a family of size n_s, around the s^{th} location. Note that in many problems $n_r \neq n_s$, leading the pair of families f_r and f_s to be unbalanced. Also note that if the locations r and s are not so far apart from each other, it is likely that y_r and y_s will be correlated. To reflect this situation, let there be n^*_{rs} common locations between the two families f_r and f_s. Finding the correlations between y_r and y_s, however, requires one to know the structure of all $n_r + n_s - n^*_{rs}$ locations within the two families f_r and f_s, which makes the computation of pairwise correlations difficult. These correlations are however needed for any reliable estimation of the variance components σ^2_γ and σ^2_ϵ, and also they are needed for efficient estimation of the regression effects β in the model (2.6) and (2.11).

Under the assumption that the correlations between y_r and y_s should decay as the distance between r^{th} and s^{th} location increases, Cressie (1991, p. 85–86) suggests for using various correlation structure such as exponential covariance function, Gaussian covariance function and a reciprocal covariance function. But these correlation structures do not accommodate the combined influence of member locations within two families f_r and f_s. Jones and Vecchia (1993) use a modified Bessel function of the second kind in terms of the distance between two objects which also decays as distance increases. The family based spatial correlations are however reflected in some studies where higher order ARMA (autoregressive moving average) models such as ARMA(3,3) are used for the construction of the pairwise correlations, see for example, Basu and Reinsel (1993b). However, one of the major difficulties with this ARMA model fitting is that the $n_r + n_s - n^*_{rs}$ locations are considered to be equi-spaced, which may not hold in many practical situations.

A recent study by Kang et al. (2010, Section 2.1.1), in a spatial-temporal set up, have used the influence of family based random effects to develop spatial-temporal correlations. At a given time $t = 1$, say, their spatial model (Kang, et al. (2010) Eq. 7, p. 274) has the form

$$\begin{aligned} y_s &= \mu_s + \upsilon_s + \epsilon^*_s \\ &= x_s'\beta + w_s'\gamma + \epsilon^*_s, \quad \text{for} \quad s = 1, \cdots, S, \end{aligned} \tag{2.12}$$

where, x'_s is the p dimensional covariate collected for s^{th} location, w_s is a vector of q dimensional known deterministic spatial basis functions and γ is a q dimensional vector of random effects with zero mean vector and $q \times q$ covariance matrix. Note that $\mu_s = x'_s\beta$ is usually referred to as a trend function and $\upsilon_s = w'_s\gamma$ is a function of unobservable random effects. Also in (2.12), $\epsilon^*_s \overset{iid}{\sim} (0, \sigma^2_\epsilon)$ is referred to as the measurement (or model) error. Further, notice from (2.12) that in Kang et al.'s (2010) approach each and every response y_s for all $s = 1, \ldots, S$, are affected by a common q dimensional random effects vector γ, whereas in our model (2.11), y_s is affected by n_s correlated random effects arising from n_s locations of the s^{th} family. However, if $n_s = q$ for all s and our covariance matrix $\Gamma_{ss} = \Gamma$ (say) is the same as the covariance matrix for γ considered by Kang et al. (2010), then both models will reduce to the same model. However, our covariance structure is new which is quite different than that of Kang et al. (2010). More importantly, our approach accommodates variable or unbalanced families of random effects. The construction of the correlation structure under this general situation is, however, not so straightforward, but manageable. We demonstrate this construction below.

2.1.1. Pair-Wise Family Structure. Note that the response y_r from the r^{th} location is likely to be influenced by the random effects of the n_r locations belonging to the family f_r with r^{th} location as the center. Similarly, y_s is expected to be influenced by the random effects of the n_s locations belonging to f_s. This marginal family property causes the variance of a response, say y_r, to be a function of the variances and covariances of the random effects from the member locations within the family f_r. By the same token, the covariance or correlation between y_r and y_s is bound to be affected by the covariances among all $n_r + n_s - n^*_{rs}$ random effects belonging to both f_r and f_s.

Now to compute the covariances among these $n_r + n_s - n^*_{rs}$ random effects, we first decompose the locations in both f_r and f_s in 2 segments as

$$n_r = n^*_r + n^*_{rs} \quad \text{and} \quad n_s = n^*_s + n^*_{rs}, \tag{2.13}$$

where n^*_{rs} is the number of common locations between the two families f_r and f_s. In (2.4) n^*_r is the number of locations from f_r those are non-overlapping with n^*_s locations belonging to f_s. Next, we decompose each group of n^*_r and n^*_s locations into 2 segments as

$$n^*_r = n^*_{r(1)} + n^*_{r(2)} \quad \text{and} \quad n^*_s = n^*_{s(1)} + n^*_{s(2)}. \tag{2.14}$$

For convenience of interpretation of further relationships among the random effects from these segments, we display the decomposition for the locations in f_r and f_s as in Fig. 1 below. Note that the decomposition of n_r^* and n_s^* locations shown by (2.14) must be done such that

1. None of the $n_{r(1)}^*$ random effects from f_r are correlated with any of the n_s^* random effects in f_s, and similarly none of the $n_{s(1)}^*$ random effects from f_s are correlated with any of the n_r^* random effects in f_r.

2. $n_{r(2)}^*$ is the number random effects in f_r that are correlated to the random effects in f_s which are not common with f_r. The number of such random effects in f_s is $n_{s(2)}^*$. Let n_{rs} denote the number of uncommon pairs of random effects from two families those are correlated.

2.1.2. Finding 3 Regions in Each Family. To be clear, one can follow the steps below to find these regions.

Step 1: For a selected location r, count all member locations around the r^{th} location satisfying

$$d_{r,r\pm j_r}^* \leq d^*,$$

where, for two positive integer values $j_{r\ell}$ and j_{ru} such that

$$j_{r\ell} \leq j_r \leq j_{ru}, \text{ with } [j_{r\ell} + j_{ru}] = n_r - 1,$$

$d_{r,r\pm j_r}^*$ is the Euclidian distance between r^{th} and $(r \pm j_r)^{\text{th}}$ neighboring location. Note that these neighboring locations do not necessarily make a linear

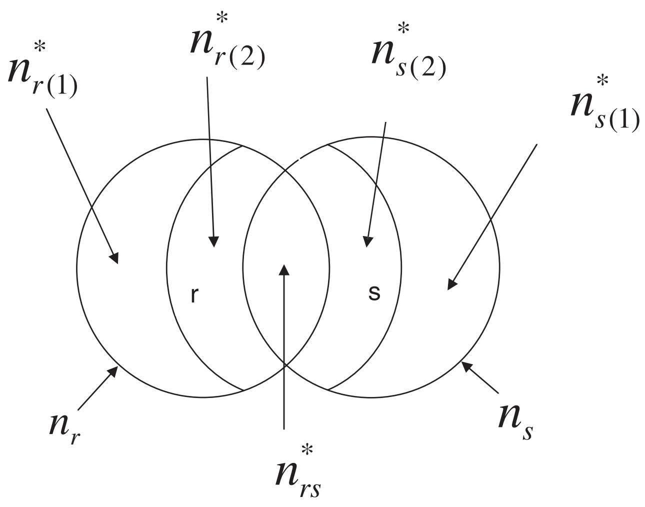

Figure 1: A general graphical display of locations belonging to two families f_r and f_s

sequence, that is, they can be anywhere around the r^{th} location satisfying the desired distance constraint. Here d^* is the same pre-specified Euclidian distance as in (2.1). Thus, one obtains

$$n_r = \#\left\{ \text{ of } \quad \text{locations} \in f_r \text{ satisfying } d^*_{r,r\pm j_r} \leq d^* \right\} \tag{2.15}$$

Step 2: Using similar notation, for a selected location $s \ \ (r \neq s)$, count n_s, that is,

$$n_s = \#\left\{ \text{ of } \quad \text{locations} \in f_s \text{ satisfying } d^*_{s,s\pm j_s} \leq d^* \right\}. \tag{2.16}$$

Step 3: Count the number of locations common between f_r and f_s. This gives n^*_{rs}.

Step 4: Find the region 1 in f_s such that the random effects in this region (not belong to f_r) are not correlated to any members in f_r except those common n^*_{rs} random effects. This gives three regions in f_s with component numbers $n^*_{s(1)}$, $n^*_{s(2)}$, and n^*_{rs}.

Step 5: Similar to Step 4, find the region 1 in f_r such that the random effects in this region (not belong to f_s) are not correlated to any members in f_s except those common n^*_{rs} random effects. This determines $n^*_{r(1)}$, and $n^*_{r(2)}$.

Note that all $n^*_{r(2)}$ random effects from location f_r and $n^*_{s(2)}$ random effects from location f_s may not be correlated. The number of such uncommon pairs of random effects from two families but those are correlated, was denoted by n_{rs}.

2.1.3. Decomposition of the Random Effects. Recall from (2.6) that the family f_r with r^{th} location as the center contains n_r locations each with a random effect. Further it follows from Fig. 1 (see also (2.13)–(2.14)) that these n_r random effects belong to 3 groups of size $n^*_{r(1)}$, $n^*_{r(2)}$, and n^*_{rs}. Similarly, the n_s random effects of the s^{th} family f_s belong to 3 groups of size $n^*_{s(1)}$, $n^*_{s(2)}$, and n^*_{rs}. One may thus express the random effect vectors γ_r in (2.6), and γ_s in (2.11), as

$$\gamma_r = (\gamma'^*_{r(1)}, \gamma'^*_{r(2)}, \gamma'^*_{rs})' : n_r \times 1 \quad \text{and} \quad \gamma_s = (\gamma'^*_{s(1)}, \gamma'^*_{s(2)}, \gamma'^*_{rs})' : n_s \times 1, \tag{2.17}$$

where, for example, the 3 component vectors of γ_r have the forms

$$\begin{aligned} \gamma'^*_{r(1)} &= (\gamma_{r1}, \cdots, \gamma_{rn^*_{r(1)}}), \gamma'^*_{r(2)} = (\gamma_{r(n^*_{r(1)}+1)}, \cdots, \gamma_{r(n^*_{r(1)}+n^*_{r(2)})})', \\ \gamma'^*_{rs} &= (\gamma_{r(n^*_{r(1)}+n^*_{r(2)}+1)}, \cdots, \gamma_{r(n^*_{r(1)}+n^*_{r(2)}+n^*_{rs})})'. \end{aligned} \tag{2.18}$$

The 3 component vectors of γ_s may be written similarly.

2.1.4. Decomposition of the Weights. Note that each of the random effects from two families f_r and f_s, that is, each component of γ_r and γ_s shown by (2.17) will carry a suitable weight and the resulting weighted sum of the components of γ_r will influence the response y_r. Similarly a suitable weighted sum of the components of γ_s will influence the response y_s. For the purpose, by matching the components of γ_r and γ_s, one may in general express the weight vectors w_r and w_s by

$$w_r = (w'_{r(1)}, w'_{r(2)}, w'_{rs})' : n_r \times 1 \text{ and } w_s = (w'_{s(1)}, w'_{s(2)}, \tilde{w}'_{rs})' : n_s \times 1, \quad (2.19)$$

respectively. Note that even though the random effect components in the third segment of γ_r and γ_s in (2.6) are the same, the corresponding weights for w_r and w_s in (2.19) are shown to be different. This is justified because the weights are constructed based on their respective families. For example, when $n_r \neq n_s$, by re-expressing the user chosen known weights w_r and w_s from (2.9) in terms of (2.19), as

$$w_r = \left(\frac{1}{\sqrt{n_r}} 1'_{n^*_{r(1)}}, \frac{1}{\sqrt{n_r}} 1'_{n^*_{r(2)}}, \frac{1}{\sqrt{n_r}} 1'_{n^*_{rs}} \right)', \quad (2.20)$$

and

$$w_s = \left(\frac{1}{\sqrt{n_s}} 1'_{n^*_{s(1)}}, \frac{1}{\sqrt{n_s}} 1'_{n^*_{s(2)}}, \frac{1}{\sqrt{n_s}} 1'_{n^*_{rs}} \right)', \quad (2.21)$$

it is clearly seen that

$$w_{rs} = \frac{1}{\sqrt{n_r}} 1_{n^*_{rs}}, \text{ and } \tilde{w}_{rs} = \frac{1}{\sqrt{n_s}} 1_{n^*_{rs}},$$

which are different, as in general, $n_r \neq n_s$. To be more clear, when the common random effects γ_{rs} will be considered as a part of the r^{th} family f_r, one would use their weights as $w_{rs} = \frac{1}{\sqrt{n_r}} 1_{n^*_{rs}}$, but if they will be treated as a part of the family f_s, one would then use $\tilde{w}_{rs} = \frac{1}{\sqrt{n_s}} 1_{n^*_{rs}}$. Similar interpretation holds for the weights of these common random effects when the weights are different such as in (2.10).

2.2. Formulation of Pair-wise Correlations. Similar to Γ_{rr} defined in (2.7), let Γ_{rs} denote the covariance matrix of γ_r and γ_s. That is,

$$\Gamma_{rs} = \text{cov}[\gamma_r, \gamma'_s] : n_r \times n_s.$$

It then follows from (2.6) and (2.11) that the covariance between two responses y_r and y_s ($r \neq s$), collected from r^{th} and s^{th} locations has the formula

$$\begin{aligned}\text{cov}[Y_r, Y_s] &= \text{cov}[(x'_r\beta + w'_r\gamma_r + \epsilon_r), (x'_s\beta + w'_s\gamma_s + \epsilon_s)] \\ &= w'_r\text{cov}[\gamma_r, \gamma'_s]w_s \\ &= w'_r\Gamma_{rs}w_s. \end{aligned} \tag{2.22}$$

Thus by (2.7) and (2.22), one writes the correlation between y_r and y_s ($r \neq s$) as

$$\text{corr}[Y_r, Y_s] = \frac{w'_r\Gamma_{rs}w_s}{[\{w'_r\Gamma_{rr}w_r + \sigma^2_\epsilon\}\{w'_s\Gamma_{ss}w_s + \sigma^2_\epsilon\}]^{\frac{1}{2}}}. \tag{2.23}$$

Note that similar to Γ_{rr} given by (2.7), the covariance matrix Γ_{rs} will also be a function of σ^2_γ and the random effects correlation index parameter ϕ. The covariance or correlation structure in (2.23) involving ϕ has, therefore, to be exploited to find any consistent estimates of the main variance component parameters σ^2_γ and σ^2_ϵ, as well as for ϕ. Further note that even though this correlation structure does not involve the regression parameter β, the use of such correlation matrix will however produce efficient estimate for β involved in the mean function $E[Y_r] = x'_r\beta$ (2.6) for all $r = 1, \ldots, S$.

2.2.1. Computation of Variances and Pair-Wise Covariances.

(a) **Computation of variances**
To compute the $\text{var}[Y_r] = w'_r\Gamma_{rr}w_r + \sigma^2_\epsilon$ given in (2.7), one needs to simplify the quadratic form $w'_r\Gamma_{rr}w_r$. Now because $\Gamma_{rr} = \sigma^2_\gamma C_{n_r n_r}(\phi)$, where $C_{n_r n_r}(\phi)$ is the $n_r \times n_r$ correlation matrix for the elements of the random effect vector γ_r which belongs to the family f_r, for w_r given by (2.20) for example, one may easily compute $w'_r\Gamma_{rr}w_r$ as

$$\begin{aligned} w'_r\Gamma_{rr}w_r &= \sigma^2_\gamma w'_r C_{n_r n_r}(\phi) w_r \\ &= \frac{\sigma^2_\gamma}{n_r}[n_r + \sum_{j \neq k}^{n_r} c_{jk}(\phi)], \end{aligned} \tag{2.24}$$

where, for $j \neq k$, $c_{jk}(\phi)$ is the $(j,k)^{\text{th}}$ element of the correlation matrix $C_{n_r n_r}(\phi)$.

If the weights are, however, unequal such as weights given by (2.10), or even say the weights in $w_{r(1)}$ are same but they are different than the weights in $w_{r(2)}$ and in w_{rs}, and so on, it is useful to have a general formula reflecting these variable weights and sizes, to compute the value of $w_r' C_{n_r n_r}(\phi) w_r$. For this general formulation, following (2.17) (see also Fig. 1), it is convenient to express γ_r as $\gamma_r = (\gamma_{r(1)}'^*, \gamma_{r(2)}'^*, \gamma_{rs}'^*)' : n_r \times 1$, along with their corresponding weights $w_r = (w_{r(1)}', w_{r(2)}', w_{rs}')'$. One may then write the general formula as

$$\begin{aligned}
w_r' \Gamma_{rr} w_r &= \sigma_\gamma^2 w_r' C_{n_r n_r}(\phi) w_r \\
&= \sigma_\gamma^2 \Big[w_{r(1)}' \{ C_{n_{r(1)}^* n_{r(1)}^*}(\phi) w_{r(1)} + C_{n_{r(1)}^* n_{r(2)}^*}(\phi) w_{r(2)} \\
&\quad + C_{n_{r(1)}^* n_{rs}^*}(\phi) w_{rs} \} + w_{r(2)}' \{ C_{n_{r(2)}^* n_{r(1)}^*}(\phi) w_{r(1)} \\
&\quad + C_{n_{r(2)}^* n_{r(2)}^*}(\phi) w_{r(2)} + C_{n_{r(2)}^* n_{rs}^*}(\phi) w_{rs} \} \\
&\quad + w_{rs}' \{ C_{n_{rs}^* n_{r(1)}^*}(\phi) w_{r(1)} + C_{n_{rs}^* n_{r(2)}^*}(\phi) w_{r(2)} \\
&\quad + C_{n_{rs}^* n_{rs}^*}(\phi) w_{rs} \} \Big] , \qquad (2.25)
\end{aligned}$$

where, for example, $C_{n_{r(1)}^* n_{r(1)}^*}(\phi)$ is the $n_{r(1)}^* \times n_{r(1)}^*$ correlation matrix among the elements of $\gamma_{r(1)}^*$, and $C_{n_{r(2)}^* n_{rs}^*}(\phi)$ is the $n_{r(2)}^* \times n_{rs}^*$ correlation matrix between the elements of $\gamma_{r(2)}^*$ and γ_{rs}^*, and so on.

(b) **Computation of pair-wise covariances**

Using similar notation as in (2.7), by (2.6) and (2.11), we first express the $\text{cov}[Y_r, Y_s]$ as

$$\text{cov}[Y_r, Y_s] = \sigma_\gamma^2 w_r' C_{n_r n_s}(\phi) w_s, \qquad (2.26)$$

where $C_{n_r n_s}(\phi) = \text{corr}[\gamma_r, \gamma_s']$ is the $n_r \times n_s$ correlation matrix between the elements of $\gamma_r = (\gamma_{r(1)}'^*, \gamma_{r(2)}'^*, \gamma_{rs}'^*)' : n_r \times 1$ and $\gamma_s = (\gamma_{s(1)}'^*, \gamma_{s(2)}'^*, \gamma_{rs}'^*)' : n_s \times 1$. By similar algebras as in (2.25), one may simplify (2.26) as

$$\begin{aligned}
w_r' \Gamma_{rs} w_s &= \sigma_\gamma^2 w_r' C_{n_r n_s}(\phi) w_s \\
&= \sigma_\gamma^2 \Big[w_{r(1)}' \{ C_{n_{r(1)}^* n_{s(1)}^*}(\phi) w_{s(1)} + C_{n_{r(1)}^* n_{s(2)}^*}(\phi) w_{s(2)} \\
&\quad + C_{n_{r(1)}^* n_{rs}^*}(\phi) \widetilde{w}_{rs} \} + w_{r(2)}' \{ C_{n_{r(2)}^* n_{s(1)}^*}(\phi) w_{s(1)} \\
&\quad + C_{n_{r(2)}^* n_{s(2)}^*}(\phi) w_{s(2)} + C_{n_{r(2)}^* n_{rs}^*}(\phi) \widetilde{w}_{rs} \} \\
&\quad + w_{rs}' \{ C_{n_{rs}^* n_{s(1)}^*}(\phi) w_{s(1)} + C_{n_{rs}^* n_{s(2)}^*}(\phi) w_{s(2)} \\
&\quad + C_{n_{rs}^* n_{rs}^*}(\phi) \widetilde{w}_{rs} \} \Big] . \qquad (2.27)
\end{aligned}$$

Note that the regions in Fig. 1 are constructed so that the random effects from the region $r(1)$ are always uncorrelated with those from the regions $s(2)$ and $s(1)$. Similarly, the random effects from the region

$s(1)$ are always uncorrelated with those from the regions $r(2)$ and $r(1)$. This leads to

$$C_{n^*_{r(1)}n^*_{s(1)}}(\phi) = 0 1_{n^*_{r(1)}} 1'_{n^*_{s(1)}},\ C_{n^*_{r(1)}n^*_{s(2)}}(\phi) = 0 1_{n^*_{r(1)}} 1'_{n^*_{s(2)}};$$

$$C_{n^*_{r(2)}n^*_{s(1)}}(\phi) = 0 1_{n^*_{r(2)}} 1'_{n^*_{s(1)}},$$

which reduces the covariance in (2.27) to

$$\begin{aligned} w'_r \Gamma_{rs} w_s &= \sigma^2_\gamma w'_r C_{n_r n_s}(\phi) w_s \\ &= \sigma^2_\gamma \Big[w'_{r(1)} \{ C_{n^*_{r(1)}n^*_{rs}}(\phi) \widetilde{w}_{rs} \} \\ &\quad + w'_{r(2)} \{ C_{n^*_{r(2)}n^*_{s(2)}}(\phi) w_{s(2)} + C_{n^*_{r(2)}n^*_{rs}}(\phi) \widetilde{w}_{rs} \} \\ &\quad + w'_{rs} \{ C_{n^*_{rs}n^*_{s(1)}}(\phi) w_{s(1)} + C_{n^*_{rs}n^*_{s(2)}}(\phi) w_{s(2)} \\ &\quad + C_{n^*_{rs}n^*_{rs}}(\phi) \widetilde{w}_{rs} \} \Big]. \end{aligned} \tag{2.28}$$

Further note that some of the correlations in the correlation matrix $C_{n^*_{r(2)}n^*_{s(2)}}(\phi)$ may be zero. To be specific, $n^*_{r(2)} n^*_{s(2)} - n_{rs}$ number of elements of the $C_{n^*_{r(2)}n^*_{s(2)}}(\phi)$ matrix would be zero. This is because, as assumed in Section 2.1.1, n_{rs} is the number of correlated pairs of random effects between the region $r(2)$ and $s(2)$.

2.2.2. Illustration of Variance-Covariance Computations.

(a) **Example 1.** Independent random effects

Suppose that

$$\widetilde{\gamma}_r \overset{iid}{\sim} (0, \sigma^2_\gamma),\ \text{for all } r = 1, \ldots, S.$$

In this case, any two random effects, irrespective of their positions in the families f_r and f_s, will be uncorrelated. Consequently, $C_{n_r n_r}(\phi = 0) = I_{n_r}$ in (2.24) or (2.25) [see also (2.7)], yielding

$$w'_r C_{n_r n_r}(\phi = 0) w_r = 1. \tag{2.29}$$

Note that to use the general formula in (2.25), one observes that in this independence setup, all elements of any rectangular size correlation matrices will be zero, but the remaining 3 square matrices will have the form

$$C_{n^*_{r(1)}n^*_{r(1)}}(\phi = 0) = I_{n^*_{r(1)}},\ C_{n^*_{r(2)}n^*_{r(2)}}(\phi = 0) = I_{n^*_{r(2)}},$$

$$C_{n^*_{rs}n^*_{rs}}(\phi = 0) = I_{n^*_{rs}}. \tag{2.30}$$

Now by using the weights from (2.20) [see also (2.9)], that is, by using

$$w_{r(1)} = \frac{1}{\sqrt{n_r}} 1_{n^*_{r(1)}}, \; w_{r(2)} = \frac{1}{\sqrt{n_r}} 1_{n^*_{r(2)}}, \text{ and } w_{rs} = \frac{1}{\sqrt{n_r}} 1_{n^*_{rs}},$$

by (2.25), one obtains

$$\begin{aligned} w_r' C_{n_r n_r}(\phi = 0) w_r &= \frac{n^*_{r(1)}}{n_r} + \frac{n^*_{r(2)}}{n_r} + \frac{n^*_{rs}}{n_r} \\ &= 1.0, \end{aligned} \tag{2.31}$$

yielding the var$[Y_r]$ in (2.7) as

$$\text{var}[Y_r] = w_r' \Gamma_{rr} w_r + \sigma^2_\epsilon = \sigma^2_\gamma + \sigma^2_\epsilon. \tag{2.32}$$

Next to simplify the covariance formula (2.28) for the present independence case, we first observe that all correlation matrices are zero except $C_{n^*_{rs} n^*_{rs}}(\phi = 0) = I_{n^*_{rs}}$. Now because the component vectors of w_s from (2.21) have the form

$$w_{s(1)} = \frac{1}{\sqrt{n_s}} 1_{n^*_{s(1)}}, \; w_{s(2)} = \frac{1}{\sqrt{n_s}} 1_{n^*_{s(2)}}, \text{ and } \widetilde{w}_{rs} = \frac{1}{\sqrt{n_s}} 1_{n^*_{rs}},$$

by using these weights and the correlation matrices from (2.30), it follows from (2.26) and (2.28) that the covariance between y_r and y_s has the formula

$$\text{cov}[Y_r, Y_s] = w_r' \Gamma_{rs} w_s = \sigma^2_\gamma \frac{1}{\sqrt{n_r}} 1'_{n^*_{rs}} I_{n^*_{rs}} \frac{1}{\sqrt{n_s}} 1_{n^*_{rs}} = \sigma^2_\gamma \frac{n^*_{rs}}{\sqrt{n_r n_s}}. \tag{2.33}$$

(b) **Example 2.** (Truncated) Equi-correlated random effects case
Suppose that the original individual random effects at the r^{th} and s^{th} locations have the truncated equi-correlation structure defined (2.5), yielding the correlation matrix $C_{n_r n_r}(\phi)$ as

$$C_{n_r n_r}(\phi) = \phi 1_{n_r} 1'_{n_r} + (1 - \phi) I_{n_r}, \tag{2.34}$$

[see also (2.8)]. Consequently, by using the weights from (2.20), one obtains the variance by (2.7) as

$$\begin{aligned} \text{var}[Y_r] &= \sigma^2_\gamma w_r' \left[\phi 1_{n_r} 1'_{n_r} + (1 - \phi) I_{n_r}\right] w_r + \sigma^2_\epsilon \\ &= \frac{1}{n_r} \left[\phi 1'_{n_r} 1_{n_r} 1'_{n_r} 1_{n_r} + (1 - \phi) 1'_{n_r} 1_{n_r}\right] + \sigma^2_\epsilon \\ &= \sigma^2_\gamma \left[1 + (n_r - 1)\phi\right] + \sigma^2_\epsilon. \end{aligned} \tag{2.35}$$

Note that to compute this variance by using the general weights based formula (2.25), one first writes the component correlation matrices as

$$\begin{aligned}
C_{n^*_{r(1)}n^*_{r(1)}}(\phi) &= \left[\phi 1_{n^*_{r(1)}} 1'_{n^*_{r(1)}} + (1-\phi) I_{n^*_{r(1)}}\right], \\
C_{n^*_{r(1)}n^*_{r(2)}}(\phi) &= \phi 1_{n^*_{r(1)}} 1'_{n^*_{r(2)}}, \\
C_{n^*_{r(1)}n^*_{rs}}(\phi) &= \phi 1_{n^*_{r(1)}} 1'_{n^*_{rs}}; \\
C_{n^*_{r(2)}n^*_{r(1)}}(\phi) &= \phi 1_{n^*_{r(2)}} 1'_{n^*_{r(1)}}, \\
C_{n^*_{r(2)}n^*_{r(2)}}(\phi) &= \left[\phi 1_{n^*_{r(2)}} 1'_{n^*_{r(2)}} + (1-\phi) I_{n^*_{r(2)}}\right], \\
C_{n^*_{r(2)}n^*_{rs}}(\phi) &= \phi 1_{n^*_{r(2)}} 1'_{n^*_{rs}}; \\
C_{n^*_{rs}n^*_{r(1)}}(\phi) &= \phi 1_{n^*_{rs}} 1'_{n^*_{r(1)}},\ C_{n^*_{rs}n^*_{r(2)}}(\phi) = \phi 1_{n^*_{rs}} 1'_{n^*_{r(2)}}, \\
C_{n^*_{rs}n^*_{rs}}(\phi) &= \left[\phi 1_{n^*_{rs}} 1'_{n^*_{rs}} + (1-\phi) I_{n^*_{rs}}\right]. \qquad (2.36)
\end{aligned}$$

Next by using these matrices (2.36), as well as the weights $w_{r(1)}$, $w_{r(2)}$, and w_{rs}, from (2.20), the variance of y_r (2.7) is computed by (2.25) as

$$\begin{aligned}
\text{var}[Y_r] &= \frac{\sigma^2_\gamma}{n_r}\left[n^*_{r(1)}(1-\phi) + n^*_{r(1)}\{n^*_{r(1)} + n^*_{r(2)} + n^*_{rs}\}\phi\right] \\
&+ \frac{\sigma^2_\gamma}{n_r}\left[n^*_{r(2)}(1-\phi) + n^*_{r(2)}\{n^*_{r(1)} + n^*_{r(2)} + n^*_{rs}\}\phi\right] \\
&+ \frac{\sigma^2_\gamma}{n_r}\left[n^*_{rs}(1-\phi) + n^*_{rs}\{n^*_{r(1)} + n^*_{r(2)} + n^*_{rs}\}\phi\right] \\
&+ \sigma^2_\epsilon, \qquad (2.37)
\end{aligned}$$

which reduces to (2.35) because $n_r = n^*_{r(1)} + n^*_{r(2)} + n^*_{rs}$.

Remark that the formula in (2.35) is much easier to derive than (2.37), but this demonstration by (2.37) should be helpful to compute the variances in a complex case where weights will be variable such as in (2.10). By this token, because the computation of $\text{cov}[Y_r, Y_s]$ depends on two different weight vectors given by (2.20) and (2.21), it would be appropriate to use the general weights based covariance formula (2.28). Now to use this formula we first notice that the formulas for all these correlation matrices except for one, namely $C_{n^*_{r(2)}n^*_{s(2)}}(\phi)$, are already given in (2.36). Further notice that the total number of elements in this special matrix is $n^*_{r(2)}n^*_{s(2)}$, but some of these elements may be zero. To be more specific, as mentioned in (2.28), there will be $n^*_{r(2)}n^*_{s(2)} - n_{rs}$

zero elements in this matrix, where n_{rs} is the number of correlated pairs between the random effects from regions $r(2)$ and $s(2)$. Thus, by (2.20) and (2.21), we write

$$\begin{aligned} w'_{r(2)}C_{n^*_{r(2)}n^*_{s(2)}}(\phi)w_{s(2)} &= \frac{1}{\sqrt{n_r n_s}}1'_{n^*_{r(2)}}C_{n^*_{r(2)}n^*_{s(2)}}(\phi)1_{n^*_{s(2)}} \\ &= \phi\frac{n_{rs}}{\sqrt{n_r n_s}}. \end{aligned} \tag{2.38}$$

Now by using (2.38) and (2.36), and further weights $w_{r(1)}$, w_{rs} from (2.20) and $w_{s(1)}$, $\widetilde{w}_{rs}$ from (2.21), into (2.28), one obtains the covariance between y_r and y_s by (2.26) as

$$\begin{aligned} \mathrm{cov}[Y_r, Y_s] &= \frac{\sigma^2_\gamma}{\sqrt{n_r n_s}}\Big[\phi 1'_{n^*_{r(1)}}1_{n^*_{r(1)}}1'_{n^*_{rs}}1_{n^*_{rs}} + \phi n_{rs} \\ &\quad +\phi 1'_{n^*_{r(2)}}1_{n^*_{r(2)}}1'_{n^*_{rs}}1_{n^*_{rs}} \\ &\quad +\phi 1'_{n^*_{rs}}1_{n^*_{rs}}1'_{n^*_{s(1)}}1_{n^*_{s(1)}} + \phi 1'_{n^*_{rs}}1_{n^*_{rs}}1'_{n^*_{s(2)}}1_{n^*_{s(2)}} \\ &\quad + 1'_{n^*_{rs}}\left\{\phi 1_{n^*_{rs}}1'_{n^*_{rs}} + (1-\phi)I_{n^*_{rs}}\right\}1_{n^*_{rs}}\Big] \\ &= \frac{\sigma^2_\gamma}{\sqrt{n_r n_s}}\Big[\phi\{n^*_{r(1)}n^*_{rs} + n_{rs} + n^*_{r(2)}n^*_{rs} + n^*_{rs}n^*_{s(1)} \\ &\quad +n^*_{rs}n^*_{s(2)}\} + \{\phi n^*_{rs}(n^*_{rs}-1) + n^*_{rs}\}] \\ &= \frac{\sigma^2_\gamma}{\sqrt{n_r n_s}}\Big[\phi\left\{n_{rs}+n^*_{rs}\left((n^*_{r(1)}+n^*_{r(2)}) + (n^*_{s(1)}+n^*_{s(2)})\right)\right\} \\ &\quad + \{\phi n^*_{rs}(n^*_{rs}-1) + n^*_{rs}\}] \\ &= \frac{\sigma^2_\gamma}{\sqrt{n_r n_s}}[\phi\{n_{rs} + n^*_{rs}(n^*_r + n^*_s)\} \\ &\quad + \{\phi n^*_{rs}(n^*_{rs}-1) + n^*_{rs}\}]. \end{aligned} \tag{2.39}$$

Notice that when $\phi = 0$, this covariance (2.39) reduces to (2.33) under the independent random effects, the responses are still being correlated.

2.2.3. Illustration for Construction of Pair-Wise Families. Note that for the computation of the correlations of the pairwise responses by (2.33) for the independent random effects case, and by (2.39) for the random effects with truncated equi-correlation structure, require the knowledge of the size for 3 regions in each of the two families involved (see Section 2.1.2), that is the values of

$$n_r, n^*_{r(1)}, \ n^*_{r(2)}, \ \ n_s, \ n^*_{s(1)}, \ n^*_{s(2)}, \ n^*_{rs}, \text{ and } n_{rs}.$$

These are in fact the design based data. They may, however, be easily obtained by following the general construction rule described in Section 2.1.1. We illustrate, for example, obtaining them for a linear sequence scheme displayed in Fig. 2, where locations are at equi-distant on a line. In this example, both f_r and f_s are constructed following the distance criterion in (2.1) with $d^* = 4$. Suppose the linear sequence consists of a spatial region with S locations. In the simulation study in Section 4, we will consider $S = 500$. Notice that, except for two extreme locations in the beginning and two extreme locations in the end of the sequence, the size of the family f_r for other locations would be $n_r = 5$. Thus, for all S locations, n_r takes the values as

$$n_r = \begin{cases} 3 & \text{for} \quad r = 1 \text{ and } S, \\ 4 & \text{for} \quad r = 2 \text{ and } S - 1, \\ 5 & \text{for all} \quad r = 3, \cdots, S - 2. \end{cases} \tag{2.40}$$

Next for pairwise locations (r, s) for $r \neq s$, $r, s = 1, 2, \cdots, S$, we follow the pair-wise family construction rule given in Section 2.1.1, and provide the component numbers within and between families as follows. More specifically, the values for n^*_{rs}, the common number of locations (see Fig. 1) between 2 families for all $r \neq s$ are given in Table 1. The values for other two components of each family, i.e., for $n^*_{r(1)}$ and $n^*_{r(2)}$ within f_r, and the values for $n^*_{s(1)}$ and $n^*_{s(2)}$ within f_s, are given in Table 2. For convenience, we also computed the values for n_{rs} which represents the number of distinct pairs of locations, where a pair is formed between two locations, one from the region $r(2)$ and the other from $s(2)$. Thus, as pointed out earlier, n_{rs} essentially

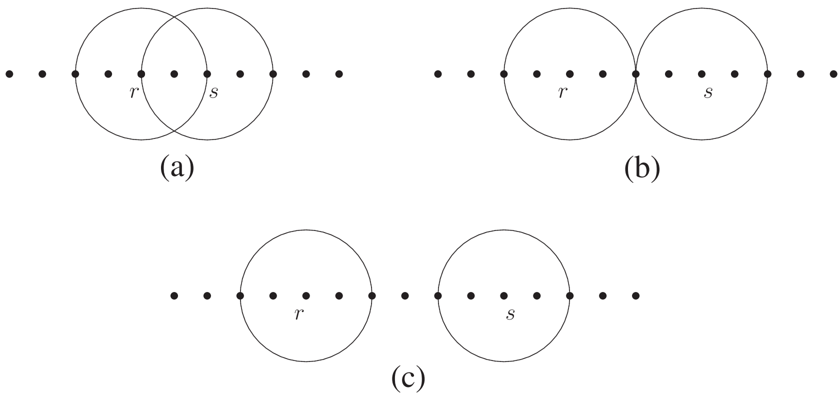

Figure 2: Illustrations for locations in a linear sequence

Table 1: Values of n^*_{rs} for Fig. 1 following the pattern of Fig. 2

$\|s-r\|$	$r=1,$ $s=2,\cdots,S$	$r=2,$ $s=3,\cdots,S$	$r=3,\cdots,S-2,$ $s=4,\cdots,S-1$	$r=1,\cdots,S,$ $s=S$
1	3	4	4	3
2	3	3	3	3
3	2	2	2	2
4	1	1	1	1
≥ 5	0	0	0	0

gives the number of pairs of correlated random effects between these two regions. These values of n_{rs} are also given in the same Table 2.

3. Parameter Estimation

Note that the main objective of the present study is to develop a general spatial correlation model which we have done in the last section. We now exploit the proposed correlation structure and demonstrate how to estimate both regression and scale (variance and correlation) parameters of the model. For the purpose, we re-express the variance and covariances of the pair-wise responses as follows.

When random effects are independent, it follows from Section 2.1.1(a), the variance and covariances of the pairwise responses involve the scale parameters σ^2_γ and σ^2_ϵ. One may then reexpress these variance and covariances from (2.32)–(2.33) as

$$\mathrm{cov}[Y_r, Y_s] = \begin{cases} \sigma_{rr}(\sigma^2_\gamma, \sigma^2_\epsilon) & \text{for } r = s,\ r = 1, \ldots, S \\ \sigma_{rs}(\{\sigma^2_\gamma\} \mid n_r, n_s, n^*_{rs}) & \text{for } r \neq s. \end{cases} \tag{3.1}$$

However, when the random effects are equi-correlated within prescribed distance d^* with equi-correlation coefficient ϕ, we re-express the variance and covariances of the pair-wise responses, from (2.35) and (2.39), respectively, as

$$\mathrm{cov}[Y_r, Y_s] = \begin{cases} \sigma_{rr}(\{\sigma^2_\gamma, \sigma^2_\epsilon, \phi\} \mid n_r) & \text{for } r = s, \\ & \quad r = 1, \ldots, S \\ \sigma_{rs}(\{\sigma^2_\gamma, \phi\} \mid n^*_{r(1)}, n^*_{r(2)}, n^*_{rs}, n^*_{s(1)}, n^*_{s(2)}, n_{rs}) & \text{for } r \neq s. \end{cases} \tag{3.2}$$

In this section, we deal with the estimation of the regression parameters β involved in the mean function $E[Y_r] = x'_r\beta$ (2.7), under the wider covariance setup (3.2). When $\phi = 0$, (3.2) reduces to the covariance setup in (3.1).

Table 2: Values of $n^*_{r(1)}$, $n^*_{r(2)}$, $n^*_{s(1)}$, $n^*_{s(2)}$ and n_{rs} for Fig. 1 following the pattern of Fig. 2

$\vert s-r\vert$	$r=1, s=2,\cdots,S$					$r=2, s=3,\cdots,S$					$r=3,\cdots,S-3, s=4,\cdots,S-2$				
	$n^*_{r(1)}$	$n^*_{r(2)}$	$n^*_{s(1)}$	$n^*_{s(2)}$	n_{rs}	$n^*_{r(1)}$	$n^*_{r(2)}$	$n^*_{s(1)}$	$n^*_{s(2)}$	n_{rs}	$n^*_{r(1)}$	$n^*_{r(2)}$	$n^*_{s(1)}$	$n^*_{s(2)}$	n_{rs}
1	0	0	0	1	0	0	0	0	1	0	1	0	1	0	0
2	0	0	2	0	0	0	1	1	1	1	1	1	1	1	1
3	0	1	1	2	2	0	2	1	2	3	1	2	1	2	3
4	0	2	1	3	5	0	3	1	3	6	1	3	1	3	6
5	0	3	1	4	9	0	4	1	4	10	1	4	1	4	10
6	0	3	2	3	6	1	3	3	2	6	2	3	2	3	6
7	1	2	3	2	3	2	2	3	2	3	3	2	3	2	3
8	2	1	4	1	1	3	1	4	1	1	4	1	4	1	1
≥ 9	3	0	5	0	0	4	0	5	0	0	5	0	5	0	0

$\vert s-r\vert$	$r=1,\cdots,S-2, s=S-1$					$r=1,\cdots,S-1, s=S$				
	$n^*_{r(1)}$	$n^*_{r(2)}$	$n^*_{s(1)}$	$n^*_{s(2)}$	n_{rs}	$n^*_{r(1)}$	$n^*_{r(2)}$	$n^*_{s(1)}$	$n^*_{s(2)}$	n_{rs}
1	0	1	0	0	0	0	1	0	0	0
2	1	1	0	1	1	2	0	0	0	0
3	1	2	0	2	3	1	2	0	1	2
4	1	3	0	3	6	2	2	0	2	5
5	1	4	0	4	10	1	4	0	3	9
6	3	2	1	3	6	2	3	0	3	6
7	3	2	2	2	3	3	2	1	2	3
8	4	1	3	1	1	4	1	2	1	1
≥ 9	5	0	4	0	0	5	0	3	0	0

3.1. A Moment-GLS Hybrid Technique.

3.1.1. GLS Estimation for β. Let $y = (y_1, \cdots, y_r, \cdots, y_S)'$ be the $S \times 1$ vector of response for all S locations. Combining the expectations $E[Y_r] = x_r'\beta$ from (2.7) for all $r = 1, \ldots, S$, one writes $E[Y] = X\beta$, where $X = (x_1, \cdots, x_r, \cdots, x_S)'$ is the $S \times p$ covariate matrix with $x_r = (x_{r1}, \cdots, x_{rp})'$. Notice from (3.2) that the elements of the $S \times S$ covariance matrix of the response vector y are functions of the scale parameters $\sigma^2_\gamma, \sigma^2_\epsilon, \phi$. By writing

$$\text{cov}(Y) = (\sigma_{rs}) = \Sigma(\{\sigma^2_\gamma, \sigma^2_\epsilon, \phi)\}|\{n^*_{r(1)}, n^*_{r(2)}, n^*_{rs}, n^*_{s(1)}, n^*_{s(2)}, n_{rs}\}, \quad (3.3)$$

in the proposed moment-GLS (generalized least squared) hybrid technique, for known values of the scale parameters $\sigma^2_\gamma, \sigma^2_\epsilon, \phi$, we obtain $\hat{\beta}_{GLS}$, the GLS estimate of β such that

$$\min_\beta \left[(Y - X\beta)'\Sigma^{-1}(Y - X\beta)\right] = \left[(Y - X\hat{\beta}_{GLS})'\Sigma^{-1}(Y - X\hat{\beta}_{GLS})\right],$$

yielding

$$\hat{\beta}_{GLS} = \left[X'\Sigma^{-1}X\right]^{-1}\left[X'\Sigma^{-1}Y\right]. \quad (3.4)$$

Note that the GLS estimator of β given by (3.4) is consistent for β. In fact, an OLS (ordinary least square) estimator constructed based on $\phi = 0$ in the covariance matrix Σ will also be consistent for β. However, it is understandable that such an OLS estimator will be less efficient than the GLS estimator. However, as in the present spatial setup, it is of main interest to estimate the variance parameters to understand the variation in the data, any estimators such as moment estimators of variance components σ^2_γ and σ^2_ϵ will be biased if ϕ is treated to be zero when in fact ϕ is a non-zero correlation index parameter. For this reason, in the next section, we demonstrate how to develop moment estimators for the variance and correlation index parameters which will be consistent for the respective parameters.

3.1.2. MM Estimation for the Variance and Correlation Parameters. Turning back to the GLS estimation of β by (3.4), it was assumed that Σ matrix is known. However as Σ is unknown, that is, the scale parameters $\sigma^2_\gamma, \sigma^2_\epsilon, \phi$, are unknown, we estimate them by using the well known moment approach. For the purpose, one first constructs three moment functions

$$\psi_j(y_1, \ldots, y_r, \ldots, y_S; \sigma^2_\gamma, \sigma^2_\epsilon, \phi, \beta)$$

so that for known β,

$$E\left[\psi_j(y_1, \ldots, y_r, \ldots, y_S; \sigma^2_\gamma, \sigma^2_\epsilon, \phi, \beta)\right] = 0, \text{ for all } j = 1, 2, 3, \quad (3.5)$$

and then solve all three moment equations

$$\psi_j(y_1,\ldots,y_r,\ldots,y_S;\sigma^2_\gamma,\sigma^2_\epsilon,\phi,\beta)=0, \tag{3.6}$$

for $\sigma^2_\gamma, \sigma^2_\epsilon$, and ϕ.

Note however that one of the major problems with the moment approach for the estimation of the scale and correlation parameters is that finding suitable unbiased moment functions for a large number of parameters may be challenging. For this reason, in the next section, we discuss the well known likelihood approach under the assumption that the errors and random effects in the linear mixed model follow the Gaussian distribution (Kang et al., 2010), where the construction of the likelihood estimating equations depends on the covariance or correlation structure only. In the present setup, even though the construction of the spatial correlation structure of the responses has been a challenge, we have however tackled this problem by constructing the moving band type spatial correlation structures as shown in (2.26)–(2.27).

By exploiting the fact that in the present familial spatial setup, the responses from the neighboring locations are likely to be highly correlated, whereas the responses from two far distant locations will be uncorrelated, we now use responses from locations with spatial lags 0, 1 and 2, to construct three unbiased moment functions satisfying (3.5). These three moment functions are:

$$\begin{aligned}
\psi_1(\cdot) &= \frac{1}{S}\sum_{r=1}^{S}(y_r-\mu_r)^2-\lambda_1\\
\psi_2(\cdot) &= S(S-1)^{-1}\left(\sum_{r=1}^{S-1}(y_r-\mu_r)(y_{r+1}-\mu_{r+1})\right)\left(\sum_{r=1}^{S}(y_r-\mu_r)^2\right)^{-1}\\
&\quad -\lambda_2\\
\psi_3(\cdot) &= S(S-2)^{-1}\left(\sum_{r=1}^{S-2}(y_r-\mu_r)(y_{r+2}-\mu_{r+2})\right)\left(\sum_{r=1}^{S}(y_r-\mu_r)^2\right)^{-1}\\
&\quad -\lambda_3,
\end{aligned} \tag{3.7}$$

where μ_r for all $r=1,2,\cdots,S$ are functions of β, because $\mu_r=x_r'\beta$, and where

$$\lambda_1=\sigma^2_\gamma+\sigma^2_\epsilon+\frac{\phi\sigma^2_\gamma}{S}\left(\sum_{r=1}^{S}n_r-S\right),\ \lambda_2=\sigma^2_\gamma\frac{\lambda_{21}}{\lambda_1},\ \lambda_3=\sigma^2_\gamma\frac{\lambda_{31}}{\lambda_{31}}, \tag{3.8}$$

with

$$\lambda_{21} = \sum_{r=1}^{S-1} \frac{1}{\sqrt{n_r n_{r+1}}} \Big[n^*_{r(1)} n^*_{r,r+1}\phi + n^*_{r+1(2)}(n^*_{r(1)} + n^*_{r,r+1})\phi + n^*_{r,r+1}\Big[1 \,+\, (n^*_{r+1(1)} \,+\, n^*_{r+1(2)} \,+\, (n^*_{r,r+1} - 1))\phi\Big]\Big],$$

and

$$\lambda_{31} = \sum_{r=1}^{S-2} \frac{1}{\sqrt{n_r n_{r+2}}} \Big[n^*_{r(1)} n^*_{r,r+2}\phi + n^*_{r+2(2)}(n^*_{r(1)} + n^*_{r,r+2})\phi + n^*_{r,r+2}\Big[1 \,+\, (n^*_{r+2(1)} \,+\, n^*_{r+2(2)} \,+\, (n^*_{r,r+2} - 1))\phi\Big]\Big].$$

One may then solve the moment estimating equations (3.6) to obtain an estimate for each of σ^2_γ, σ^2_ϵ and ϕ. These estimates are then used in Σ in (3.4) to obtain an improved estimate for β, which in turn is used in (3.6) to obtain improved estimates for σ^2_γ, σ^2_ϵ and ϕ. This constitute a cycle, and the cycles of iteration continues until convergence. These moment estimators are denoted by $\hat{\sigma}^2_{\gamma,MM}$, $\hat{\sigma}^2_{\epsilon,MM}$, $\hat{\phi}_{MM}$, whereas, in (3.4), the GLS estimator of β was denoted by $\hat{\beta}_{GLS}$.

3.2. Likelihood Estimation. Note that even though the moment-GLS hybrid technique discussed in the last section provides consistent estimators for the scale parameters, it may not be easy to construct unbiased moment functions specially when the number of correlation parameters is large. For example, in some situations one may find it more appropriate to use truncated AR(2) type correlations with two parameters ϕ_1 and ϕ_2, instead of truncated equi-correlation structure with correlation ϕ for the random effects. This will require to use four unbiased moment functions which may not be easy to construct. Furthermore, even though the MMs produces unbiased and consistent estimates for the scale parameters, these estimators, as expected, can however be inefficient, mainly because of the fact that the moment estimating equations are not developed by accommodating the variances of the unbiased functions. When such variances are accommodated, the MM approach becomes the generalized MM (GMM) approach (Hansen, 1982), but the calculations for the variances (involving fourth or higher order moments) of any second order unbiased functions will require some 'working' distribution assumption, such as 'working' normal errors. See, for example, Rao, Sutradhar, and Pandit (2012), for the construction of a 'working' distribution based GMM estimation approach, in the panel data setup. However, as the normality assumption for the random effects and model errors is quite reasonable, see for example, Kang et al. (2010, Eqs. (7), (15)), in this section we concentrate on normality assumption based likelihood estimation, rather than developing any GMM techniques. The resulting estimators will be consistent and highly efficient.

Under the normality assumption, we write

$$Y \sim N_S(X\beta, \Sigma), \text{ where } \Sigma \equiv \Sigma(\sigma^2_\gamma, \sigma^2_\epsilon, \phi) = \text{cov}[Y] = (\sigma_{rs}),$$

by (3.3). Then, for $\xi = (\xi_1, \xi_2, \xi_3)' \equiv (\sigma^2_\gamma, \sigma^2_\epsilon, \phi)'$, the log likelihood function is given by

$$l(\beta, \psi \mid y) = -\frac{S}{2} ln(2\pi) - \frac{1}{2} ln|\Sigma| - \frac{1}{2}(y - X\beta)'\Sigma^{-1}(y - X\beta), \quad (3.9)$$

yielding the likelihood estimating equations for β and $\xi_i, (i = 1, 2, 3)$ as

$$\begin{aligned} \frac{\partial l}{\partial \beta} &= X'\Sigma^{-1}Y - X'\Sigma^{-1}X\beta = 0 \\ \frac{\partial l}{\partial \xi_i} &= -\frac{1}{2} tr\left[\Sigma^{-1}\Sigma_{(i)}\right] - \frac{1}{2}(Y - X\beta)'\Sigma^{(i)}(Y - X\beta) = 0, \quad (3.10) \end{aligned}$$

where

$$\Sigma_{(i)} = \frac{\partial \Sigma}{\partial \xi_i}, \; \Sigma^{(i)} = \frac{\partial \Sigma^{-1}}{\partial \xi_i} = -\Sigma^{-1}\Sigma_{(i)}\Sigma^{-1}.$$

It then follows that for known ξ, the maximum likelihood (ML) estimator of β is given by

$$\hat{\beta}_{ML} = \left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}y,$$

which is the same as the GLS estimator shown in Section 3.1. However, for known β, the ML estimate of ξ would be different than the MM estimate found in the last section. The ML estimate of ξ can be obtained by using the Fisher scoring iterative algorithm. More specifically, given the value of $\hat{\xi}_{ML}(t)$ at the t^{th} iteration, $\hat{\xi}_{ML}(t+1)$ is obtained by solving

$$\hat{\xi}_{ML}(t+1) = \hat{\xi}_{ML}(t) + \left[B_\xi^{-1}\left(\frac{\partial l}{\partial \xi}\right)\right]_{(t)}, \quad (3.11)$$

where $[.]_{(t)}$ denotes that the expression within brackets is evaluated at $\hat{\xi}_{ML}(t)$, $\frac{\partial l}{\partial \xi}$ is evaluated from (3.10) and B_ξ^{-1} is the inverse of the Fisher information matrix B_ξ. This Fisher information matrix has the formula

$$\begin{aligned} B_\xi &= -E\left(l^{(2)}(\beta,\xi \mid y)\right) \\ &= -E\left[\frac{\partial l}{\partial\xi\partial\xi'}\right] = \left(\frac{1}{2}tr[\Sigma^{-1}\Sigma_{(i)}\Sigma^{-1}\Sigma_{(j)}]\right) \\ &= (t_{ij}),\ i,j = 1,2,3, \end{aligned} \tag{3.12}$$

because

$$\frac{\partial l}{\partial\xi_i\partial\xi_j} = -\frac{1}{2}\left(tr\left[\Sigma^{-1}\Sigma_{(ij)} + \Sigma^{(j)}\Sigma_{(i)}\right] + (y - X\beta)'\Sigma^{(ij)}(y - X\beta)\right), \tag{3.13}$$

with

$$\begin{aligned} \Sigma_{(ij)} &= \frac{\partial\Sigma_{(i)}}{\partial\xi_j} \\ \Sigma^{(ij)} &= \frac{\partial\Sigma^{(i)}}{\partial\xi_j} = \Sigma^{-1}\left[\Sigma_{(j)}\Sigma^{-1}\Sigma_{(i)} + \Sigma_{(i)}\Sigma^{-1}\Sigma_{(j)} - \Sigma_{(ij)}\right]\Sigma^{-1}, \end{aligned}$$

yielding $-E\left[\frac{\partial l}{\partial\xi_i\partial\xi_j}\right] = t_{ij} = \frac{1}{2}tr[\Sigma^{-1}\Sigma_{(i)}\Sigma^{-1}\Sigma_{(j)}]$. The final ML estimates for β and ξ are obtained by using cycles of iteration until convergence.

4 An Empirical Illustration Through Simulations

Recall that $r \in \mathcal{S}$ denotes a location of events belonging to a spatial region $\mathcal{S}$. Also recall from (2.6) that y_r is the associated measurement from the r^{th} spatial location given by

$$\begin{aligned} y_r &= x_r'\beta + w_r'\gamma_r + \epsilon_r \\ &\equiv u_r'\alpha + z_r'\theta + w_r'\gamma_r + \epsilon_r, \end{aligned} \tag{4.1}$$

where $u_r = (u_{r1}, \cdots, u_{rp_1})'$ is a p_1-dimensional fixed covariate vector containing for example, the epidemiological or demographic information from the r^{th} location, and $z_r = (z_{r1}, \cdots, z_{rp_2})'$ is a p_2-dimensional deterministic (or location dependent) vector of covariate containing the environmental information from the r^{th} location. Here α and θ are the fixed regression effects of u_r and z_r on y_r, respectively, that is $\beta = (\alpha', \theta')'$ is the effect of $x_r' = (u_r', z_r')$ on y_r. Also in (4.1) the components of γ_r vector are random effects of n_r locations belonging to the r^{th} family, f_r. Furthermore, as mentioned before ϵ_r are model errors and we assume that $\epsilon_r \overset{iid}{\sim} N(0, \sigma_\epsilon^2)$.

4.1. Simulation Design.

4.1.1. Selection of Fixed Covariates. In this simulation study, we choose $S = 500$ locations. With regard to the fixed covariates, we choose $p_1 = 3$ and the associated covariates $u_r = (u_{r1}, u_{r2}, u_{r3})'$ as follows:

1. Intercept covariate:
$$u_{r1} = 1, \quad \text{for} \quad r = 1, 2, \cdots, S$$

2. Fixed epidemiological binary covariate (such as old or new spatial location)
$$u_{r2} = \begin{cases} 1 & \text{if } r \text{ is in old category,} \\ 0 & \text{if } r \text{ is in new category,} \end{cases}$$
and

3. Another epidemiological covariate (Geographical, say)
$$u_{r3} = \begin{cases} 0 & \text{if } 1 \leq r \leq S/8, \quad \text{(locations are on high ground, for example),} \\ 1 & \text{if } S/8 + 1 \leq r \leq 3S/4, \quad \text{(on plane ground),} \\ 0 & \text{if } 3S/4 + 1 \leq r \leq S, \quad \text{(on high ground).} \end{cases}$$

For environmental type covariate z_r such as to understand the wind effects due to relative positions, we choose two sets of categorical variables each with three categories which may be represented by two categorical variables. We thus choose $p_2 = 4$. To be specific to accommodate for example, the winds from backward (or left) side of a location covering 180^0 we consider the first set of categorical variables represented by two dummy variables (z_{r1}, z_{r2}) defined as:
$$(z_{r1}, z_{r2}) = \begin{cases} (1, 0) & \text{if } 135^0 < \omega < 225^0, \\ (0, 1) & \text{if } 90^0 < \omega < 135^0, \\ (0, 0) & \text{if } 225^0 < \omega < 270^0, \end{cases}$$
where, ω is the angle between r^{th} and its neighboring (backward) locations of events.

Similarly to accommodate for example, the winds from forward (or right) side of a location covering 180^0 we consider the second set of categorical variables represented by two other dummy variables (z_{r3}, z_{r4}) defined as:
$$(z_{r3}, z_{r4}) = \begin{cases} (1, 0) & \text{if } 315^0 < \omega < 360^0, \quad \& \quad 0 < \psi < 45^0 \\ (0, 1) & \text{if } 45^0 < \omega < 90^0, \\ (0, 0) & \text{if } 270^0 < \omega < 315^0, \end{cases}$$

for which, ω is the angle between r^{th} and its neighboring (forward) locations of events.

As far as the parameter values are concerned, for the fixed regression effects, we chose $\alpha = (\alpha_1, \alpha_2, \alpha_3)'$ and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)'$, that is,

$$\beta = (\alpha', \theta')' \equiv (0.3,\ 0.5, -0.5,\ 0.6,\ 0.4,\ 0.5,\ 0.2)' \tag{4.2}$$

Note that, we have chosen these components of β from some practical point of views. For example, $\alpha_2 = 0.5$ indicates positive effects of older or aged plants on the yields.

4.1.2. Selection of Variance Components. We assume that $\epsilon_r \overset{iid}{\sim} N(0, \sigma_\epsilon^2)$ and the random effects marginally follow the normal distribution given by $\widetilde{\gamma}_r \sim N(0, \sigma_\gamma^2)$. In the simulation study, we consider for example two sets of values for the variance components σ_ϵ^2 (model error variance) and σ_γ^2 (location based random effects variance):

Independent case : $\phi = 0$

$$\sigma_\epsilon^2 \equiv 1.0,\ \text{and}\ \sigma_\gamma^2 \equiv (0.25,\ 0.75,\ 1.0,\ 2.0); \tag{4.3}$$

Correlation case : $\phi \neq 0$

$$\sigma_\epsilon^2 \equiv (0.75,\ 1.0),\ \text{and}\ \sigma_\gamma^2 \equiv (0.75,\ 1.0,\ 1.5,\ 2.0). \tag{4.4}$$

4.1.3. Generation of Random Effects.

(a) **Generation of independent** ($\phi = 0$) **random effects: Study 1**
As mentioned in the last section, in this case, random effects will be generated following $\widetilde{\gamma}_r \overset{iid}{\sim} N(0, \sigma_\gamma^2)$ with $\sigma_\gamma^2 \equiv (0.25,\ 0.75,\ 1.0,\ 2.0)$.

(b) **Generation of (truncated) equi-correlated** ($\phi \neq 0$) **random effects: Study 2**
Under this scheme, we generate $S = 500$ original random effects $\widetilde{\gamma}_r (r = 1, \ldots, 500)$ (before grouping them into families) such that marginally

$$\widetilde{\gamma}_r \sim N(0, \sigma_\gamma^2),$$

and pair-wise they follow the truncated equi-correlation structure as in (2.5), that is,

$$\text{corr}(\widetilde{\gamma}_r, \widetilde{\gamma}_s) = \begin{cases} 1 & \text{for}\ \ d_{rs}^* = 0 \\ \phi & \text{for}\ \ d_{rs}^* \leq d^* \\ 0 & \text{for}\ \ d_{rs}^* > d^*, \end{cases}$$

which generates a band correlation matrix with pairwise correlation ϕ within the band, where the band width is determined by the spatial distance (lag) d^*. We choose $d^* = 4$ reflecting Fig. 2. For correlation parameter, we choose, for example, $\phi = 0.3$.

To be specific, for $S = 500$ random effects to satisfy the aforementioned marginal and truncated equi-correlation structure, we generate them, that is, $\widetilde{\gamma}_r (r = 1, \ldots, 500)$, in a sequence as follows:

$$\begin{aligned}
\widetilde{\gamma}_1 &\sim N\left(0, \sigma_\gamma^2\right) \\
\widetilde{\gamma}_2 \mid \widetilde{\gamma}_1 &\sim N\left(\phi\widetilde{\gamma}_1, \sigma_\gamma^2(1-\phi^2)\right) \\
\vdots \quad & \vdots \quad \vdots \\
\widetilde{\gamma}_r \mid \widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_{r-1} &\sim N\left[\Lambda_{21}^{(r)}\left(\Lambda_{11}^{(r)}\right)^{-1}\gamma^*_{(r-1)},\ \ \Lambda_{22}^{(r)} - \Lambda_{21}^{(r)}\left(\Lambda_{11}^{(r)}\right)^{-1}\Lambda_{12}^{(r)}\right],
\end{aligned} \tag{4.5}$$

where $\gamma^*_{(r-1)} = (\tilde{\gamma}_1, \cdots, \tilde{\gamma}_{r-1})'$. In (58),

$$\begin{aligned}
\Lambda_{11}^{(r)} &= 1_{r-1}1'_{r-1}\phi\sigma_\gamma^2 + \sigma_\gamma^2(1-\phi)I_{r-1}, \\
\Lambda_{21}^{(r)} &= \phi\sigma_\gamma^2 1'_{r-1},\ \ \Lambda_{22}^{(r)} = \sigma_\gamma^2,\ \ \Lambda_{12}^{(r)} = \left(\Lambda_{21}^{(r)}\right)',\ \text{for } r = 2, \ldots, 5,
\end{aligned}$$

whereas for $r = 6, \cdots, 500$ their formulas are

$$\begin{aligned}
\Lambda_{11}^{(r)} &= \left(\lambda_{uv}^{(r)}\right) = \begin{cases} \sigma_\gamma^2 & \text{when } \ u = v \\ \phi\sigma_\gamma^2 & \text{when } \ |u-v| = 1, \cdots, d^* \\ 0 & \text{for } \ |u-v| > d^*, \end{cases} \\
\Lambda_{21}^{(r)} &= \phi\sigma_\gamma^2\left[0 \times 1'_{r-1-d^*}, 1'_{d^*}\right],\ \ \Lambda_{22}^{(r)} = \sigma_\gamma^2,\ \ \Lambda_{12}^{(r)} = \left(\Lambda_{21}^{(r)}\right)'.
\end{aligned}$$

(c) **Identification of the random effects in all families**
Now to use (4.1) to generate y_r, we need to identify $\gamma_r = (\gamma_{r1}, \cdots, \gamma_{rn_r})'$ under f_r. Under the present linear sequence with $d^* = 4$, for convenience, we present these family based random effects in a tabular form as in Table 3.

4.1.4. Selection of Random Effects Design Weights (RDW). Remark that the random effects design weights w_r for the rth family of random effects (f_r) are chosen by the user/experimenter based on the spatial filed

Table 3: Familial random effects corresponding to spatial random effects under the linear sequence with $d^* = 4$.

Family	Family Random Effects	Corresponds to original random effects
f_1	$[\gamma_{11}, \gamma_{12}, \gamma_{13}]$	$[\widetilde{\gamma}_1(=\gamma_{11}), \widetilde{\gamma}_2, \widetilde{\gamma}_3]$
f_2	$[\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}]$	$[\widetilde{\gamma}_2(=\gamma_{21}), \widetilde{\gamma}_1, \widetilde{\gamma}_3, \widetilde{\gamma}_4]$
f_3	$[\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}, \gamma_{35}]$	$[\widetilde{\gamma}_3(=\gamma_{31}), \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_4, \widetilde{\gamma}_5]$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
f_{498}	$[\gamma_{498,1}, \gamma_{498,2}, \gamma_{498,3}, \gamma_{498,4}, \gamma_{498,5}]$	$[\widetilde{\gamma}_{498}(=\gamma_{498,1}), \widetilde{\gamma}_{496}, \widetilde{\gamma}_{497}, \widetilde{\gamma}_{499}, \widetilde{\gamma}_{500}]$
f_{499}	$[\gamma_{499,1}, \gamma_{499,2}, \gamma_{499,3}, \gamma_{499,4}]$	$[\widetilde{\gamma}_{499}(=\gamma_{499,1}), \widetilde{\gamma}_{500}, \widetilde{\gamma}_{497}, \widetilde{\gamma}_{498}]$
f_{500}	$[\gamma_{500,1}, \gamma_{500,2}, \gamma_{500,3}]$	$[\widetilde{\gamma}_{500}(=\gamma_{500,1}), \widetilde{\gamma}_{499}, \widetilde{\gamma}_{498}]$

condition. Thus, these weights are known. We consider two sets of design weights as follows:

RDW1: Equal Weights for Each Random Effects in the Family In this case, the random effects in a family of locations get equal weights irrespective of situations whether they are independent or correlated. A suitable equal weights design is given (2.9) [see also (2.20)–(2.21)], i.e,

$$w_r = \left(\frac{1}{\sqrt{n_r}} 1'_{n^*_{r(1)}}, \frac{1}{\sqrt{n_r}} 1'_{n^*_{r(2)}}, \frac{1}{\sqrt{n_r}} 1'_{n^*_{rs}} \right)',$$

with n_r as in (2.40), and $n^*_{r(1)},\ n^*_{r(2)},$; and n^*_{rs} are given in Tables 1 and 2.

RDW2: Distance Based Exponentially Decaying Weights for Random Effects in the Family We consider this exponentially decaying weights in spirit of (2.10). However we consider a symmetric design weights scheme, namely, for an odd n_r, we choose

$$\begin{aligned} w_r &= [w_{r1}, w_{r2}, \cdots, w_{r,(n_r+1)/2-1}, w_{r,(n_r+1)/2}, w_{r,(n_r+1)/2+1}, \cdots, w_{r,n_r-1}, w_{rn_r}]' \\ &= [c^{(n_r-1)/2}/m, c^{(n_r-1)/2-1}/m, \cdots, c^1/m, c^0/m, c^1/m, \cdots, c^{(n_r-1)/2-1}/m, c^{(n_r-1)/2}/m]', \end{aligned}$$

where c is a constant fraction, say $c = \frac{1}{2}$, and $m = [1 + 2\sum_{j=1}^{(n_r-1)/2}\{c^j\}^2]^{\frac{1}{2}}$. Some adjustments for the weights would be necessary for the extreme end families. The values of n_r and other sizes would remain the same as in the RDW1 case.

4.2. Data Generation. We consider $S = 500$ spatial locations. Under the aforementioned study 1, to generate data, i.e., the responses $y_1, \ldots, y_r, \ldots, y_{500}$, we

Study 1

1. choose $p_1 = 3$ epidemiological and $p_2 = 4$ environmental type fixed covariates as described in Section 4.1.1, along with their effects $\beta = (\alpha', \theta')'$ chosen as in (4.2);

2. follow Section 4.1.3(a), and generate 500 independent random effects $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_r, \ldots, \widetilde{\gamma}_{500}$, from $N(0, \sigma^2_\gamma)$ for selected small and large values of σ^2_γ as in (4.3);

3. group these random effects in families as in Table 3 following the family structure of locations described in Tables 1 and 2, constructed from a linear sequence of locations with $d^* = 4$;

4. generate independent model errors $\epsilon_r (r = 1, \ldots, 500)$ from $N(0, \sigma^2_\epsilon)$;

5. choose w_r as in the RDW1 case;

6. finally use them in (4.1).

Study 2 In this study, except for 2 and 5, the simulation designs remain the same as in Study 1. Here we follow Section 4.1.3(b), and generate 500 random effects $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_r, \ldots, \widetilde{\gamma}_{500}$, following the truncated equi-correlation structure with a selected value of σ^2_γ from (4.4) and $\phi = 0.3$, for example. For the random effects design weights w_r, we use both RDW1 and RDW2 cases.

4.2.1. Simulation Results. All simulations are repeated for 500 times. Because modeling the spatial correlations was the main objective of the paper, in this section, we present the empirical results such that the effect of correlation index parameter ϕ is understood clearly on the estimation of the parameters, including the main variance component parameters. Thus, we interpret the results under Studies 1 and 2 as follows:

Study 1 ($\phi = 0$) **Using RDW1 (Equal Design Weights)** In this case spatial responses are generated with $\phi = 0$, and the regression and variance component parameters are first estimated applying the GLS and MM approaches discussed in Section 3.1.1 and 3.1.2, respectively. We then estimated these variance component and regression parameters using the ML

Table 4: Equal design weights (RDW1) based Study 1 (Independent random effects : $\phi = 0$)

		Parameter estimates								
σ_γ^2	Quantity	σ_ϵ^2	σ_γ^2	α_1	α_2	α_3	θ_1	θ_2	θ_3	θ_4
0.25	SM	0.983	0.241	0.287	0.499	−0.499	0.606	0.408	0.506	0.205
	SSE	0.087	0.084	0.226	0.116	0.135	0.178	0.113	0.171	0.212
0.75	SM	0.985	0.727	0.287	0.500	−0.498	0.607	0.409	0.505	0.202
	SSE	0.096	0.150	0.257	0.120	0.195	0.188	0.117	0.181	0.229
1.0	SM	0.986	0.971	0.287	0.500	−0.498	0.608	0.409	0.505	0.201
	SSE	0.100	0.183	0.270	0.122	0.219	0.191	0.118	0.185	0.234
1.5	SM	0.988	1.457	0.287	0.500	−0.498	0.608	0.409	0.505	0.200
	SSE	0.110	0.250	0.295	0.125	0.259	0.196	0.120	0.190	0.242

The simulated means (SMs) and simulated standard errors (SSEs) of the GLS estimates of the regression parameters $\beta = (\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3, \theta_4)' \equiv (0.3, 0.5, -0.5, 0.6, 0.4, 0.5, 0.2)'$, and the MM estimates for selected values of σ_γ^2 and model error variance $\sigma_\epsilon^2 = 1.0$, based on 500 simulations

approach from Section 3.2. The simulated means (SMs) and simulated standard error (SSEs) of the GLS-MM estimates are reported in Table 4, whereas the SMs and SSEs of the ML estimates are presented in Table 5. In this $\phi = 0$ case, as expected, both GLS-MM and ML methods appear to estimate the parameters including the variance components very well. For example, Table 4 shows that the large true value of $\sigma_\gamma^2 = 1.5$ is estimated as 1.457

Table 5: Equal design weights (RDW1) based Study 1 (Independent random effects : $\phi = 0$)

		Parameter estimates								
σ_γ^2	Quantity	σ_ϵ^2	σ_γ^2	α_1	α_2	α_3	θ_1	θ_2	θ_3	θ_4
0.25	SM	0.983	0.242	0.290	0.500	−0.498	0.602	0.407	0.501	0.196
	SSE	0.082	0.079	0.229	0.119	0.132	0.177	0.112	0.175	0.204
0.75	SM	0.998	0.714	0.287	0.500	−0.498	0.607	0.409	0.505	0.202
	SSE	0.124	0.175	0.257	0.120	0.195	0.188	0.117	0.181	0.229
1.0	SM	0.981	0.979	0.286	0.500	−0.496	0.602	0.408	0.502	0.195
	SSE	0.095	0.173	0.271	0.124	0.214	0.190	0.116	0.188	0.226
1.5	SM	0.980	1.471	0.283	0.500	−0.495	0.601	0.407	0.503	0.195
	SSE	0.101	0.232	0.295	0.127	0.253	0.195	0.119	0.233	0.242

The simulated means (SMs) and simulated standard errors (SSEs) of the maximum likelihood (ML) estimates of the regression parameters $\beta = (\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3, \theta_4)' \equiv (0.3, 0.5, -0.5, 0.6, 0.4, 0.5, 0.2)'$, random effects variance parameter σ_γ^2, and model error variance $\sigma_\epsilon^2 = 1.0$, based on 500 simulations with standard error 0.250 by the method of moments, and Table 5 shows that this parameter value is estimated by the likelihood approach as 1.471 with standard error 0.232. Between the two estimation approaches, the likelihood method produces the estimates in general with slightly smaller mean squared errors as compared to the moment methods.

Note the proposed correlation model generates pairwise spatial correlations those decay as the distance between the locations for the responses increases but this decaying pattern depends on the family composition around these two locations. For example, when random effects associated to each location in the spatial region are independent, under the present linear spatial sequence with $d^* = 4$, the true correlation between two responses y_r and y_s, by (2.32)–(2.33) is given as

$$\rho_{y_r,y_s} \equiv \rho_{rs} = \frac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\epsilon^2} \frac{n_{rs}^*}{\sqrt{n_r n_s}}, \tag{4.6}$$

which can be computed by using family composition from Table 1, and these correlations now can be estimated by using the estimates for σ_γ^2 and σ_ϵ^2 from Tables 4 and 5. For example, we display, a few spatial correlations and their estimates as follows.

Illustration of spatial correlations and their estimates with $\sigma_\gamma^2 = 1.5$, $\sigma_\epsilon^2 = 1.0$, and family sizes $\{n_r = n_s = 5\}$

	Spatial correlations ρ_{rs}					
Quantity	ρ_{34}	ρ_{35}	ρ_{36}	ρ_{37}	ρ_{38}	ρ_{39}
True	0.480	0.360	0.240	0.120	0.000	0.000
Estimated (by MMs)	0.477	0.358	0.238	0.119	0.000	0.000
Estimated (Likelihood based)	0.480	0.360	0.240	0.120	0.000	0.000

Study 2 ($\phi = 0.3$) **Using RDW1 (Equal Design Weights), but Estimation is done Ignoring** ϕ **(i.e., Using** $\hat{\phi} = 0$**); Correlation Mis-Specification Effect** In this case, we generate the spatial responses with $\phi = 0.3$, but obtain the ML estimates of the regression and variance component parameters by treating ϕ as $\phi = 0$. The SMs and SSEs of the ML estimates are provided in Table 6. It is clear from the results of the table that the regression parameters are still estimated well as expected, but variance component estimates became highly biased. For example, $\sigma_\epsilon^2 = 1.0$ and $\sigma_\gamma^2 = 1.5$ are estimated as 0.580 and 2.731, respectively, showing that estimators are inconsistent.

Study 2 ($\phi = 0.3$) **Using RDW1 (Equal Design Weights)** Because the correlation mis-specification has serious consequence on the estimation

Table 6: Equal design weights (RDW1) based Study 2 (Correlated random effects: $\phi = 0.3$)

			Parameter estimates									
σ^2_ϵ	σ^2_γ	Quantity	ϕ	σ^2_ϵ	σ^2_γ	α_1	α_2	α_3	θ_1	θ_2	θ_3	θ_4
1.0	1.0	SM	0.0	0.841	1.930	0.294	0.494	−0.507	0.597	0.402	0.506	0.199
		SSE	–	0.057	0.256	0.366	0.124	0.369	0.203	0.121	0.193	0.246
1.0	1.5	SM	0.0	0.841	2.731	0.292	0.494	−0.509	0.597	0.402	0.507	0.199
		SSE	–	0.089	0.336	0.419	0.128	0.442	0.209	0.124	0.199	0.255
0.75	2.0	SM	0.0	0.580	3.340	0.289	0.496	−0.510	0.597	0.402	0.507	0.199
		SSE	–	0.070	0.375	0.450	0.118	0.494	0.189	0.111	0.180	0.232

The simulated means (SMs) and simulated standard errors (SSEs) of the so-called ‘working’ independence ($\hat{\phi} = 0$) assumption based maximum likelihood (ML) estimates of the regression parameters $\beta = (\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3, \theta_4)' \equiv (0.3, 0.5, -0.5, 0.6, 0.4, 0.5, 0.2)'$, random effects variance parameter $\sigma^2_\gamma = 1.0,\ 1.5,\ 2.0$ and model error variance $\sigma^2_\epsilon = 0.75,\ 1.0$; based on 500 simulations of variance parameters, it is necessary to accommodate the correlations in estimation. We now estimate all parameters including ϕ by using the ML approach and report the simulation results in Table 7. Note that we have done this for more combinations of variance parameters. The results of the table show that all parameters including variance components and correlation parameters are estimated well. For example, unlike in Table 6, the variance parameters $\sigma_\epsilon^2 = 1.0$ and $\sigma_\gamma^2 = 1.5$ are now estimated as 0.979 and 1.475, respectively, with correlation parameter estimate as 0.278 for $\phi = 0.3$.

Note that by (2.37) and (2.39), similar to (4.6), one can compute the correlations for the pairwise responses as

$$\rho_{y_r,y_s} \equiv \rho_{rs} = \frac{\sigma_\gamma^2}{\left[(\sigma_\gamma^2[1+(n_r-1)\phi]+\sigma_\epsilon^2)(\sigma_\gamma^2[1+(n_s-1)\phi]+\sigma_\epsilon^2)\right]^{\frac{1}{2}}} \times \frac{1}{\sqrt{n_r n_s}}\left[\phi\left\{n_{rs}+n_{rs}^*\left(n_r^*+n_s^*\right)\right\}\left\{\phi n_{rs}^*(n_{rs}^*-1)+n_{rs}^*\right\}\right]. \quad (4.7)$$

To have an idea about the correlation parameter effect on the spatial lag correlations of the responses, we display below true correlations and their likelihood estimates for this $\phi = 0.3$ case.

Illustration of spatial correlations and their estimates with $\sigma_\gamma^2 = 0.75$, $\sigma_\epsilon^2 = 1.0$, $\phi = 0.3$, and family sizes $\{n_r = n_s = 5\}$

Quantity	Spatial correlations ρ_{rs}					
	ρ_{34}	ρ_{35}	ρ_{36}	ρ_{37}	ρ_{38}	ρ_{39}
True	0.566	0.492	0.402	0.294	0.170	0.102
Estimated (Likelihood based)	0.586	0.511	0.418	0.307	0.178	0.107

Notice that even though we have considered $\sigma_\gamma^2 = 0.75$ here, random effects correlation $\phi = 0.3$ appears to cause significant increase in correlations between two responses, as compared to the independent case. Thus, these correlations have to be estimated well, through estimating all scale parameters including this random effects correlation parameter ϕ, which we have done as shown in Table 7.

Study 2 ($\phi = 0.3$) **using RDW2 (exponentially decaying design weights)** We have further conducted another simulation study by changing the design weights. More specifically, using the exponentially decaying weights RDW2 with $c = 0.5$ from Section 4.1.4, we have repeated the aforementioned simulation study 2 by replacing RDW1 with RDW2. The new simulation results are reported in Table 8. The results of the table show that similar to that of Table 7, all parameters including the variance and

Table 7: Equal design weights (RDW1) based Study 2 (Correlated random effects: $\phi = 0.3$)

			Parameter estimates									
σ_ϵ^2	σ_γ^2	Quantity	ϕ	σ_ϵ^2	σ_γ^2	α_1	α_2	α_3	θ_1	θ_2	θ_3	θ_4
1.0	0.75	SM	0.310	0.982	0.768	0.286	0.495	−0.495	0.600	0.403	0.507	0.209
		SSE	0.146	0.034	0.213	0.330	0.109	0.328	0.197	0.112	0.185	0.257
1.0	1.0	SM	0.309	0.988	1.005	0.286	0.494	−0.495	0.602	0.404	0.506	0.209
		SSE	0.126	0.097	0.245	0.362	0.111	0.373	0.200	0.111	0.188	0.260
1.0	1.5	SM	0.278	0.979	1.475	0.287	0.500	−0.502	0.597	0.394	0.515	0.215
		SSE	0.085	0.096	0.302	0.418	0.126	0.430	0.205	0.119	0.194	0.247
0.75	2.0	SM	0.289	0.747	2.033	0.292	0.496	−0.485	0.596	0.387	0.505	0.201
		SSE	0.058	0.096	0.476	0.429	0.117	0.473	0.178	0.108	0.180	0.230

The simulated means (SMs) and simulated standard errors (SSEs) of the maximum likelihood (ML) estimates of the regression parameters $\beta = (\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3, \theta_4)' \equiv (0.3, 0.5, -0.5, 0.6, 0.4, 0.5, 0.2)'$, random effects variance parameter $\sigma_\gamma^2 = 0.75,\ 1.0,\ 1.5,\ 2.0$; model error variance $\sigma_\epsilon^2 = 0.75,\ 1.0$; and correlation index parameter $\phi = 0.3$; based on 500 simulations

Table 8: Exponentially decaying design weights (RDW2) based Study 2 (Correlated random effects: $\phi = 0.3$)

			Parameter estimates									
σ^2_ϵ	σ^2_γ	Quantity	ϕ	σ^2_ϵ	σ^2_γ	α_1	α_2	α_3	θ_1	θ_2	θ_3	θ_4
1.0	0.75	SM	0.299	0.994	0.753	0.301	0.497	−0.507	0.597	0.412	0.497	0.201
		SSE	0.092	0.070	0.214	0.332	0.128	0.298	0.200	0.118	0.110	0.244
1.0	1.0	SM	0.286	0.977	1.000	0.303	0.492	−0.510	0.597	0.402	0.499	0.197
		SSE	0.098	0.091	0.234	0.335	0.125	0.324	0.202	0.121	0.900	0.249
1.0	1.5	SM	0.291	0.992	1.440	0.312	0.491	−0.520	0.593	0.401	0.499	0.198
		SSE	0.078	0.095	0.204	0.380	0.129	0.382	0.212	0.125	0.199	0.263
0.75	2.0	SM	0.289	0.748	1.900	0.293	0.490	−0.514	0.604	0.412	0.491	0.200
		SSE	0.054	0.077	0.274	0.449	0.119	0.456	0.188	0.123	0.179	0.283

The simulated means (SMs) and simulated standard errors (SSEs) of the maximum likelihood (ML) estimates of the regression parameters $\beta = (\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3, \theta_4)' \equiv (0.3, 0.5, -0.5, 0.6, 0.4, 0.5, 0.2)'$, random effects variance parameter $\sigma^2_\gamma = 0.75,\ 1.0,\ 1.5,\ 2.0$; model error variance $\sigma^2_\epsilon = 0.75,\ 1.0$; and correlation index parameter $\phi = 0.3$; with $c = 0.5$ in RDW2, based on 500 simulations random effects correlation parameters are estimated well. For example, the variance parameters $\sigma_\epsilon^2 = 1.0$ and $\sigma_\gamma^2 = 1.5$ are estimated as 0.992 and 1.440, respectively, with correlation parameter estimate as 0.291 for $\phi = 0.3$. Thus, the ML approach appears to estimate the parameters well irrespective of the random effects design weights. Furthermore, remark that the presence of spatial correlations due to correlated random effects play a big role in variance component and correlation parameter estimation as well as in efficient regression estimation. These efficiencies for regression estimates are, in general, reflected through SSEs when compared to the $\hat{\phi} = 0$ case.

5. Concluding Remarks

Spatial data analysis is a highly important research topic in many practical fields such as forestry, environment and ecology, and biomedical studies including disease mapping. Over the last two decades, a substantial progress is made toward this spatial analysis, and the research has also been extended to deal with spatial-temporal data. See for example, a recent text book by Cressie and Wikle (2011) and the references therein. However, as we discussed in the paper, it has been a challenge in the context of spatial data analysis, to develop a viable spatial correlation model to accommodate the influences of unbalanced family of locations on all pair-wise responses in the whole spatial region. In the paper we have developed a correlation model as a resolution to this problem through introducing moving familial random effects. The proposed spatial correlation model has been illustrated through two specialized spatial series of locations. Likelihood inference has been used for the regression as well as all scale parameters including the correlations of the familial random effects. A correlation mis-specification effect on the estimation of the parameters is studied through monte carlo simulations. The proposed correlation model may be extended to the binary and count data setup, which is however, beyond the scope of the present paper.

Acknowledgments. The authors thank the referee for comments and suggestions leading to improvement of the paper.

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Hensley H Mariathas
Brajendra C Sutradhar
Department of Mathematics
and Statistics, Memorial
University of Newfoundland,
St.John's, NL A1C 5S7, Canada
E-mail: mhhubert@mun.ca

Paper received: 17 May 2013.