Sankhyā : The Indian Journal of Statistics
2016, Volume 78-B, Part 1, pp. 39-65

Inferences in Longitudinal Count Data Models with Measurement Errors in Time Dependent Covariates

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Abstract

Unlike in the independent setup, the measurement error analysis in longitudinal setup especially for discrete responses is not adequately addressed in the literature. In linear longitudinal setup, recently Fan, Sutradhar, and Rao (*Sankhya B,* **74**, 126-148 2012) have introduced a bias corrected generalized quasi-likelihood (BCGQL) approach for the estimation of the regression effects after accommodating both measurement errors in time dependent covariates and correlations of the repeated responses. In longitudinal setup for repeated count data, a similar BCGQL estimating equation for the regression effects is provided by Sutradhar (2013) under the assumption that longitudinal correlation index parameter and measurement error variances are known. In this paper, we offer three main contributions. First, because the BCGQL estimation approach for discrete longitudinal data is complex and less familiar, we provide a complete derivation for this BCGQL estimating equation under the longitudinal count data model subject to measurement errors in time dependent covariates. Second, because the longitudinal correlation index parameter and measurement error variances involved in the model are unknown in practice, and because the main regression parameters can not be estimated without knowing them, we estimate these nuisance parameters consistently by solving appropriate unbiased estimating equations for these parameters. Next, the basic asymptotic properties of the estimators of main regression parameters are indicated.

AMS (2000) subject classification. Primary 62F10, 62F12, Secondary 62J12.
Keywords and phrases. Bias correction, Consistency, Correlations among repeated counts, Dynamic model, Generalized quasi-likelihood estimation, Linear measurement error model for the covariates, Measurement error variances, Method of moments estimation

1 Introduction

In the longitudinal setup, repeated responses along with time dependent covariates are collected from a large number of independent individuals. When the covariates in this setup are subject to measurement errors, the consistent estimation of the regression effects of the true covariates on the responses becomes difficult specially when the responses are discrete such as binary or counts.

In the independent setup, that is, when responses along with multi-dimensional covariates subject to measurement errors are collected at a cross sectional level from a large number of independent individuals, the bias corrected estimation for the regression effects involved in generalized linear measurement-error models (GLMEMs) with normal measurement errors in covariates has been studied extensively in the literature. See for example, the text books by Fuller (1987), Carroll et al. (2006), and Buonaccorsi (2010), and the references in these books. Some of these studies also addressed measurement error problems in various complicated situations such as when the data also contain outliers, and regression function is partly specified. As far as the longitudinal setup is concerned, some attention is given for measurement error analysis in linear model setup involving continuous responses. See, for example, Wansbeek (2001) [see also Wansbeek and Meijer (2000)], Xiao et al. (2007), and Fan et al. (2012). However, not much attention is given to the measurement error models for longitudinal count and binary data. Wang et al. (1996) considered a measurement error model in a generalized linear regression setup where covariates are replicated and the measurement errors for replicated covariates are assumed to be correlated with a stationary correlation structure such as Gaussian auto-regressive of order 1 (AR(1)) structure. As far as the responses are concerned, they were assumed to be independent, collected at a cross sectional level from a large number of independent individuals. Thus this study does not address the measurement error problems in the longitudinal setup where responses are supposed to be collected repeatedly from a large number of independent individuals. Sutradhar and Rao (1996) study, however, dealt with repeated binary responses, but their bias correction estimation approach, similar to Stefanski (1985), was developed under a restricted assumption that the measurement error variances are small in magnitude.

In this paper, following Sutradhar (2013), we consider a measurement error model in the longitudinal setup for repeated count data. In this setup, repeated count responses along with a vector of multi-dimensional time dependent covariates are collected from a large number of independent individuals, where the covariates are subject to measurement errors. Let $y_i = (y_{i1}, \ldots, y_{it}, \ldots, y_{iT})'$ denote the T repeatedcount responses from the ith $(i = 1, \ldots, K)$ individual, and $x_{it} = (x_{it1}, \ldots, x_{itu}, \ldots, x_{itp})'$ denote the $p-$dimensional covariate vector corresponding to the scalar response y_{it} recorded at time point t. Suppose that in the absence of measurement errors, $z_{it} = (z_{it1}, \ldots, z_{itu}, \ldots, z_{itp})'$ denote the true but invisible covariate vector corresponding to y_{it}. It is of interest to find the effect of z_{it} on y_{it}, say β, but by using observed x_{it} in stead of z_{it} as this later true covariate vector is not available. Under the assumption that repeated count responses $y_{i1}, \ldots, y_{it}, \ldots, y_{iT}$ follow an AR(1) (auto-regressive order 1) dynamic model, recently Sutradhar (2013) has suggested a bias corrected generalized quasi-likelihood (BCGQL) estimator of β. However this estimator was derived under the assumption that the correlation index parameter involved in the AR(1) dynamic model and the measurement error variance parameters involved in the relationship between z_{it} and x_{it}, are known. Also, the BCGQL estimating equation was given without any proof or derivation. Note that because the BCGQL estimation for longitudinal count data model in the presence of measurement errors in covariates is complex and less familiar, in Section 3, we first provide a detailed derivation of the estimating equation through several lemmas. The proofs for the lemmas are given in the Appendix. To save space, in the same section, we only state the asymptotic properties of the BCGQL estimator of the main regression parameters. The dynamic AR(1) correlation model for repeated count responses involving true covariates, and the measurement error model relating true and observed covariates are briefly discussed in Section 2. A simulation study is carried out in Section 4 to examine the finite sample performance of the proposed BCGQL estimators for the main regression parameters, provided in Section 3. In the simulation study, a various selection of correlation index parameter and measurement error variance values is considered. These correlation and variance parameters were assumed to be known.

However, because the longitudinal correlation index parameter and measurement error variances are unknown in practice, and because the main regression parameters can not be estimated without knowing them, in Section 5, we provide a moment estimation approach which yields consistent estimators for these nuisance parameters. More specifically, the moment estimators are obtained by solving appropriate unbiased estimating equations for these nuisance parameters. The paper concludes in Section 6.

2 A Dynamic Model for Repeated Counts with Covariates Subject to Measurement Errors

In the absence of measurement errors, regression models for time series of count data have been studied by many authors. For example, one may refer to McKenzie (1986), Zeger (1988), and Mallick and Sutradhar (2008). Similarly, dynamic regression models for longitudinal count data in the absence of measurement errors have been discussed by Montalvo (1997), Wooldridge (1999), Sutradhar (2010), and Sutradhar et al. (2014), for example. However, as indicated in the last section, except the study by Sutradhar (2013), not much attention is paid to deal with longitudinal count data with time dependent covariates subject to measurement error. One of the main inference related difficulties arises from the fact that on top of the mean and variances, the correlations of the repeated count data also contain time dependent covariates which are now subject to measurement errors, making the bias correction for measurement errors extremely difficult. To understand these issues clearly, in the following subsection, we present an auto-regressive order 1 (AR(1)) type dynamic model for repeated counts which clearly exhibits the involvement of time dependent covariates in the correlations. In Section 2.2, we then present a measurement error model relating the true and observed time dependent covariates.

2.1. Auto-regressive Order 1 (AR(1)) Type Dynamic Model. Suppose that conditional on the true covariate vector z_{it}, the count response y_{it} for all $t = 1, \ldots, T$, marginally follow the well known Poisson distribution with mean

$$E(Y_{it}|z_{it}) = \mu_{iz,t} = \exp(z'_{it}\beta). \tag{2.1}$$

Denote this distribution as $y_{it} \sim Poi(\mu_{iz,t})$. It is also known that variance of y_{it} is the same as its mean. That is

$$\text{var}(Y_{it}|z_{it}) = \sigma_{iz,tt} = \exp(z'_{it}\beta) = \mu_{iz,t}. \tag{2.2}$$

Next as far as the relationships among repeated count responses$y_{i1}, \ldots,$ $y_{it}, \ldots, y_{iT}$, are concerned, we assume that they follow the dynamic relationship given by:

$$\begin{aligned} y_{i1} &\sim Poi(\mu_{iz,1}) y_{it} \\ &= \rho * y_{i,t-1} + d_{it} = \sum_{j=1}^{y_{i,t-1}} b_j(\rho) + d_{it},\ t = 2, \ldots, T, \end{aligned} \tag{2.3}$$

[Sutradhar (2010)] where for given counts $y_{i,t-1}$ at time point $t-1$, $\sum_{j=1}^{y_{i,t-1}}$ $b_j(\rho)$ denotes the sum of $y_{i,t-1}$ independent binary values with $Pr[b_j(\rho) =$

$1] = \rho$ and $Pr[b_j(\rho) = 0] = 1 - \rho$, ρ being the longitudinal correlation index parameter. Now under the assumptions that $y_{i,t-1} \sim Poi(\mu_{iz,t-1})$, $d_{it} \sim Poi(\mu_{iz,t} - \rho\mu_{iz,t-1})$, for $t = 2, \ldots, T$, and d_{it} and $y_{i,t-1}$ are independent, for $r < t$, one may follow (2.3) and compute the covariance between y_{ir} and y_{it} by using

$$\begin{aligned} cov(Y_{ir}, Y_{it}|z_{ir}, z_{it}) &= E(Y_{ir}Y_{it}|z_{ir}, z_{it}) - E(Y_{ir}|z_{ir})E(Y_{it}|z_{it}) \\ &= E_{Y_{ir}}Y_{ir}E_{Y_{i,r+1}} \ldots E_{Y_{i,t-1}}E[Y_{it}|y_{i,t-1}, y_{i,t-2}, \ldots, y_{i,r+1}] \\ &\quad -\mu_{iz,r}\mu_{iz,t}, \end{aligned} \tag{2.4}$$

which subsequently yields the lag $t - r$ correlation between y_{ir} and y_{it} as

$$\text{corr}(Y_{ir}, Y_{it}) = c_{iz,rt} = \begin{cases} \rho^{t-r}[\mu_{iz,r}\mu_{iz,t}^{-1}]^{\frac{1}{2}}, & \text{for } r < t \\ \rho^{r-t}[\mu_{iz,t}\mu_{ir,t}^{-1}]^{\frac{1}{2}}, & \text{for } r > t. \end{cases} \tag{2.5}$$

Note that the lag correlations given by (2.5) are non-stationary by nature as they depend on the time dependent variances through the covariates z_{it} and z_{iu}, whereas in the stationary case when $z_{it} = z_{iu}$ for all $u \neq t$, they reduce to ρ^{t-u}, a Gaussian type $AR(1)$ correlation structure. Further note that because $E[Y_{it}] = \mu_{iz,t} = \exp(z'_{it}\beta)$ by (2.1), the regression parameters vector β measures the effects of z_{it} on y_{it} for all $t = 1, \ldots, T$. But in the present set up, z_{it}'s are unobservable, and hence they can not be used to estimate β. In stead, one must use the observed covariates x_{it}, which are, however, subject to certain measurement error. In the following section, we consider a widely used measurement error model relating x_{it} and z_{it}.

Remark that in time series setup, Staudenmayer and Buonaccorsi (2005) have studied a measurement error model for time series responses, where these responses are assumed to follow the Gaussian auto-regressive order 1 (AR(1)) correlation process subject to measurement errors. Thus, even though the correlation structure under their dynamic AR(1) model is similar to (2.3), these authors have considered a different measurement error model with time series responses $\{y_t\}$ subject to measurement errors whereas in the present study the time dependent covariates z_{it} in (2.2) and (2.5) are not observed causing measurement errors due to the replacement of z_{it} with observed covariates x_{it}, (say).

In the next section, we present a measurement error model relating the observed covariates x_{it} and the true but invisible covariates z_{it}.

2.2. Measurement Error Model: Relationship Between Observed and True Covariates. Suppose that the p-dimensional observed covariate vector x_{it} is related to the p-dimensional true covariate vector z_{it} through the relationship

$$x_{it} = z_{it} + v_{it}, \tag{2.6}$$

where $v_{it} = (v_{it1}, \ldots, v_{itu}, \ldots, v_{itp})'$ satisfies the following two assumptions:
A1. $v_{it} \sim N(0, \Lambda = \text{diag}[\sigma_1^2, \ldots, \sigma_u^2, \ldots, \sigma_p^2])$ for all $t = 1, \ldots, T$.
A2. Also,

$$\text{corr}[v_{iru}, v_{itm}] = \begin{cases} \phi_u, & \text{for } m = u; r \neq t, r, t = 1, \ldots, T \\ 0, & \text{for } m \neq u; r, t = 1, \ldots, T. \end{cases}$$

These two assumptions imply that the uth covariate has the measurement error variance σ_u^2 for $u = 1, \ldots, p$, at a given time t for all $t = 1, \ldots, T$. Also, the covariate values for the same uth covariate recorded at two different times r and t are equally correlated with correlation ϕ_u for all $r \neq t$. Note that the above two assumptions A1 and A2 imply by (2.6) that

$$\begin{pmatrix} x_{ir} \\ x_{it} \end{pmatrix} \sim N_{2p} \left[\begin{pmatrix} z_{ir} \\ z_{it} \end{pmatrix}, \begin{pmatrix} \Lambda & \Lambda_\phi \\ \Lambda_\phi & \Lambda \end{pmatrix} \right], \tag{2.7}$$

where $\Lambda_\phi = \text{cov}(v_{ir}, v'_{it}) = \text{diag}[\phi_1 \sigma_1^2, \ldots, \phi_u \sigma_u^2, \ldots, \phi_p \sigma_p^2]$.

Now because our objective is to estimate β in (2.2), the effect of z_{it} on y_{it}, but by using x_{it} and y_{it}, it is clear from (2.7) that the variance and correlation parameters (σ_u^2, ϕ_u) of measurement errors will play a significant role in obtaining the bias corrected estimate for the regression parameter β. The longitudinal correlation index parameter ρ [see (2.5)] for the repeated responses will also play a significant role in the bias correction process. In the next section, we develop a BCGQL (bias corrected GQL) approach for the estimation of β, where, as expected, the bias corrected estimator of β appears to be dependent on the above mentioned all three parameters, namely, $\rho, \sigma_u^2,$ and ϕ_u, for $u = 1, \ldots, p$. In Section 5, we conduct a simulation based empirical study to examine the role of these parameters in producing the bias corrected regression estimates.

3 Bias Corrected Estimation for the Regression Parameters

In Section 2.1 we have described a correlation model for repeated count responses $y_{i1}, \ldots, y_{it}, \ldots, y_{iT}$. The marginal mean and the variance of y_{it} for all $t = 1, \ldots, T$, are functions of the true time dependent covariate vector z_{it}. Similarly, for $r < t$, the correlations between y_{ir} and y_{it} are the functions of the covariate vectors z_{ir} and z_{it}. Note however that the true covariate vectors

z_{it} for all $t = 1, \ldots, T$, are fixed but unobserved. Consequently, one can not use these potential z_{it}'s to construct the mean, variance and covariances in order to develop an estimating equation for the regression effect β. Remark however that if z_{it} were known, one could then exploit them to compute the mean vector and covariance matrix of y_i given by

$$\begin{aligned} E[Y_i] &= \mu_{iz} = [\mu_{iz,1}, \ldots, \mu_{iz,t}, \ldots, \mu_{iz,T}]' : T \times 1 \\ \mathrm{cov}[Y_i] &= \Sigma_{iz} = (c_{iz,rt}\sqrt{\mu_{iz,r}\mu_{iz,t}}) : T \times T, \end{aligned} \tag{3.1}$$

with mean $\mu_{iz,t}$ as given in (2.1) and correlation $c_{iz,rt}$ as given in (2.5). Consequently, one could solve the so called generalized quasi-likelihood (GQL) estimating equation

$$\sum_{i=1}^{K} \frac{\partial \mu'_{iz}}{\partial \beta} \Sigma_{iz}^{-1}(y_i - \mu_{iz}) = 0 \tag{3.2}$$

[Sutradhar et al. (2008)] to obtain a consistent and highly efficient estimate of β [see also Sutradhar (2011, Chapter 6)], where $\frac{\partial \mu'_{iz}}{\partial \beta}$ is the $p \times T$ first derivative matrix of μ'_{iz} with respect to β.

Note, however, that because the true covariates z_{it} are unobservable in the present setup, one can not use (3.1) to construct the estimating equation (3.2) for the estimation of β.

3.1. Naive GQL Estimation for Regression Parameters β and Inconsistency. Suppose that by using the observed covariates x_{it}, one writes a naive GQL (NGQL) estimating equation given by

$$\sum_{i=1}^{K} \frac{\partial \mu'_{ix}}{\partial \beta} \Sigma_{ix}^{-1}(y_i - \mu_{ix}) = 0, \tag{3.3}$$

where

$$\begin{aligned} \mu_{ix,t} &= \mu_{iz,t}|_{z_{it}=x_{it}}, \text{ and} \\ \Sigma_{ix} &= (\sigma_{ix,rt}) = \Sigma_{iz|_{z=x}} = ([c_{iz,rt}\sqrt{\mu_{iz,r}\mu_{iz,t}}]|_{z_{it}=x_{it}}). \end{aligned}$$

But, this NGQL estimating equation (3.3) will yield biased and hence inconsistent estimate for β. This is because the NGQL estimating function in the left hand side of the equation (3.3) is not unbiased for the true covariates based GQL estimating function in the left hand side of the equation (3.2). That is,

$$E_x \left[\sum_{i=1}^{K} \frac{\partial \mu'_{ix}}{\partial \beta} \Sigma_{ix}^{-1}(y_i - \mu_{ix}) \right] \neq \sum_{i=1}^{K} \frac{\partial \mu'_{iz}}{\partial \beta} \Sigma_{iz}^{-1}(y_i - \mu_{iz}). \tag{3.4}$$

In the simulation study in Section 5, we demonstrate the biased performance of the NGQL estimator of β obtained by solving the NGQL estimating equation (3.3), where x_{it} would be related to z_{it} through the measurement error relationship (2.6).

3.2. BCGQL Estimator of β. Notice from (3.4) that the NGQL estimator obtained by solving the GQL estimating equation (3.3) produces biased and hence inconsistent estimate for β. The purpose of this section is to modify the NGQL estimating function in (3.3) so that the modified GQL estimating function becomes unbiased for true covariates based GQL estimating function $\sum_{i=1}^{K} \frac{\partial \mu'_{iz}}{\partial \beta} \Sigma_{iz}^{-1}(y_i - \mu_{iz})$. Let this modified GQL estimating function be denoted by $g_x(x, \beta, \rho, \Lambda, \phi_1, \ldots, \phi_p | y)$, satisfying

$$E_x\left[g_x(x, \beta, \rho, \Lambda, \phi_1, \ldots, \phi_p | y)\right] = \sum_{i=1}^{K} \frac{\partial \mu'_{iz}}{\partial \beta} \Sigma_{iz}^{-1}(y_i - \mu_{iz}), \tag{3.5}$$

where the observed covariates $\{x \equiv x_{it}\}$ follow the measurement error model (2.6) with their distributions as shown by (2.7). One may then obtain the bias corrected GQL (BCGQL) estimate of β by solving the BCGQL estimating equation yielding the BCGQL

$$g_x(x, \beta, \rho, \Lambda, \phi_1, \ldots, \phi_p | y) = 0. \tag{3.6}$$

Notice that once $g_x(\cdot)$ is computed, the solution of (3.6) would provide unbiased and hence consistent estimate for β under some mild conditions on the observed covariates. This consistency of the BCGQL estimator for β is discussed in brief in Section 3.3. The finite sample performance of the BCGQL estimator is examined through a simulation study in Section 4.

A formula for the estimating function $g_x(\cdot)$ is available from Sutradhar (2013, Section 3.2.1) without any derivations. In order to write this formula, it is convenient to use the following notations.

1. Let $X'_i = (x_{i1}, \ldots, x_{it}, \ldots, x_{iT})$ be the $p \times T$ observed covariates matrix, and
2. $A_{ix} = \text{diag}[\mu_{ix,1}, \ldots, \mu_{ix,t}, \ldots, \mu_{ix,T}]$, with $\mu_{ix,t} = \exp(x'_{it}\beta)$ for $t = 1, \ldots, T$.
3. Consider the $p \times T$ matrix $\beta \otimes 1'_T$ where $1'_T = (1 \ldots, 1)$ is the $1 \times T$ vector of unity, $\otimes$ denotes the well known Kronecker or direct product, so that the matrix contains $\beta = (\beta_1 \ldots, \beta_p)'$ in each column.

4. Suppose that $\tilde{C}_{ix}(\rho) = (\tilde{c}_{ix,rt})$ is an unbiased correlation matrix for the true correlation matrix $C_{iz}(\rho) = (c_{iz,rt})$, where the matrix element $c_{iz,rt}$ is defined in (2.5). That is, $E_x[\tilde{c}_{ix,rt}] = c_{iz,rt}$, where the formula for the (r,t)-th element of the unbiased correlation matrix is given by

$$\tilde{c}_{ix,rt} = \rho^{t-r}\left[\exp\left(x_{ir} - x_{it}\right)'\frac{\beta}{2} - \frac{1}{4}\beta'(\Lambda - \Lambda_\phi)\beta\right], \qquad (3.7)$$

with Λ and Λ_ϕ given as in (2.7).

5. Let $B_{1\phi} = \frac{1}{2}(\Lambda - \Lambda_\phi)$, and $B_{2\phi} = \frac{1}{2}(\Lambda + \Lambda_\phi)$.

6. For $m_1 = \exp\{-\frac{1}{4}\beta'(\Lambda - \Lambda_\phi)\beta\}$, and $m_2 = \exp\left\{-\frac{1}{4}\beta'(\Lambda + \Lambda_\phi)\beta\right\}$, define $M_{1\phi} = \text{diag}[m_1, \ldots, m_1] : p \times p$; and $M_{2\phi} = \text{diag}[m_2, \ldots, m_2] : p \times p$.

7. Let $\tilde{Q}_{ix}(\rho) = \tilde{C}_{ix}^{-1}(\rho)$.

Following (Sutradhar, 2013), one may then write the formula for the unbiased estimating function $g_x(\cdot)$ as

$$\begin{aligned} g_x(x, \beta, \rho, \Lambda, \phi_1, \ldots, \phi_p|y) &= \sum_{i=1}^{K}\Big[\left\{M_{1\phi}X_i' - M_{1\phi}B_{1\phi}(\beta \otimes 1_T')\right\}\left\{A_{ix}^{\frac{1}{2}}\tilde{Q}_{ix}(\rho)A_{ix}^{-\frac{1}{2}}\right\}y_i \\ &\quad - \left\{M_{2\phi}X_i' - M_{2\phi}B_{2\phi}(\beta \otimes 1_T')\right\}\left\{A_{ix}^{\frac{1}{2}}\tilde{Q}_{ix}(\rho)A_{ix}^{-\frac{1}{2}}\right\}\mu_{ix}\Big] \\ &= \sum_{i=1}^{K} g_{ix}(x_i, \beta, \rho, \Lambda, \phi_1, \ldots, \phi_p|y_i), \end{aligned} \qquad (3.8)$$

satisfying (3.5), that is, $E_x\left[g_x(x, \beta, \rho, \Lambda, \phi_1, \ldots, \phi_p|y)\right] = \sum_{i=1}^{K}\frac{\partial\mu_{iz}'}{\partial\beta}\Sigma_{iz}^{-1}(y_i - \mu_{iz})$.

Remark that the bias corrected estimating function $g_x(\cdot)$ as shown in (3.8) was provided by Sutradhar (2013) without any derivations. Now because the unbiased function in (3.8) is long and less familiar, for the sake of completeness, we provide a step by step derivation for this function. For convenience, the main results for the derivation are given below but the steps in detailed are presented through several lemmas in the Appendix.

It is desired to construct the formula for $g_x(\cdot)$ such that $E_x\left[g_x(\cdot)\right] = \sum_{i=1}^{K}\frac{\partial\mu_{iz}'}{\partial\beta}\Sigma_{iz}^{-1}(y_i - \mu_{iz})$. For the purpose, following Lemma 1 from the

Appendix, we first re-express the true covariates-based GQL estimating function in the right hand side as

$$\sum_{i=1}^{K}\frac{\partial\mu'_{iz}}{\partial\beta}\Sigma_{iz}^{-1}(y_i-\mu_{iz}) = \sum_{i=1}^{K}\sum_{r=1}^{T}\sum_{t=1}^{T}q_{iz,rt}\left[y_{ir}z_{ir}\exp\left\{(z_{ir}-z_{it})'\frac{\beta}{2}\right\} - z_{ir}\exp\left\{(z_{ir}+z_{it})'\frac{\beta}{2}\right\}\right], \quad (3.9)$$

where

$$Q_{iz}(\rho)=(q_{iz,rt})=[C_{iz}(\rho)]^{-1}=[(c_{iz,rt}(\rho))]^{-1}, \quad (3.10)$$

with $c_{iz,rt}=\rho^{t-r}[\mu_{iz,r}\mu_{iz,t}^{-1}]^{\frac{1}{2}}$ as in (2.5). Next, following Lemma 3 in the Appendix, we construct two functions $g_1(x)$ and $g_2(x)$ such that

$$E_x[g_1(x)] = z_{ir}\exp\left\{(z_{ir}-z_{it})'\frac{\beta}{2}\right\}, \text{ and}$$
$$E_x[g_1(x)] = z_{ir}\exp\left\{(z_{ir}+z_{it})'\frac{\beta}{2}\right\}. \quad (3.11)$$

These two functions have the formulas

$$g_1(x_{ir},x_{it};\beta,\Lambda,\phi)=\left[x_{ir}-(\Lambda-\Lambda_\phi)\frac{\beta}{2}\right]\exp\left\{(x_{ir}-x_{it})'\frac{\beta}{2}-\frac{1}{4}\beta'(\Lambda-\Lambda_\phi)\beta\right\}, \quad (3.12)$$

and

$$g_2(x_{ir},x_{it};\beta,\Lambda,\phi)=\left[x_{ir}-(\Lambda+\Lambda_\phi)\frac{\beta}{2}\right]\exp\left\{(x_{ir}+x_{it})'\frac{\beta}{2}-\frac{1}{4}\beta'(\Lambda+\Lambda_\phi)\beta\right\}, \quad (3.13)$$

respectively. By using (3.11) to (3.9), for known correlation elements $q_{iz,rt}$, it then follows that

$$E_x\left[\sum_{i=1}^{K}\sum_{r=1}^{T}\sum_{t=1}^{T}q_{iz,rt}\left[y_{ir}g_1(x)-g_2(x)\right]\right]=\sum_{i=1}^{K}\frac{\partial\mu'_{iz}}{\partial\beta}\Sigma_{iz}^{-1}(y_i-\mu_{iz}). \quad (3.14)$$

Now by Lemma 4 from the Appendix, we re-express the quantity within the square bracket in the left hand side of (3.14) as

$$\sum_{i=1}^{K}\sum_{r=1}^{T}\sum_{t=1}^{T}q_{iz,rt}\left\{y_{ir}g_1(x)-g_2(x)\right\}$$
$$=\sum_{i=1}^{K}\left[\left\{M_{1\phi}X'_i-M_{1\phi}B_{1\phi}(\beta\otimes 1'_T)\right\}\left\{A_{ix}^{\frac{1}{2}}Q_{iz}(\rho)A_{ix}^{-\frac{1}{2}}\right\}y_i\right.$$

$$- \quad \{M_{2\phi}X_i' - M_{2\phi}B_{2\phi}(\beta \otimes 1_T')\} \left\{ A_{ix}^{\frac{1}{2}} Q_{iz}(\rho) A_{ix}^{-\frac{1}{2}} \right\} \mu_{ix} \Big]. \qquad (3.15)$$

Next because $Q_{iz}(\rho)$ in (3.15) has the form $Q_{iz}(\rho) = C_{iz}^{-1}(\rho)$, one may obtain an approximate unbiased estimate for this matrix by using an unbiased estimate for $C_{iz}(\rho)$ defined by (3.10). By Lemma 5, the unbiased estimate for $C_{iz}(\rho)$ is obtained from

$$E[\tilde{C}_{ix}(\rho)] = C_{iz}(\rho) = (c_{iz,rt}), \qquad (3.16)$$

where $\tilde{C}_{ix}(\rho) = (\tilde{c}_{ix,rt})$, with

$$\tilde{c}_{ix,rt} = \rho^{t-r} \left[\exp{(x_{ir} - x_{it})' \frac{\beta}{2}} - \frac{1}{4}\beta'(\Lambda - \Lambda_\phi)\beta \right].$$

Consequently, the desired biased corrected estimating function in (3.8) is obtained from (3.15) by replacing $Q_{iz}(\rho)$ with $\tilde{Q}_{ix} = \tilde{C}_{ix}^{-1}(\rho)$.

3.2.1. Some Remarks on Special Cases.

BCGQL reduces to the GQL Estimation : A Pedagogical Virtue When there is no measurement error, that is, when $\sigma_u^2 = 0$, for all $u = 1, \ldots, p$, or equivalently $X_i = Z_i$, we examine whether the BCGQL estimating equation (3.6) with $g_x(\cdot)$ given by (3.8) reduces to the standard GQL estimating equation (3.3). For the purpose, when $\sigma_u^2 = 0$, both Λ and Λ_ϕ matrices in (2.7) reduce to the null matrix. Consequently, $M_{1\phi}$ and $M_{2\phi}$ used in (3.8) reduce to the unit matrix I_p. Similarly, $B_{1\phi}$ and $B_{2\phi}$ matrices reduce to the $p \times p$ null matrix. Furthermore, $\tilde{c}_{ix,rt}$ in (3.7) reduces to

$$c_{ix,rt} = \rho^{t-r} \left[\exp{(x_{ir} - x_{it})' \frac{\beta}{2}} \right], \qquad (3.17)$$

which, following (2.5), is simply the (r, t)-th element of the correlation matrix $C_{ix}(\rho)$. Thus, in the absence of measurement error, $\tilde{Q}_{ix}(\rho)$ in (3.8) reduces to $Q_{ix}(\rho) = C_{ix}^{-1}(\rho)$. Hence the BCGQL estimating function in (3.8) reduces to

$$\sum_{i=1}^{K} \left[X_i' \left\{ A_{ix}^{\frac{1}{2}} Q_{ix}(\rho) A_{ix}^{-\frac{1}{2}} \right\} y_i - X_i' \left\{ A_{ix}^{\frac{1}{2}} Q_{ix}(\rho) A_{ix}^{-\frac{1}{2}} \right\} \mu_{ix} \right]. \qquad (3.18)$$

Now by using $\Sigma_{ix}(\rho) = A_{ix}^{\frac{1}{2}} C_{ix}(\rho) A_{ix}^{\frac{1}{2}}$, the estimating function in (3.18) yields the estimating equation

$$\sum_{i=1}^{K} \left[X_i' A_{ix} \left\{ A_{ix}^{-\frac{1}{2}} Q_{ix}(\rho) A_{ix}^{-\frac{1}{2}} \right\} (y_i - \mu_{ix}) \right] = \sum_{i=1}^{K} \frac{\partial \mu_{ix}'}{\partial \beta} \Sigma_{ix}^{-1}(\rho)(y_i - \mu_{ix}) = 0, \qquad (3.19)$$

which is (3.3). Thus, the BCGQL estimating equation (3.6) has the pedagogical virtue of reducing to the standard GQL estimating equation (3.3) in the absence of measurement error.

BCGQL reduces to the BCQL Estimation: Another Pedagogical Virtue In this section, we demonstrate that the BCGQL estimating equation (3.6) has the pedagogical virtue of reducing to the simpler BCQL estimating equation under the independence model. To be specific, suppose that on top of the assumption that K individuals are independent, we also assume that the repeated responses $y_{i1}, \ldots, y_{it}, \ldots, y_{iT}$ are independent. This implies that $\rho = 0$ in the AR(1) model (2.3) yielding zero correlations by (2.5). Thus, the correlation matrix defined by (2.5) has the form $C_{iz}(\rho) = I_T$, where I_T is the $T \times T$ identity matrix, further implying that $Q_{iz}(\rho) = C_{iz}^{-1}(\rho) = I_T$. In such a case, one would also use $Q_{ix}(\rho) = I_T$ in the BCGQL estimating function (3.8). Furthermore, because

$$\mathrm{corr}(Y_{ir}, Y_{it}) = \rho^{t-r}\sqrt{\frac{\mu_{iz,r}}{\mu_{iz,t}}}$$

by (2.5), to understand the consequence of $\rho = 0$, i.e., of zero correlations, it is enough to consider the stationary case where $\mu_{iz,t} = \mu_{iz,r} = \exp(z_{i\cdot}'\beta)$, (say). This in turn implies that it is enough to consider that for the same observed covariate (u, say), the correlations between two observations at two different times t and r should approach to 1. That is,

$$\mathrm{corr}(x_{itu}, x_{iru}) = \phi_u \to 1.$$

This implies that $\Lambda_\phi \to \Lambda$. Thus, in (3.6) and (3.8), we use

$$M_{1\phi} \cong I_p, \quad M_{2\phi} \cong \exp\left\{-\frac{1}{2}\beta'\Lambda\beta\right\} I_p, \quad B_{1\phi} \cong 0 \ \text{ and } B_{2\phi} \cong \Lambda.$$

Furthermore, since $Q_{ix}(\rho = 0) = I_T$, the BCGQL estimating function (3.8) reduces to

$$\sum_{i=1}^{K}\left[X_i'y_i - \exp\left(-\frac{1}{2}\beta'\Lambda\beta\right)\{X_i' - \Lambda(\beta \otimes 1_T')\}\,\mu_{ix}\right], \tag{3.20}$$

yielding the estimating equation

$$\sum_{i=1}^{K}\sum_{t=1}^{T}\left[x_{it}y_{it} - (x_{it} - \Lambda\beta)\exp\left\{x_{it}'\beta - \frac{1}{2}\beta'\Lambda\beta\right\}\right] = 0, \tag{3.21}$$

which is referred to as the BCQL estimating equation.

Remark that this BCQL estimating equation (3.21) for β is the same as the bias corrected score (BCS) equation discussed by Nakamura (1990, p. 113)) under the independence set up. Thus the proposed BCGQL estimating equation (3.6) developed under the Poisson AR(1) model (2.3)-(2.5) has the pedagogical virtue of reducing to the well known BCQL estimating equation used in the independence setup.

3.3. Asymptotic Property of the BCGQL Estimator of β. Further remark that the proposed BCGQL estimator is consistent and under some regularity conditions it asymptotically ($K \to \infty$) follows a multivariate Gaussian distribution. More specifically, under the proposed model, $y_1, \ldots, y_i, \ldots, y_K$ are independent to each other but they are not identically distributed because

$$Y_i \sim (\tilde{\mu}_{iz}, \Sigma_{iz}(\beta, \rho)), \tag{3.22}$$

by (3.1). Consequently, under the assumption that multivariate version of Lindeberg's condition holds, by using the Lindeberg-Feller central limit theorem [Amemiya (1985, Theorem 3.3.6), McDonald (2005, Theorem 2.2)] one may show that

$$\lim_{K\to\infty} \hat{\beta}_{BCGQL} \to N(\beta, {V^*}_K^{-1}(\beta, \Lambda, \rho)), \tag{3.23}$$

and also it follows that

$$||[V_K^*(\beta, \Lambda, \rho)]^{\frac{1}{2}}[\hat{\beta}_{BCGQL} - \beta]|| = O_p(\sqrt{p}), \tag{3.24}$$

where $\hat{\beta}_{BCGQL}$ is the solution of the BCGQL estimating equation (3.6), and

$$\begin{aligned} V_K^*(\beta, \Lambda, \rho) &= \sum_{i=1}^{K} V_i(\beta, \Lambda, \rho) \\ &= \sum_{i=1}^{K} \left[\exp\left\{-\frac{1}{4}\beta'\Lambda\beta\right\} \sum_{i=1}^{K} \left[\left\{ X_i' - \frac{1}{2}\Lambda(\beta \otimes 1_T') \right\} A_{ix}^{\frac{1}{2}} \right] \right. \\ &\times \left. \tilde{Q}_{ix}(\rho) \left[A_{ix}^{\frac{1}{2}} \left\{ X_i - \frac{1}{2}(1_T \otimes \beta')\Lambda \right\} \right] \right]. \end{aligned} \tag{3.25}$$

Details for the proof of this asymptotic result is omitted but is available from the authors upon request.

4 A Simulation Study

It has been demonstrated in the last section through an asymptotic study that the observed covariates-based BCGQL estimator is consistent for the regression parameter β which is the effects of the true covariates on the repeated count responses. It was also found through a finite sample based ($K = 100, T = 4$) simulation study (Sutradhar (2013, Section 3.2.2)) that the same BCGQL estimator performs well in producing unbiased and hence consistent estimates for the regression effects. This simulation study was done for a limited set of parameter values, namely for examining the performance of the BCGQL estimates for the regression parameter $\beta = [\beta_1, \beta_2]' \equiv [0.3,\ 0.1]'$ when the observed covariates $x_{it} = [x_{it1}, x_{it2}]'$ differ from the true covariates $z_{it} = [z_{it1}, z_{it2}]'$ following the measurement error model (2.6) with measure error variances $\sigma_1^2 \equiv 0.1,\ 0.3$; and $\sigma_2^2 \equiv 0.3,\ 0.8$. As far as the longitudinal correlation index parameter ρ (see Eq. (2.5)) is concerned, the study considered $\rho \equiv 0.0,\ 0.5$; and furthermore two repeated covariates were assumed to hold equi-correlations (see Eq. (2.7)) with coefficients $\phi_1 \equiv 0.25, 1.0$; and $\phi_2 \equiv 0.5,\ 1.0$.

In this section, further to theempirical study by Sutradhar (2013), we carry out more simulations covering: (1) larger values of correlation index parameter ρ such as with $\rho \equiv 0.3,\ 0.8$, and an additional pair of measurement error variances $\sigma_1^2 = 0.5,\ \sigma_2^2 = 0.2$; (2) a different set of values for the main regression parameters, namely $\beta_1 = \beta_2 = 0.0$, and a wider selection of the values of ρ, namely $\rho \equiv 0.0,\ 0.3,\ 0.5,\ 0.8$; (3) similar parameter values for ρ as in Sutradhar (2013) except using $\beta_1 = \beta_2 = 0.0$, for the main regression parameters. As far as the true covariates are concerned they remain the same as in Sutradhar (2013), thus $Z_1 \sim N(0,1)$ and $Z_2 \sim \frac{\chi_4^2 - 4}{\sqrt{8}}$ and these values were kept fixed under all 500 simulations. The simulation results under the aforementioned 3 setups (1) to (3) are reported in Tables 1, 2, and 3, respectively.

In Table 1, we also provide the naive GQL (NGQL) estimates obtained from (3.3) by ignoring the measurement errors. This we have done to provide a clear feeling and/or motivation for the need of bias correction for measurement errors. As expected, for all selected parameter values, the NGQL approach appears to produce highly biased regression estimates with smaller standard errors as compared to the bias corrected GQL (BCGQL) estimates. Because a biased estimate with a small standard error indicates the convergence problem to the true parameter value, for convenience, we have also presented the percentage relative bias (**%RB**) of the estimates computed, for example, by

$$\mathbf{\%RB}(\hat{\beta}_1) = \frac{|\hat{\beta}_1 - \beta_1|}{s.e.(\hat{\beta}_1)} \times 100.$$

Table 1: Simulated estimates and standard errors (SSEs) of the regression parameters $\beta_1 = 0.3$, $\beta_2 = 0.1$, under AR(1) count data model for selected response correlation ρ, and measurement error variances σ_1^2, σ_2^2, with K=100; T=4; measurement error correlations $\phi_1 = 1.0$ and $\phi_2 = 1.0$; and true covariate values $Z_1 \sim N(0,1)$ and $Z_2 \sim \frac{\chi_4^2 - 4}{\sqrt{8}}$

			Estimates							
			NGQL				BCGQL			
ρ	σ_1^2	σ_2^2	$\hat{\beta}_1$	%RB($\hat{\beta}_1$)	$\hat{\beta}_2$	%RB($\hat{\beta}_2$)	$\hat{\beta}_1$	%RB($\hat{\beta}_1$)	$\hat{\beta}_2$	%RB($\hat{\beta}_2$)
0.3	0.1	0.3	0.2649	57.0	0.0870	28.0	0.2987	2.0	0.1048	7.0
			(0.0614)		(0.0471)		(0.0710)		(0.0553)	
	0.3	0.3	0.2254	131.0	0.0824	37.0	0.3025	3.0	0.1058	10.0
			(0.0569)		(0.0475)		(0.0809)		(0.0577)	
	0.5	0.2	0.1973	192.0	0.0827	36.0	0.3072	8.0	0.1066	11.0
			(0.0535)		(0.0486)		(0.0925)		(0.0584)	
	0.3	0.8	0.2200	144.0	0.0673	76.0	0.3041	5.0	0.1076	11.0
			(0.0555)		(0.0431)		(0.0833)		(0.0661)	
0.8	0.1	0.3	0.2696	36.0	0.0904	15.0	0.3045	5.0	0.1089	12.0
			(0.0836)		(0.0630)		(0.0967)		(0.0739)	
	0.3	0.3	0.2288	94.0	0.0853	23.0	0.3077	7.0	0.1094	13.0
			(0.0755)		(0.0626)		(0.1064)		(0.0748)	
	0.5	0.2	0.2000	143.0	0.0852	23.0	0.3122	10.0	0.1100	13.0
			(0.0701)		(0.0635)		(0.1185)		(0.0741)	
	0.3	0.8	0.2232	104.0	0.0701	52.0	0.3093	8.0	0.1127	15.0
			(0.0736)		(0.0569)		(0.1096)		(0.0823)	

Table 2: Simulated estimates and standard errors (SSEs) of the regression parameters $\beta_1 = 0.0$, $\beta_2 = 0.0$, under AR(1) count data model for selected response correlation ρ, and measurement error variances σ_1^2, σ_2^2, with K=100; T=4; measurement error correlations $\phi_1 = 1.0$ and $\phi_2 = 1.0$; and true covariate values $Z_1 \sim N(0,1)$ and $Z_2 \sim \frac{\chi_4^2 - 4}{\sqrt{8}}$

			Estimates			
ρ	σ_1^2	σ_2^2	$\hat{\beta}_1$	$SSE(\hat{\beta}_1)$	$\hat{\beta}_2$	$SSE(\hat{\beta}_2)$
0.0	0.1	0.1	0.0013	0.0508	0.0017	0.0378
	0.1	0.2	0.0013	0.0508	0.0018	0.0386
	0.1	0.3	0.0013	0.0508	0.0018	0.0393
	0.2	0.1	0.0008	0.0530	0.0018	0.0379
	0.2	0.2	0.0008	0.0530	0.0018	0.0386
	0.2	0.3	0.0008	0.0530	0.0019	0.0394
	0.3	0.1	0.0004	0.0553	0.0019	0.0379
	0.3	0.2	0.0004	0.0553	0.0019	0.0387
	0.3	0.3	0.0004	0.0553	0.0020	0.0394
0.3	0.1	0.1	0.0001	0.0662	0.0021	0.0487
	0.1	0.2	0.0001	0.0663	0.0020	0.0498
	0.1	0.3	0.0001	0.0665	0.0020	0.0509
	0.2	0.1	0.0004	0.0690	0.0022	0.0487
	0.2	0.2	0.0004	0.0692	0.0022	0.0498
	0.2	0.3	0.0004	0.0693	0.0021	0.0509
	0.3	0.1	0.0007	0.0718	0.0024	0.0487
	0.3	0.2	0.0007	0.0719	0.0023	0.0498
	0.3	0.3	0.0007	0.0721	0.0022	0.0509
0.5	0.1	0.1	−0.0003	0.0723	0.0021	0.0534
	0.1	0.2	−0.0003	0.0723	0.0020	0.0547
	0.1	0.3	−0.0003	0.0723	0.0020	0.0561
	0.2	0.1	0.0000	0.0760	0.0024	0.0536
	0.2	0.2	0.0000	0.0760	0.0023	0.0549
	0.2	0.3	0.0000	0.0760	0.0023	0.0562
	0.3	0.1	0.0003	0.0795	0.0026	0.0537
	0.3	0.2	0.0002	0.0795	0.0025	0.0550
	0.3	0.3	0.0002	0.0795	0.0025	0.0563
0.8	0.1	0.1	0.0038	0.0897	0.0118	0.0616
	0.1	0.2	0.0038	0.0897	0.0122	0.0630
	0.1	0.3	0.0038	0.0898	0.0127	0.0644
	0.2	0.1	0.0043	0.0933	0.0120	0.0617
	0.2	0.2	0.0043	0.0934	0.0125	0.0631
	0.2	0.3	0.0043	0.0935	0.0129	0.0645
	0.3	0.1	0.0048	0.0971	0.0123	0.0618
	0.3	0.2	0.0048	0.0972	0.0127	0.0632
	0.3	0.3	0.0047	0.0973	0.0131	0.0646

Table 3: Simulated estimates and standard errors (SSEs) of the regression parameters $\beta_1 = 0.0$, $\beta_2 = 0.0$, under AR(1) count data model for selected response correlation ρ, and measurement error variances σ_1^2, σ_2^2, with K=100; T=4; measurement error correlations $\phi_1 = 0.25$ and $\phi_2 = 0.5$; and true covariate values $Z_1 \sim N(0,1)$ and $Z_2 \sim \frac{\chi_4^2 - 4}{\sqrt{8}}$

			Estimates			
ρ	σ_1^2	σ_2^2	$\hat{\beta}_1$	$SSE(\hat{\beta}_1)$	$\hat{\beta}_2$	$SSE(\hat{\beta}_2)$
0.0	0.1	0.1	0.0020	0.0479	0.0012	0.0364
	0.1	0.2	0.0020	0.0479	0.0011	0.0361
	0.1	0.3	0.0020	0.0478	0.0010	0.0359
	0.2	0.1	0.0018	0.0469	0.0013	0.0364
	0.2	0.2	0.0018	0.0469	0.0011	0.0361
	0.2	0.3	0.0018	0.0469	0.0010	0.0359
	0.3	0.1	0.0016	0.0460	0.0013	0.0364
	0.3	0.2	0.0016	0.0460	0.0011	0.0361
	0.3	0.3	0.0016	0.0460	0.0010	0.0359
0.5	0.1	0.1	0.0000	0.0712	0.0026	0.0542
	0.1	0.2	0.0000	0.0712	0.0027	0.0556
	0.1	0.3	0.0000	0.0712	0.0029	0.0567
	0.2	0.1	0.0003	0.0736	0.0034	0.0546
	0.2	0.2	0.0003	0.0736	0.0035	0.0559
	0.2	0.3	0.0003	0.0737	0.0037	0.0571
	0.3	0.1	0.0005	0.0753	0.0042	0.0549
	0.3	0.2	0.0005	0.0754	0.0043	0.0563
	0.3	0.3	0.0005	0.0755	0.0045	0.0575

The results of the table show that the relative bias is quite small for the BCGQL estimates ranging from 2.0 % to 10.0 % for β_1 and 7.0 % to 15.0 % for β_2, whereas the NGQL estimates exhibit very large relative bias, namely ranging from 57 % to 192 % for the estimation of β_1 and from 15.0 % to 76 % for the estimation of β_2.

Note that unlike in Table 1, the simulation results reported in Tables 2 and 3 were obtained for a different set of values for the main regression parameters, namely for $\beta_1 = \beta_2 = 0.0$. We however do not provide the NGQL estimates any more as this NGQL estimator produces highly biased estimates as also evident by the results from Table 1. In stead we have chosen more values of correlation index parameter ρ along with various choices of the measurement error variances in order to examine the performance of the BCGQL approach in estimating $\beta_1 = \beta_2 = 0.0$. The results of Table 2 show that the BCGQL estimates are very close to the true parameter values for any value of ρ, from 0.0 to 0.8. The standard errors of these estimates are reasonably small and they are relatively smaller as compared to those for the BCGQL estimates in Table 1 for true parameter values $\beta_1 = 0.3,\ \beta_2 = 0.1$. To be specific, for correlation index parameter value $\rho = 0.8$ and measurement error variances $\sigma_1^2 = 0.3,\ \sigma_2^2 = 0.3$, the true parameter $\beta_1 = 0.0$ was estimated as 0.0047 with standard error 0.0973, and $\beta_2 = 0.0$ was estimated as 0.0131 with standard error 0.0646. Thus, the BCGQL estimates are almost unbiased showing the benefit of bias correction for correlated count data in the presence of measurement error in covariates. One may interpret the results of Table 3 similarly. The main difference between Table 2 and 3 is the choice of the values of measurement error correlations. The simulation results reported in Table 2 were computed for $\phi_1 = \phi_2 = 1.0$, whereas Table 3 results were based on $\phi_1 = 0.25$ and $\phi_2 = 0.5$. The BCGQL estimates are also seen to be almost unbiased under Table 3.

5 Estimation of Nuisance Parameters

In estimating the main regression parameter β, it was assumed in Section 3 that the longitudinal correlation index parameter ρ, the measurement error correlations ϕ_u and the measurement error variances σ_u^2, for $u = 1, \ldots, p$, are known. In practice, these parameters are, however, unknown. In this section, we provide an iterative technique to estimate these parameters by using method of moments.

5.1. A Bias Correction Estimation for the Correlation Index Parameter ρ. Note that in developing this bias corrected moment (BCM) estimate, similar to the estimation for β, we continue to assume that the measurement error variances $\sigma_1^2, \ldots, \sigma_u^2, \ldots, \sigma_p^2$ and the time dependent measurement error correlations $\phi_1, \ldots, \phi_u, \ldots, \phi_p$, are known. The estimating formulas for these variances and correlations are given in the next section.

Remark that if true covariates z_{it} were known, that is when there is no measurement error, the measurement error model for repeated count data discussed in Section 2 reduces to the non-stationary correlation model for repeated count data studied by Sutradhar (2010), for example. Consequently, in the absence of measurement error, one would have estimated the correlation index parameter ρ in (3.1)-(3.2) [see also the model (2.5)] by solving the moment estimating equation

$$\sum_{i=1}^{K}\sum_{t=1}^{T-1}\left[\frac{(y_{it}-\mu_{iz,t})(y_{i,t+1}-\mu_{iz,t+1})}{\{\mu_{iz,t}\mu_{iz,t+1}\}^{\frac{1}{2}}}\right] - \rho\sum_{i=1}^{K}\sum_{t=1}^{T-1}\left\{\frac{\mu_{iz,t}}{\mu_{iz,t+1}}\right\}^{\frac{1}{2}} = 0, \quad (5.1)$$

[see also Sutradhar (2010, eqn. (6.8), p. 189)]. In view of (5.1), by using observed covariates, that is, by replacing the true covariates with observed covariates, we first write a naive moment estimating equation for ρ as

$$\sum_{i=1}^{K}\sum_{t=1}^{T-1}\left[\frac{(y_{it}-\mu_{ix,t})(y_{i,t+1}-\mu_{ix,t+1})}{\{\mu_{ix,t}\mu_{ix,t+1}\}^{\frac{1}{2}}}\right]-\hat{\rho}_{\text{Naive}}\sum_{i=1}^{K}\sum_{t=1}^{T-1}\left\{\frac{\mu_{ix,t}}{\mu_{ix,t+1}}\right\}^{\frac{1}{2}}=0. \tag{5.2}$$

Now because x_{it} and $x_{i,t+1}$ follow the joint and conditional normal distributional properties as in (2.7) and (6.12)-(6.13) [see the Appendix], respectively, it follows from (6.10) and (6.11) in the Appendix that

$$\begin{aligned} E\left[\exp\left\{(x_{it}+x_{i,t+1})'\frac{\beta}{2}-\frac{1}{4}\beta'(\Lambda+\Lambda_\phi)\beta\right\}\right] &= \exp\left\{(z_{it}+z_{i,t+1})'\frac{\beta}{2}\right\}, \\ E\left[\exp\left\{-\frac{1}{2}(x_{it}+x_{i,t+1})'\beta-\frac{1}{4}\beta'(\Lambda+\Lambda_\phi)\beta\right\}\right] & \\ = \exp\left\{-\frac{1}{2}(z_{it}+z_{i,t+1})'\beta\right\}. & \end{aligned} \tag{5.3}$$

yielding

$$\begin{aligned} E[\{\mu_{ix,t}\mu_{ix,t+1}\}^{\frac{1}{2}}] &= m_2[\{\mu_{iz,t}\mu_{iz,t+1}\}^{\frac{1}{2}}], \text{ and} \\ E[\{\mu_{ix,t}\mu_{ix,t+1}\}^{-\frac{1}{2}}] &= m_2[\{\mu_{iz,t}\mu_{iz,t+1}\}^{-\frac{1}{2}}], \end{aligned} \tag{5.4}$$

respectively, where m_2 is defined in (3.8). Similarly

$$\begin{aligned} E\left[\exp\left\{(x_{it}-x_{i,t+1})'\frac{\beta}{2}-\frac{1}{4}\beta'(\Lambda-\Lambda_\phi)\beta\right\}\right] &= \exp\left\{(z_{it}-z_{i,t+1})'\frac{\beta}{2}\right\}, \\ E\left[\exp\left\{-\frac{1}{2}(x_{it}-x_{i,t+1})'\beta-\frac{1}{4}\beta'(\Lambda-\Lambda_\phi)\beta\right\}\right] & \\ = \exp\left\{-\frac{1}{2}(z_{it}-z_{i,t+1})'\beta\right\}. & \end{aligned} \tag{5.5}$$

yielding

$$\begin{aligned} E\left[\left\{\frac{\mu_{ix,t}}{\mu_{ix,t+1}}\right\}^{\frac{1}{2}}\right] &= m_1\left[\left\{\frac{\mu_{iz,t}}{\mu_{iz,t+1}}\right\}^{\frac{1}{2}}\right], \text{ and} \\ E\left[\left\{\frac{\mu_{ix,t}}{\mu_{ix,t+1}}\right\}^{-\frac{1}{2}}\right] &= m_1\left[\left\{\frac{\mu_{iz,t}}{\mu_{iz,t+1}}\right\}^{-\frac{1}{2}}\right], \end{aligned} \tag{5.6}$$

respectively, with m_1 as defined in (3.8).

Next, by combining (5.4) and (5.6), and using the bias corrected moment estimate $\hat{\rho}_{\text{BCM}}$ for $\hat{\rho}_{\text{Naive}}$ in (5.2), we write

$$\begin{aligned} E_{x|y}\Bigg[\sum_{i=1}^{K}\sum_{t=1}^{T-1}\Bigg[&\frac{m_2 y_{it} y_{i,t+1}}{\{\mu_{ix,t}\mu_{ix,t+1}\}^{\frac{1}{2}}} - m_1 y_{it}\left\{\frac{\mu_{ix,t+1}}{\mu_{ixt}}\right\}^{\frac{1}{2}} - m_1 y_{i,t+1}\left\{\frac{\mu_{ixt}}{\mu_{ix,t+1}}\right\}^{\frac{1}{2}} \\ + \quad & m_2\{\mu_{ix,t}\mu_{ix,t+1}\}^{\frac{1}{2}}\Bigg] - \hat{\rho}_{BCM} m_1 \sum_{i=1}^{K}\sum_{t=1}^{T-1}\left\{\frac{\mu_{ix,t}}{\mu_{ix,t+1}}\right\}^{\frac{1}{2}}\Bigg] \\ = \quad & \sum_{i=1}^{K}\sum_{t=1}^{T-1}\left[\frac{(y_{it}-\mu_{iz,t})(y_{i,t+1}-\mu_{iz,t+1})}{\{\mu_{iz,t}\mu_{iz,t+1}\}^{\frac{1}{2}}}\right] - \rho\sum_{i=1}^{K}\sum_{t=1}^{T-1}\left\{\frac{\mu_{iz,t}}{\mu_{iz,t+1}}\right\}^{\frac{1}{2}} = 0. \end{aligned} \tag{5.7}$$

Consequently, the BCM estimate of ρ has the formula

$$\begin{aligned} \hat{\rho}_{BCM} \quad = \quad & \left[\sum_{i=1}^{K}\sum_{t=1}^{T-1}\left[\frac{m_2 y_{it} y_{i,t+1}}{\{\mu_{ixt}\mu_{ix,t+1}\}^{\frac{1}{2}}} - m_1 y_{it}\left\{\frac{\mu_{ix,t+1}}{\mu_{ixt}}\right\}^{\frac{1}{2}}\right.\right. \\ - \quad & \left.\left. m_1 y_{i,t+1}\left\{\frac{\mu_{ixt}}{\mu_{ix,t+1}}\right\}^{\frac{1}{2}} + m_2\left\{\frac{\mu_{ixt}}{\mu_{ix,t+1}}\right\}^{\frac{1}{2}}\right]\right] \\ \div \quad & \left[m_1\sum_{i=1}^{K}\sum_{t=1}^{T-1}\left\{\frac{\mu_{ixt}}{\mu_{ix,t+1}}\right\}^{\frac{1}{2}}\right]. \end{aligned} \tag{5.8}$$

Note that the aforementioned computation ensures that $E[\hat{\rho}_{BCM}] = \rho$. Thus, $\hat{\rho}_{BCM} \to \rho$ in probability.

5.2. Moment Estimation for Measurement Error Variances and Correlations. Recall from the the measurement error model (2.6)-(2.7) that

$$x_{itu} \sim N(z_{itu}, \sigma_u^2), \text{ for } u = 1, \ldots, p,$$

and

$$corr(x_{itu}, x_{iru}) = \phi_u.$$

These variance and correlation parameters were assumed to be known while computing the BCGQL estimate of β by (3.6) [see also (3.8)], and the BCM estimate of ρ by using (5.8). Note however that the estimation of these parameters has nothing to do with the repeated responses as they are simply measurement error model (2.6)-(2.7) parameters. As far as the repeated time dependent observed covariates are concerned, their variance-covariance matrix may be written as

$$\mathrm{cov}[(x_{i1u}, \ldots, x_{itu}, \ldots, x_{iTu})'] = \sigma_u^2[\phi_u U_T - \phi_u I_T], \tag{5.9}$$

where U_T is the $T \times T$ unit matrix and I_T is the $T \times T$ identity matrix. One may consequently use the repeated measurements for the selected covariates to obtain the unbiased estimates of the variance and their correlation. Thus, we estimate ϕ_u by using the method of moments(MM) as

$$\hat{\phi}_{u,MM} = \frac{\sum_{i=1}^{K}\sum_{t\neq r}^{T}(x_{itu} - \bar{x}_{i.u})(x_{iru} - \bar{x}_{i.u})/KT(T-1)}{\hat{\sigma}^2_{u,MM}}, \tag{5.10}$$

where$\hat{\sigma}^2_{u,MM}$ is the MM estimate of the variance σ_u^2 given by

$$\hat{\sigma}^2_{u,MM} = \sum_{i=1}^{K}\sum_{t=1}^{T}(x_{itu} - \bar{x}_{i.u})^2/KT, \tag{5.11}$$

with $\bar{x}_{i.u} = \sum_{t=1}^{T} x_{itu}/T$. Note that the above two unbiased moment estimators obtained from (5.10) and (5.11) would be consistent provided KT is sufficiently large, K being already large in the longitudinal setup.

6 Concluding Remarks

In this paper, we have used an AR(1) type Poisson correlation model for repeated count responses and discussed an observed covariates based BCGQL estimation approach for the main regression parameters of interest. A part of the paper was devoted to the justification of the formula for the BCGQL estimating equation for the regression parameters. The paper also examined the asymptotic and finite sample properties of the BCGQL estimator for the regression effects. It is demonstrated that the bias corrected GQL approach produces unbiased and hence consistent estimates under mild regularity conditions. Notice that the correlation model used in the paper is quite practical as it allows exponential decaying among correlations as lag between repeated observations increases. Similar correlation structure was used by Staudenmayer and Buonaccorsi (2005) in a Gaussian time series measurement error setup. For correlations among repeated covariates, we have used equi-correlation structure which is similar but different than Wang et al. (1996). These secondary parameters, namely the longitudinal correlations, measurement error variances, and the correlations among time dependent observed covariates are required for the BCGQL estimation of the regression parameters. In this paper, these secondary parameters are estimated consistently by using the method of moments.

Note that the development of a similar BCGQL estimation approach for longitudinal binary data subject to measurement error in covariates would be much more complex. This is because as opposed to the count data model, the binary logit form, for example, contains measurement error based observed covariates in a complicated way. We leave such a development to further research.

Acknowledgments. A significant part of this research was done in Sri Sathya Sai University many years ago. The authors are grateful to Bhagawan Sri Sathya Sai Baba for the opportunities to carry out this research in Sri Sathya Sai University. The authors thankfully acknowledges the valuable comments and suggestions made by the referee that has significantly improved the presentation of the paper. Also, the authors thank Rafiqul Islam Chowdhury for his computational help.

Appendix

Lemma 1. *The true GQL estimating function in the right hand side of* (3.5) *or in the left hand side of* (3.1) *has the simplified form given by*

$$\sum_{i=1}^{K} \frac{\partial \mu'_{iz}}{\partial \beta} \Sigma_{iz}^{-1}(y_i - \mu_{iz}) = \sum_{i=1}^{K}\sum_{r=1}^{T}\sum_{t=1}^{T} q_{iz,rt}\left[y_{ir} z_{ir} \exp\left\{(z_{ir} - z_{it})'\frac{\beta}{2}\right\} - z_{ir}\exp\left\{(z_{ir} + z_{it})'\frac{\beta}{2}\right\}\right]. \tag{6.1}$$

Proof. *We first express the true covariates* $\{z_{it}\}$ *based GQL function given in the right hand side of* (3.5) *as*

$$\begin{aligned} \sum_{i=1}^{K} \frac{\partial \mu'_{iz}}{\partial \beta} \Sigma_{iz}^{-1}(y_i - \mu_{iz}) &= \sum_{i=1}^{K} Z_i' A_{iz}\left[A_{iz}^{\frac{1}{2}} C_{iz}(\rho) A_{iz}^{\frac{1}{2}}\right]^{-1}(y_i - \mu_{iz}) \\ &= \sum_{i=1}^{K} Z_i' A_{iz}^{\frac{1}{2}} C_{iz}^{-1}(\rho) A_{iz}^{-\frac{1}{2}}(y_i - \mu_{iz}) \end{aligned} \tag{6.2}$$

where $Z_i' = [z_{i1}, \ldots, z_{it}, \ldots, z_{iT}] : \ p \times T; A_{iz} = diag[\mu_{iz,1}, \ldots, \mu_{iz,t}, \ldots, \mu_{iz,T}]$, *and* $C_{iz}(\rho) = (c_{iz,rt}(\rho))$ *is the* $T \times T$ *correlation matrix of* y_i *with its* (r,t)*-th elements given by* $c_{iz,rt} = \rho^{t-r}[\mu_{iz,r}\mu_{iz,t}^{-1}]^{\frac{1}{2}}$ *as in* (2.5). *Next because* $\mu_{iz} = [\mu_{iz,1}, \ldots, \mu_{iz,r}, \ldots, \mu_{iz,T}]'$ *with* $\mu_{iz,r} = \exp(z_{ir}'\beta)$, *the GQL estimating* *function in* (6.2) *may be expressed in the form of* (6.1) *by writing* $C_{iz}^{-1}(\rho) = Q_{iz}(\rho) = (q_{iz,rt}) : T \times T$.

Lemma 2. *For notational convenience, the GQL function in* (6.1) *under Lemma 1 may be re-expressed as*

$$\sum_{i=1}^{K} \frac{\partial \mu_{iz}'}{\partial \beta} \Sigma_{iz}^{-1}(y_i - \mu_{iz}) = \sum_{i=1}^{K}\sum_{r=1}^{T}\sum_{t=1}^{T} q_{iz,rt}[y_{ir} G_{1,rt}(z_{ir}, z_{it}; \beta) - G_{2,rt}(z_{ir}, z_{it}; \beta)], \tag{6.3}$$

where

$$G_{1,rt}(z_{ir}, z_{it}; \beta) = z_{ir} \exp\left\{(z_{ir} - z_{it})' \frac{\beta}{2}\right\} : p \times 1$$
$$G_{2,rt}(z_{ir}, z_{it}; \beta) = z_{ir} \exp\left\{(z_{ir} + z_{it})' \frac{\beta}{2}\right\} : p \times 1. \tag{6.4}$$

Lemma 3. *Under the assumption that the correlation matrix* $Q_{iz}(\rho) = (q_{iz,rt})$ *is known for the time being, the GQL estimating function in* (6.3) *may be unbiased estimated by using the function*

$$\sum_{i=1}^{K}\sum_{r=1}^{T}\sum_{t=1}^{T} q_{iz,rt}\left[y_{ir} g_1(x_{ir}, x_{it}; \beta, \Lambda, \phi) - g_2(x_{ir}, x_{it}; \beta, \Lambda, \phi)\right], \tag{6.5}$$

where

$$g_1(x_{ir}, x_{it}; \beta, \Lambda, \phi) = \left[x_{ir} - (\Lambda - \Lambda_\phi)\frac{\beta}{2}\right] \exp\left\{(x_{ir} - x_{it})'\frac{\beta}{2} - \frac{1}{4}\beta'(\Lambda - \Lambda_\phi)\beta\right\}, \tag{6.6}$$

$$g_2(x_{ir}, x_{it}; \beta, \Lambda, \phi) = \left[x_{ir} - (\Lambda + \Lambda_\phi)\frac{\beta}{2}\right] \exp\left\{(x_{ir} + x_{it})'\frac{\beta}{2} - \frac{1}{4}\beta'(\Lambda + \Lambda_\phi)\beta\right\}. \tag{6.7}$$

Proof. *Because* $x_{it} \sim N(z_{it}, \Lambda)$, *it can be shown that*

$$E[\exp(x_{it}'\beta)] = \exp\left[z_{it}'\beta + \frac{1}{2}\beta'\Lambda\beta\right] \tag{6.8}$$
$$E[x_{it}\exp(x_{it}'\beta)] = [z_{it} + \Lambda\beta] \exp\left[z_{it}'\beta + \frac{1}{2}\beta'\Lambda\beta\right]. \tag{6.9}$$

Also because $[x_{ir}', x_{it}']'$ *follows the* $2p$*-dimensional normal distribution as in* (2.7) *with* Λ_ϕ *as the covariance matrix between* x_{ir} *and* x_{it}, *it then follows by using the the multi-normal moment generating function* (6.8) *that*

$$E_x\left[\exp\left\{\frac{\beta}{2}(x_{ir} + x_{it})'\right\}\right] = \exp\left[\frac{\beta}{2}(z_{ir} + z_{it})' + \frac{1}{4}\beta'(\Lambda + \Lambda_\phi)\beta\right], \tag{6.10}$$

$$E_x\left[\exp\{-\frac{\beta}{2}(x_{ir}+x_{it})'\}\right]=\exp[-\frac{\beta}{2}(z_{ir}+z_{it})'+\frac{1}{4}\beta'(\Lambda+\Lambda_\phi)\beta]. \quad (6.11)$$

Next, using conditioning and un-conditioning principle, we write

$$E\left[x_{ir}\exp\left\{(x_{ir}+x_{it})'\frac{\beta}{2}\right\}\right]=E_{x_{it}}\exp\left(x_{it}'\frac{\beta}{2}\right)\left[E_{x_{ir}}\left\{x_{ir}\exp(x_{ir}'\frac{\beta}{2})\right\}|x_{it}\right]. \quad (6.12)$$

Because by (2.7)*, the conditional distribution of* x_{ir} *given* x_{it} *has the form*

$$x_{ir}|x_{it}\sim N_p[z_{ir}+\Lambda_\phi\Lambda^{-1}(x_{it}-z_{it}),\Lambda-\Lambda_\phi\Lambda^{-1}\Lambda_\phi],$$

it then follows that

$$\begin{aligned}&E\left[x_{ir}\exp\left(x_{ir}'\frac{\beta}{2}\right)|x_{it}\right] &(6.13)\\ =\ &\left[z_{ir}+\Lambda_\phi\Lambda^{-1}(x_{it}-z_{it})+C_\phi\frac{\beta}{2}\right]\exp\left[\{z_{ir}+\Lambda_\phi\Lambda^{-1}(x_{it}-z_{it})\}'\frac{\beta}{2}+\frac{1}{8}\beta'C_\phi\beta\right],\end{aligned}$$

where $C_\phi=\Lambda-\Lambda_\phi\Lambda^{-1}\Lambda_\phi$ *is the conditional covariance matrix of* x_{ir} *given* x_{it}*. Now by using* (6.13) *in* (6.12)*, after some algebra one obtains*

$$\begin{aligned}E\left[x_{ir}\exp\left\{(x_{ir}+x_{it})'\frac{\beta}{2}\right\}\right] &= \exp\left\{w_{irt}'\frac{\beta}{2}+\frac{1}{8}\beta'C_\phi\beta\right\}\\ &\left[\left(w_{irt}+C_\phi\frac{\beta}{2}\right)E_{x_{it}}\left\{\exp[\frac{\beta'}{2}(I_p+\Lambda_\phi\Lambda^{-1})x_{it}]\right\}[2ex]\right.\\ &+ \left.\Lambda_\phi\Lambda^{-1}E_{x_{it}}x_{it}\exp\left\{\left(\frac{\beta'}{2}\right)(I_p+\Lambda_\phi\Lambda^{-1})x_{it}\right\}\right], \quad (6.14)\end{aligned}$$

where $w_{irt}=z_{ir}-\Lambda_\phi\Lambda^{-1}z_{it}$*. Next, by evaluating the two expectations in* (6.14) *following* (6.8)-(6.9) *for any* t*, after further algebras, we obtain*

$$\begin{aligned}E\left[x_{ir}\exp\{(x_{ir}+x_{it})'\frac{\beta}{2}\right] &= \exp\left[\left\{w_{irt}'\frac{\beta}{2}+\frac{1}{8}\beta'C_\phi\beta\right\}+\frac{\beta'}{2}(I_p+\Lambda_\phi\Lambda^{-1})z_{it}+\frac{\xi^*}{8}\right]\\ &\times\left[\left(w_{irt}+C_\phi\frac{\beta}{2}\right)+\Lambda_\phi\Lambda^{-1}\left\{z_{it}+\Lambda(I_p+\Lambda_\phi\Lambda^{-1})\frac{\beta'}{2}\right\}\right]\\ &= \exp[(z_{ir}+z_{it})'\frac{\beta}{2}+\frac{1}{8}(\xi^*+\beta'C_\phi\beta)]\left\{z_{ir}+(\Lambda+\Lambda_\phi)\frac{\beta}{2}\right\}\end{aligned} \quad (6.15)$$

with $\xi^* = \beta'(\Lambda + \Lambda_\phi)(I_p + \Lambda_\phi\Lambda^{-1})\beta$, *yielding*

$$E\left[x_{ir}\exp\left\{(x_{ir}+x_{it})'\frac{\beta}{2} - \frac{1}{8}(\xi^* + \beta' C_\phi \beta)\right\}\right] = \left[z_{ir} + (\Lambda + \Lambda_\phi)\frac{\beta}{2}\right]\exp\left\{(z_{ir}+z_{it})'\frac{\beta}{2}\right\}, \tag{6.16}$$

that is,

$$E[g_2(x_{ir}, x_{it}; \beta, \Lambda, \phi)] = z_{ir}\exp\left\{(z_{ir}+z_{it})'\frac{\beta}{2}\right\} \tag{6.17}$$

showing that $g_2(x_{ir}, x_{it}; \beta, \Lambda, \phi)$ *in* (6.7) *is an unbiased estimating function for* $G_{2,rt}(z_{ir}, z_{it}; \beta)$ *in* (6.4). *In similar manner, one may show that* $g_1(x_{ir}, x_{it}; \beta, \Lambda, \phi)$ *in* (6.6) *is an unbiased estimating function for* $G_{1,rt}(z_{ir}, z_{it}; \beta) = z_{ir}\exp\left\{(z_{ir} - z'_{it})\frac{\beta}{2}\right\}$ *in* (6.4).

Lemma 4. *Turning back to the matrix and vector notation, the estimating function in* (6.5) *may be expressed as*

$$\begin{aligned}
&\sum_{i=1}^{K}\sum_{r=1}^{T}\sum_{t=1}^{T} q_{iz,rt}\left\{y_{ir}g_1(x_{ir}, x_{it}; \beta, \Lambda, \phi) - g_2(x_{ir}, x_{it}; \beta, \Lambda, \phi)\right\} \\
=\ &\sum_{i=1}^{K}\Bigg[\{M_{1\phi}X_i' - M_{1\phi}B_{1\phi}(\beta\otimes 1_T')\}\left\{A_{ix}^{\frac{1}{2}}Q_{iz}(\rho)A_{ix}^{-\frac{1}{2}}\right\}y_i \\
-\ &\{M_{2\phi}X_i' - M_{2\phi}B_{2\phi}(\beta\otimes 1_T')\}\left\{A_{ix}^{\frac{1}{2}}Q_{iz}(\rho)A_{ix}^{-\frac{1}{2}}\right\}\mu_{ix}\Bigg],
\end{aligned} \tag{6.18}$$

where all notations except for the formula for $Q_{iz}(\rho)$ *are defined in* (3.8), *and by* (3.10) $Q_{iz}(\rho)$ *is the inverse of the true correlation matrix. More specifically,* $Q_{iz}(\rho) = C_{iz}^{-1}(\rho)$, *where*

$$C_{iz}(\rho) = (c_{iz,rt}(\rho)),\ \textit{with}\ c_{iz,rt} = \rho^{t-r}[\mu_{iz,r}\mu_{iz,t}^{-1}]^{\frac{1}{2}}, \tag{6.19}$$

as in (2.5).

Lemma 5. *Recall from* (3.7) *that* $\tilde{c}_{ix,rt} = \rho^{t-r}[\exp(x_{ir} - x_{it})'\frac{\beta}{2} - 1\frac{1}{4}\beta'(\Lambda - \Lambda_\phi)\beta]$ *is an unbiased estimate of* $c_{iz,rt} = \rho^{t-r}\exp\{(z_{ir} - z_{it})'\frac{\beta}{2}\}$, *yielding*

$$E[\tilde{C}_{ix}(\rho)] = C_{iz}(\rho) = (c_{iz,rt}),$$

where $\tilde{C}_{ix}(\rho) = (\tilde{c}_{ix,rt})$.

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Paper received: 19 January 2015.