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Risk Diversifying Treaty Between Two Companies with Only One in Insurance Business

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Abstract

We consider an insurance company (Company 1) and another company (Company 2) operating under a risk diversification treaty; we assume that Company 2 does not have any insurance business of its own. Company 2 takes care of a pre-agreed fraction of any possible deficit that Company 1 may face; in return, Company 2 gets a retainer fee at a constant rate. (The situation can also be looked upon as Company 1 acting as a subsidiary of Company 2.) The joint dynamics is modelled in terms of appropriate Skorokhod problem in the quadrant. Corresponding ruin problem is studied, and advantages of the treaty are pointed out. It is shown that ruin probability decays at a faster rate under the treaty. Some numerical results are also presented to project advantages of our formal model.

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1 Introduction

Risk diversification is a strategy where part of a potential liability/ loss is transferred to another company/ agent, perhaps at a cost; an essential requirement is that no agency can transfer the entire liability to another in a riskless/ costless fashion. This is a prevalent aspect of insurance. Even the basic insurance contract between the insured (i.e., policy holder) and the insurer (insurance company) is an instance of the policy holder transferring his risk to the insurer by paying the premium regularly; the so called net profit condition, together with a loading factor, generally ensures that the insurance company is not exposed to a sure ruin in finite time. Various forms of reinsurance are also risk diversifying methods. Ruin problems in the context of one company, that is, questions and results concerning probability of ruin of the insurance company in finite time, have been a central part of the subject and extensively investigated; Asmussen and Albrecher (2010), Embrechts et al. (1997), Rolski et al. (1999) are authoritative treatises. For connections with economics see Buhlman (1970), Kaas et al. (2001) and references therein; for a recent work see Bhattacharya et al. (2013).

As Buhlman has aptly put it, though the term "ruin" is the usual terminology (while considering a single insurance company), it is a bit unfortunate. The concerned event here does not imply the insurer crashing out of business, but only highlights a "need for additional capital"; see p.133 of Buhlman (1970). It is also referred to as "capital injection by the shareholders of the company" in Dickson and Waters (2004). It is then natural to consider an optimal way for capital injection so that the company's capital does not go below 0. Well-known optimality aspects of solution to one dimensional Skorokhod problem in $[0, \infty)$ (see Harrison (1985)) suggest an optimal way.

Optimality properties of the Skorokhod problem in an orthant have been investigated in Reiman (1984), Chen and Mandelbaum (1991), Ramasubramanian (2000). In an extension, a formal multidimensional insurance model with a risk diversifying treaty has been proposed in Ramasubramanian (2006), viewing the setup as an $n-$person dynamic game; in such a case the pushing part of the solution to the Skorokhod problem in an orthant provides a (unique) Nash equilibrium. According to the treaty, when a company in the network needs an amount to prevent its surplus from getting wiped out, the required capital injection is obtained from other companies in the network, as well as from the shareholders/ external sources in pre-agreed proportions; and the optimal way to go about is provided by the Skorokhod problem. See Ramasubramanian (2011) for more details and discussion. Moreover, in this frame-work there is also a natural notion of ruin of the network, which is similar to the one dimensional analogue; see Ramasubramanian (2012). It may be mentioned that the model considered in these papers, and continued here, is only a formal model, and not an empirical model based on data; see Remark 1.1 below.

The purpose of this paper is to study the advantages of the risk diversifying treaty, and this too in a limited context. We consider the situation of one company (denoted Company 1 in the sequel) doing insurance business, and another company (say, Company 2) providing a pre-agreed proportion of capital injection to the Company 1 as and when the need arises; in return, Company 1 pays Company 2 a retainer fee at a fixed rate; Company 2 is assumed to be not having any insurance business of its own. Thejoint dynamics of the two companies are governed by the relevant Skorokhod problem in the quadrant. (The situation can also be looked upon as Company 1 acting as a subsidiary of Company 2. See Remark 3.9 and the subsequent note.) Probability of ruin of the network in finite time is to be analysed. In Section 2, besides presenting the renewal-risk-type setup, we show that the two dimensional ruin problem is equivalent to a one dimensional ruin problem in the classical setup. This is used in Section 3 to study the advantages of operating under the treaty for both companies, with particular reference to the asymptotic decay of ruin probability. For the renewal-risk-type model, we consider the case of light tailed distributions; we are able to compare the Lundberg adjustment coefficients with treaty and without treaty for Company 1. For the Cramer-Lundberg model, in addition, we compare the asymptotic behaviours in the case of subexponential claim size distributions. Also some advantages over proportional reinsurance is indicated. We take a brief look at finite horizon ruin probability in Section 4. Some numerical results are presented in Section 5. As part of concluding remarks in Section 6, some possible directions for future investigations are indicated. Proofs of all the results are given in an Appendix.

Section 5 includes a real life scenario involving theft claims having heavy-tailed Pareto distribution; the rather long remark below explains the general context in which the latter example may be viewed. As it might also give a proper perspective for the paper, it is presented at the end of this introduction.

Remark 1.1. *(i) We begin by recalling the role of one-dimensional ruin problems in classical (actuarial) risk theory. Note that an insurance set up involves statistical/ economic/ probabilistic, as well as legal aspects; the last one is due to an insurance policy being basically a legal contract, subject to approval from regulatory body; see for example Dorfman (2005). The two quantities subject to randomness are claim arrival times and claim sizes. Distributions of these, along with all the relevant parameters, are determined using statistical methods; see Boland (2007) for example. Once these distributions are decided upon, the insurer's objective is to fix an appropriate premium rate so that the set up is viable. Should the unfortunate event (against which insurance is taken) happen, note that the bulk of the financial burden falls on the insurance company; so it is natural that the set up is somewhat loaded in favour of the insurer. However, this could mean that the insured (that is, end users availing the policy, considered collectively)* *pay more than what they expect to get on the average; it is a consequence of the so called 'net profit condition' (NPC). As insurance business obviously exists, and also considered a necessity, in economics, this apparent anomaly is explained in terms of utility functions; see Subsection 3.4.1 of Ramasubramanian (2009) and references given therein. The premium rate is often taken to be a positive constant, as the legal contract has to be specified at the very beginning of the policy. Ruin problems play a major role at this stage as outlined below.*

Let $c, E(X_1), z$ *denote respectively premium rate, average claim size, initial capital; let* $\psi(z) =$ *probability of ruin in finite time. If* $ct \leq$ *[average number of claims during* $[0, t]$*]*$\times E(X_1)$*, then it can be shown that 'ruin' in finite time is certain, however large* z *might be; this indicates the basic necessity of (NPC) alluded to above in any insurance model. Typically* $0 < \psi(z) < 1$ *even if (NPC) is satisfied, for any* $z \in [0, \infty)$*; that is, risk of ruin is never completely eliminated, which is an important feature in any insurance set up. In the Cramer-Lundberg model (NPC) is* $c > \lambda E(X_1)$*, where* λ *is the rate of claim arrivals. Usually* c *is taken as* $c = (1 + \rho)\lambda E(X_1)$*, where* $\rho > 0$ *is called the safety loading factor. While larger value of* ρ *will lead to smaller ruin probability, it can also make the policy unattractive to the end user as the premium rate would then be higher. Suppose the insurer has chosen a suitable* ρ*; Let* $\epsilon > 0$ *be a fixed small number; the insurer may want to ascertain a suitable* z *so that* $\psi(z) \approx \epsilon$*. In other words, the insurer would like "ruin" to be a rare event; this might require a large initial capital* z*. Asymptotic analysis of ruin probability is quite useful here. Thus ruin probability is regarded as an objective theoretical yardstick to judge the viability of an insurance set up; see Section 1.6 of Rolski et al. (1999).*

(ii) Now we look at the multidimensional set up. A striking feature of the multidimensional scenario is captured in a comment on p.442 of Asmussen and Albrecher (2010): "Although multivariate ruin theory is a very natural extension of classical ruin theory with a lot of potential applications also in fields outside of insurance, this research field is not yet very far developed." This is in spite of interactions among insurance companies being known for more than 100 years.

The best known among these is proportional reinsurance. In such a set up, a fixed proportion of every claim is paid by the reinsurer. To prevent the primary insurer from transferring all liability to the reinsurer (that is, to avoid arbitrage opportunity for primary insurer), reinsurance premium is quite high relative to the premium income received by the primary insurer. Once the proportion and the reinsurance premium rate are decided, the dynamics of the two companies are considered separately; their joint dynamics *is generally not studied; that is, the interaction between the primary insurer and the reinsurer is just like that between an insurance policy holder and an insurer.*

Some other types of known interactions: (a) Sharing of data or information regarding catastrophe losses; see McNeil et al. (2005), p.464. (b) Even in the absence of regulatory body, a tacit understanding among companies to be slightly generous in honouring somewhat suspicious claims, so that the entire industry does not suffer from any public perception that companies are reluctant to settle claims; see Andersson and Skogh (2003). (c) Common strategies during negotiations with regulatory body/ government. Of course, it is not clear how such interactions can be quantified.

Thus data-based truly empirical multidimensional models do not seem to have been proposed in the literature. A reason for this lacuna could be the following. Because of the legal/ regulatory requirements, large initial investments may be needed in the insurance business. For a quantitative interaction to be arrived at between companies, all the parameters related to the interaction/ agreement (in addition to the parameters of the statistical distributions) should be spelt out, and the nature/ measure of risk involved for all the concerned parties must be well understood. Moreover it is desirable that the parameters related to interaction are obtained in a reasonable (if not in an optimal) manner. Present level of understanding of these aspects may be inadequate; see Remark 3 of Section 6.

In the absence of empirical models, formal models (like the ones given on pp. 435-443 of Asmussen and Albrecher (2010) and references therein) might be useful to understand the nature of the situation, and point out directions in which refinements/ tools need to be developed, so that empirical studies may be carried out in an inexpensive way. The multidimensional model developed in Ramasubramanian (2006), Ramasubramanian (2011), Ramasubramanian (2012), and continued here, is also a formal model; however, this model is based on optimal properties of Skorokhod problem, and also has a canonical notion of ruin that may also be mathematically amenable. As multidimensional problems are more difficult to handle, one seeks to identify suitable classes of such problems which can be tackled using one dimensional techniques; the present paper is one minor step in that direction.

The specific scenario considered in Section 5.2 is a hybrid of a real life situation and a hypothetical set up. For Ruin problem (N), which describes the one dimensional case (for Company 1) in the absence of risk diversification treaty, the distributional and the parametric input arise from a real life situation; the claim size distribution is Pareto distribution. Using asymptotic analysis of ruin probability, we compute a suitable initial capital z_1*,* *which makes the policy reasonably attractive, and at the same time makes ruin probability quite small. Then we consider Ruin problem (E), which is equivalent to the two dimensional situation with risk diversifying treaty; this involves parameters* a, c_2, *in addition to those in the earlier problem. Note that* a *denotes the fraction of the amount to be given by Company 2 to Company 1 as capital injection should such a contingency arise; this can depend on* z_1 *in general. Similarly* c_2 *is the fraction of premium income of Company 1 that Company 2 would get as retainer fee. These have been chosen somewhat arbitrarily, just ensuring that the net profit condition continues to hold. An objective way of selecting these might depend on a satisfactory solution to the problem mentioned in Remark 3 of Section 6; the latter could result in a cost-effective way of generating empirical data to guide further research.*

2 Reduction to One-Dimensional Setup

We consider two companies operating under a risk diversifying treaty. The first company, denoted Company 1, has insurance business, while the other company, denoted Company 2, does not have an insurance business of its own for the purposes of this paper. According to the treaty, whenever Company 1 encounters a deficit in meeting its liability, Company 2 takes care of a preassigned portion of the deficit; in return Company 1 pays Company 2 a retainer fee at a preassigned constant rate. We describe the setup in terms of a Skorokhod problem in the nonnegative quadrant as follows.

All stochastic processes are defined on a basic probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{R}^2_+ = \{(x_1, x_2) : x_1 \geq 0,\ x_2 \geq 0\}$ denote the nonnegative quadrant. Let $\{N(t) : t \geq 0\}$ denote the claim number process, and $\{X_l : l = 1, 2, ...\}$ claim sizes, of course, both for Company 1.

Put $T_0 = 0$. For $k = 1, 2, \cdots$ let $T_k(\omega) = \inf\{t \geq 0 : N(t, \omega) = k\}$; so $\{T_k\}$ is the sequence of claim arrival times. Let $A_j = T_j - T_{j-1},\ j \geq 1$ denote claim inter arrival times. We consider a general renewal risk setup; that is, we assume

(A0) $c_1 > 0,\ c_2 > 0,\ 0 < a < 1$.

(A1) $A_j,\ j \geq 1$ are independent positive random variables; so $0 < T_1 < T_2 < \cdots < T_n \cdots$, and $T_n \uparrow +\infty$ with probability 1; so in any finite time interval there are only finitely many claims.

(A2) Claim sizes $X_l,\ l \geq 1$ are i.i.d. positive random variables with $P(X_1 > x) > 0$ for all $x \in (0, \infty)$.

(A3) $\{X_l : l \geq 1\}, \{N(t) : t \geq 0\}$ are independent families of random variables

(A4) Claim sizes $X_l,\ l \geq 1$ are also absolutely continuous.

So $\{N(t) : t \geq 0\}$ is a renewal counting process independent of the claim size sequence $\{X_l\}$. According to the treaty if Company 1 faces a deficit dy_1 at some instant, then ady_1 is given by Company 2. The shortfall $(1-a)dy_1$ has to be provided by the shareholders of Company 1 as capital injection. Here c_1 is the constant premium rate for Company 1, c_2 is the constant rate at which Company 2 gets retainer fee. Write $S_1(t) = \sum_{j=1}^{N(t)} X_j,\ t \geq 0$; then $S_1(t) =$ total claim amount for Company 1 during [0, t]. Let $z_1 \geq 0,\ z_2 \geq 0$ denote the initial capitals of the respective companies.

The dynamics of the two companies under the treaty is described by two r.c.l.l. processes $(Y_1(\cdot),\ Y_2(\cdot)),\ (Z_1(\cdot),\ Z_2(\cdot))$ satisfying the following.

(SP1) Skorokhod equation holds, that is, for $t \geq 0,\ \omega \in \Omega$,

$$Z_1(t,\omega) = z_1 + c_1 t - \sum_{j=1}^{N(t,\omega)} X_j(\omega) + Y_1(t,\omega) \tag{2.1}$$

$$Z_2(t,\omega) = z_2 + c_2 t + Y_2(t,\omega) - aY_1(t,\omega). \tag{2.2}$$

(SP2) $Z_1(t,\omega) \geq 0,\ Z_2(t,\omega) \geq 0$ for all $t \geq 0,\ \omega \in \Omega$, that is, $(Z_1(\cdot), Z_2(\cdot))$ takes value in $\mathbb{R}_+^2$.

(SP3) For $i = 1, 2,\ Y_i(0) = 0,\ Y_i(\cdot,\omega)$ is nondecreasing, and $Y_i(\cdot,\omega)$ can increase only when $Z_i(\cdot,\omega) = 0$, that is, for $0 \leq s < t$

$$Y_i(t,\omega) - Y_i(s,\omega) = \int_{(s,t]} I_{\{0\}}(Z_i(u,\omega))dY_i(u,\omega) \tag{2.3}$$

Note that requirement (SP2) is a constraint, while (2.3) in (SP3) is a minimality condition. Write $R = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$; then Skorokhod equation above can be written in the vector form as

$$\begin{pmatrix} Z_1(t,\omega) \\ Z_2(t,\omega) \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + t\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \begin{pmatrix} S_1(t,\omega) \\ 0 \end{pmatrix} + R\begin{pmatrix} Y_1(t,\omega) \\ Y_2(t,\omega) \end{pmatrix} \tag{2.4}$$

Note that we can write $R = I - W$, with $W = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$; by (A0) spectral radius of W is less than 1. Hence by the seminal result of Harrison and Reiman (1981), for fixed $\omega \in \Omega, z_1, z_2 \geq 0$ there exist unique

$(Y_1(\cdot,\omega), Y_2(\cdot,\omega))$, $(Z_1(\cdot,\omega), Z_2(\cdot,\omega))$ satisfying (SP1)-(SP3); that is, the deterministic Skorokhod problem is well posed. Thus required processes $(Y_1(\cdot), Y_2(\cdot))$, $(Z_1(\cdot), Z_2(\cdot))$ can be obtained by solving the deterministic Skorokhod problem pathwise. In such a case $(Z_1(\cdot), Z_2(\cdot))$ is called the regulated/ reflected part, while $(Y_1(\cdot), Y_2(\cdot))$ is called the pushing part of the solution.

Write

$$\begin{pmatrix} H_1(t,\omega) \\ H_2(t,\omega) \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + t \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \begin{pmatrix} S_1(t,\omega) \\ 0 \end{pmatrix}, \quad t \geq 0 \tag{2.5}$$

By solving the deterministic Skorokhod problem (in the quadrant) corresponding to $(H_1(\cdot), H_2(\cdot))$ path-by-path, note that $(Y_1(\cdot), Y_2(\cdot)), (Z_1(\cdot), Z_2(\cdot))$ is obtained.

We define ruin as the event $\{(Z_1(t), Z_2(t)) = (0,0) \text{ for some } t > 0\}$; see Ramasubramanian (2012) for similarities with the notion of ruin in classical one dimensional models. In this section we want to show that the two dimensional ruin problem in the above setup can be investigated using a one dimensional problem.

Theorem 2.1. *Let $z_1 \geq 0$, $z_2 \geq 0$; assume (A0)-(A4). Let $(Y_1(\cdot), Y_2(\cdot)), (Z_1(\cdot), Z_2(\cdot))$ be the unique pair of processes satisfying (SP1)-(SP3). Then the following hold:*

(i) Ruin can possibly occur only at a claim arrival time, and $Y_i(\cdot), i = 1, 2$ can possibly increase only at a claim arrival time.

(ii) If $Y_2(\cdot)$ is strictly increasing at t_0, then so is $Y_1(\cdot)$ at t_0.

(iii) Also $Z_2(t_0) = 0$ implies $Z_1(t_0) = 0$.

(iv) For $i = 1, 2$, $Z_i(t_0) = 0$ if and only if $Y_i(t_0) - Y_i(t_0-) > 0$ with probability one.

Note. *Absolute continuity of claim sizes X_l has been used only in the proof of necessity in assertion (iv) of the preceding theorem; it may be noted that the other assertions are valid even without the assumption .*

With notation as above, for $\omega \in \Omega$ define

$$H(t,\omega) = (az_1 + z_2) + (ac_1 + c_2)t - aS_1(t,\omega), \quad t \geq 0. \tag{2.6}$$

Theorem 2.2. *Let the notation be as in Theorem 2.1. Assume (A0)-(A3), and let $z_1 \geq 0$, $z_2 \geq 0$. Fix $\omega \in \Omega$. Let $Y_2(\cdot,\omega)$, $Z_1(\cdot,\omega), Z_2(\cdot,\omega)$ be as in Theorem 2.1. Then the following are true.*

(S1) *One dimensional Skorokhod equation holds, that is,*

$$aZ_1(t,\omega) + Z_2(t,\omega) = H(t,\omega) + Y_2(t,\omega), \ \ t \geq 0. \tag{2.7}$$

(S2) *(Constraint)* $aZ_1(t,\omega) + Z_2(t,\omega) \geq 0, \ \ t \geq 0.$

(S3) *(Minimality)* $Y_2(0,\omega) = 0$, $Y_2(\cdot,\omega)$ *is nondecreasing, and* $Y_2(\cdot,\omega)$ *can increase only when* $aZ_1(\cdot,\omega) + Z_2(\cdot,\omega) = 0$.

In other words, $Y_2(\cdot,\omega)$, $aZ_1(\cdot,\omega) + Z_2(\cdot,\omega)$ *is the unique solution pair to the one dimensional Skorokhod problem in* $[0, \infty)$ *corresponding to* $H(\cdot,\omega)$.

Remark 2.3. *For Skorokhod problem in* $[0,\infty)$ *the notion of minimality given in (S3) in the preceding theorem coincides with the following more conventional notion of minimality. Fix* $\omega \in \Omega$; *so* $H(\cdot,\omega)$ *is fixed. A nondecreasing function* $\xi(\cdot)$ *on* $[0,\infty)$ *with* $\xi(0) = 0$ *is called a feasible control if* $H(t,\omega)+\xi(t) \geq 0$ *for all* $t \geq 0$. *Then the pushing part* $Y_2(\cdot,\omega)$ *of the solution to Skorokhod problem above is known to be the unique minimal feasible control in the sense that* $Y_2(t,\omega) \leq \xi(t), \ t \geq 0$ *for any feasible control* $\xi(\cdot)$. *See Harrison (1985), Harrison and Reiman (1981), Karatzas and Shreve (1991). In fact, such a notion was generalized in Ramasubramanian (2006) to Skorokhod problem in an orthant in terms of a Nash equilibrium, and to argue that a multidimensional insurance setup can be modelled in terms of Skorokhod problem; see Ramasubramanian (2011) and also references therein.*

Remark 2.4. *Note that* $H(\cdot)$ *given by (2.6) is the one dimensional risk process (without capital injection) corresponding to claim number process* $N(\cdot)$, *premium rate* $(ac_1 + c_2)$, *initial capital* $(az_1 + z_2)$, *and claim sizes* (aX_l), $l \geq 1$. *For this classical risk process, recall that ruin is defined as the event* $\{H(t) < 0$ *for some* $t \geq 0\}$. *Clearly ruin can occur only at a claim arrival time. At such a time the concerned company faces a deficit. So capital injection by the shareholders of the company is needed to meet its obligations. The preceding remark indicates the way to go about capital injection in an optimal way using Skorokhod problem. If the claim sizes are continuous random variables, then it is not difficult to see that the first time* $H(\cdot)$ *enters* $(-\infty,0)$ *is the same as the first time the regulated part* $aZ_1(\cdot) + Z_2(\cdot)$ *hits 0, with probability 1.*

The preceding remarks now suggest the next result. Define

$$\tau_0 = \inf\{t > 0 : (Z_1(t), Z_2(t)) = (0,0)\} \tag{2.8}$$

$$\tau_1 = \inf\{t \geq 0 : H(t) < 0\} \tag{2.9}$$

$$\begin{aligned}\tau_2 &= \inf\{t > 0 : Z_2(t) = 0\} \\ \tau_3 &= \inf\{t > 0 : H(t) \leq 0\} \\ \tau_4 &= \inf\{t > 0 : Y_i(t) - Y_i(t-) > 0,\ i = 1, 2\}.\end{aligned}$$

In the above, note that while τ_1 is an entrance time, the rest are all hitting times.

Theorem 2.5. *Let $z_1 \geq 0,\ z_2 \geq 0$. Assume (A0)-(A4). Then $\tau_0 = \tau_i,\ i = 1, 2, 3, 4$ with probability 1. In particular, for any $z_1 \geq 0, z_2 \geq 0$,*

$$P(\tau_0 < \infty) = P(\tau_1 < \infty) \tag{2.10}$$

$$P(\tau_0 \leq t) = P(\tau_1 \leq t),\ t \geq 0. \tag{2.11}$$

Note that the left hand sides of (2.10), (2.11) are the ruin probabilities of interest in our two dimensional risk diversifying setup, while the corresponding right hand sides are the ruin probabilities in the classical one dimensional model associated with (2.6).

Remark 2.6. *We describe a $d-$dimensional scenario when the problem can be reduced to the one dimensional case as above. Suppose there are d companies, with Company i, $1 \leq i \leq (d-1)$ doing insurance business, with Company d providing pre-agreed proportion of capital injection to Company i, $1 \leq i \leq d-1$ as and when the need arises; in return Company i, $1 \leq i \leq (d-1)$ pays Company d a retainer fee at a constant rate; Company d does not have any insurance business of its own. Assume that claim sizes are i.i.d. $(d-1)-$dimensional random variables $(X_\ell^{(1)}, \cdots, X_\ell^{(d-1)}), \ell \geq 1$ arriving according to (scalar) renewal counting process $N(t), t \geq 0$; claim arrival times and claim sizes are assumed to be independent. For example Company i, $1 \leq i \leq (d-1)$ can be coinsurers of potentially large claims; in such a case $X_\ell^{(i)}, 1 \leq i \leq (d-1)$ will be certain fixed fractions of a scalar claim X_ℓ. As another example, Company i, $1 \leq i \leq (d-1)$ can be subsidiaries of Company d handling different components of a vector claim, like earthquakes/ floods triggering property loss as well as immediate medical expenses. The joint dynamics of the d companies can be described in terms of appropriate Skorokhod problem in the $d-$dimensional nonnegative orthant. Let $c_i > 0$ denote the premium rate for Company i, $1 \leq i \leq (d-1)$; let $c_d > 0$ denote the cumulative constant rate at which Company d gets retainer fee form the other $(d-1)$ companies. For $1 \leq j \leq (d-1)$, let $0 < a_j < 1$ denote the fraction of the deficit facing Company j at some instant that is met by Company d; the shortfall $(1-a_j)dy_j$ has to be provided as capital injection by the shareholders* *of Company j. It is assumed that there is no risk diversification arrangement among the first* $(d-1)$ *companies. In this case the* $d \times d$ *reflection matrix is of the form* $R = I - W$ *with* $W_{dj} = a_j, 1 \leq j \leq (d-1),\ W_{ij} = 0$ *otherwise for* $1 \leq i, j \leq d$. *Analogous to (2.1),(2.2), Skorokhod equations are*

$$Z_i(t,\omega) = z_i + c_i t - \sum_{\ell=1}^{N(t,\omega)} X_\ell^{(i)}(\omega) + Y_i(t,\omega),\ 1 \leq i \leq (d-1), \tag{2.12}$$

$$Z_d(t,\omega) = z_d + c_d t + Y_d(t,\omega) + \sum_{j=1}^{d-1} R_{dj} Y_j(t,\omega). \tag{2.13}$$

Constraint and minimality are similar, and ruin of the network can be defined analogously. As in Theorem 2.1, it can be shown that $Z_d(t_0) = 0$ *implies* $Z_i(t_0) = 0, 1 \leq i \leq (d-1)$, *and that if* $Y_d(\cdot)$ *is strictly increasing at* t_0 *then so are* $Y_i(\cdot)$ *at* t_0. *Consequently, putting*

$$H(t,\omega) = \left(\sum_{j=1}^{d-1} a_j z_j + z_d\right) + \left(\sum_{j=1}^{d-1} a_j c_j + c_d\right) t - \sum_{\ell=1}^{N(t,\omega)} \sum_{j=1}^{d-1} a_j X_\ell^{(j)}(\omega),\ t \geq 0, \omega \in \Omega,$$

it can be shown as in Theorem 2.2 that $Y_d(\cdot,\omega)$, $\sum_{j=1}^{d-1} a_j Z_j(\cdot,\omega) + Z_d(\cdot,\omega)$ *is the unique solution pair to the one dimensional Skorokhod problem in* $[0,\infty)$ *corresponding to* $H(\cdot,\omega)$. *That is, the* $d-$*dimensional ruin problem can be basically reduced to a one dimensional ruin problem in the classical set up. Therefore, once the nature of the distribution of the one dimensional claim size random variable* $\sum_{j=1}^{d-1} a_j X_1^{(j)}$ *is given, the analysis to be developed in the sequel can as well be applied with obvious modifications.*

It should, however, be pointed out that if the claim arrivals for the $(d-1)$ *companies are governed by different renewal processes, then our methods may not be applicable, as reduction to the one dimensional case is not possible in general.*

3 Advantages of the Treaty

In this section we want to analyse the advantages of the risk diversifying treaty for each company.

We make the following hypothesis in our setup, in addition to the assumption made in the preceding section.

(A5) $E(A_1) < \infty$, $E(X_1) < \infty$, $A_j, j \geq 1$ are i.i.d. random variables; moreover

$$E(X_1 - c_1 A_1) < 0. \tag{3.1}$$

Remark 3.1. *Note that (3.1) is the net profit condition for Company 1. As $c_2 > 0$ and Company 2 does not have insurance business of its own, note that the net profit condition for Company 2 is trivially satisfied. It is a basic assumption in Ramasubramanian (2012) that net profit condition holds for all the companies. So, under (A0)-(A5), by Proposition 2.2 in Ramasubramanian (2012) it follows that $Z_i(t, \omega) \to +\infty$ as $t \to \infty$ for a.e. $\omega,\ i = 1, 2$.*

Consider the ruin probability as a function of the initial capital, that is, write

$$\psi(z_1, z_2) = P(\tau_0 < \infty | Z_1(0) = z_1, Z_2(0) = z_2). \tag{3.2}$$

In the preceding section we had shown that the ruin problem for the two dimensional risk diversifying setup is equivalent to the following ruin problem.

Ruin problem (E) This is the one dimensional ruin problem associated with (2.6). In this case initial capital $= (az_1 + z_2)$, premium rate $= (ac_1 + c_2)$, claim sizes are $(aX_j), j \geq 1$; the appropriate ruin time is $\tau^{(E)} = \tau_1$ given by (2.9). Define

$$\psi^{(E)}(y) = P(\tau^{(E)} < \infty | H(0) = y). \tag{3.3}$$

From the preceding section we know that

$$\psi(z_1, z_2) = \psi^{(E)}(az_1 + z_2),\ z_1 \geq 0, z_2 \geq 0. \tag{3.4}$$

We will see first how the risk diversifying treaty could be advantageous to Company 1. Of course, some advantages are obvious. Under the treaty, whenever there is a deficit Company 1 need not obtain all the required amount through capital injection from its shareholders. (A tacit assumption here is that Company 1 has been in business sufficiently long, and from experience, is confident that its shareholders can provide $(1 - a)-$fraction of the deficit without much difficulty.) Moreover, the additional capital z_2 of Company 2 becomes available should the need arise; so larger the capital z_2, longer is the survival time. To have a better picture, in this section, we look at the asymptotic behaviour of ruin probability.

To do this, let us see what the situation can be for Company 1 without the treaty. As Company 2 does not have any insurance business of its own, in the absence of the treaty there are no dealings (directly or indirectly) between the insured policy holder and Company 2. Also it is reasonable to assume that the retainer fee paid by Company 1 to Company 2, at rate c_2, comes from the premium income of Company 1. Thus in the absence of the treaty, ruin problem for Company 1 may be taken as the following.

Ruin problem (N) In this case initial capital is z_1, premium rate= $c = c_1 + c_2$, claim sizes are $X_j,\ j \geq 1$. So the corresponding one dimensional risk process is given by $H^{(N)}(0) = z_1$ and

$$H^{(N)}(t) = H^{(N)}(0) + (c_1 + c_2)t - \sum_{j=1}^{N(t)} X_j,\ t \geq 0. \tag{3.5}$$

Appropriate ruin probability is the function

$$\psi^{(N)}(y) = P(\tau^{(N)} < \infty | H^{(N)}(0) = y), \tag{3.6}$$

where $\tau^{(N)} = \inf\{t \geq 0 : H^{(N)} < 0\}$ is the relevant ruin time.

It is now clear that our objective is to compare $\psi^{(E)}(az_1 + z_2)$ and $\psi^{(N)}(z_1)$ for given $0 < a < 1,\ z_2 \geq 0$.

We introduce the following auxiliary one dimensional ruin problem for this purpose.

Ruin problem (A) Initial capital $= az_1$, premium rate $= ac = a(c_1 + c_2)$, claim sizes are $(aX_j),\ j \leq 1$; the risk process is

$$H^{(A)}(t) = H^{(A)}(0) + act - \sum_{j=1}^{N(t)} aX_j,\ t \geq 0, \tag{3.7}$$

with initial value $H^{(A)}(0) = az_1$. Ruin time is $\tau^{(A)} = \inf\{t \geq 0 : H^{(A)}(t) < 0\}$, and ruin probability is

$$\psi^{(A)}(y) = P(\tau^{(A)} < \infty | H^{(A)}(0) = y). \tag{3.8}$$

Lemma 3.2. *Assume (A0)-(A5). Then*

$$E(X_1 - cA_1) < 0, \tag{3.9}$$

$$E(aX_1 - (ac_1 + c_2)A_1) < 0. \tag{3.10}$$

In other words, the net profit condition holds for Ruin problem (N), Ruin problem (A) and Ruin problem (E).

We now assume the following light tailed situation

(A6) A_1 has finite moment generating function in a neighbourhood of 0. There exists $0 < h_0 < \infty$ such that $E(e^{hX_1}) < \infty$ for $h < h_0$, and

$$\lim_{h \uparrow h_0} E(e^{hX_1}) = +\infty. \tag{3.11}$$

Lemma 3.3. *Assume (A0)-(A6). Then there is a unique Lundberg coefficient $r^{(N)} > 0$ for Ruin problem (N). Hence there exist positive numbers $\hat{b}_-$, $\hat{b}_+$ such that*

$$\hat{b}_- \exp(-r^{(N)} z_1) \leq \psi^{(N)}(z_1) \leq \hat{b}_+ \exp(-r^{(N)} z_1), \ z_1 \geq 0.$$

(See Remark 3.11 below; some comments there may also apply to results of the above lemma.)

Now suppose $H^{(N)}(0) = z_1, \ H^{(A)}(0) = az_1$. As the claim number process $\{N(t) : t \geq 0\}$ is the same, clearly $H^{(A)}(t) = aH^{(N)}(t), \ t \geq 0$. Since ruin can occur only at a claim arrival time $T_n, \ n \geq 1$, note that

$$\begin{aligned} & \text{Ruin occurs for Ruin problem (N) at } \ T_k \\ \Leftrightarrow \ & \sum_{j=1}^{m} (X_j - cA_j) \leq z_1, \ \text{for } m < k, \ \sum_{j=1}^{k} (X_j - cA_j) > z_1 \\ \Leftrightarrow \ & \sum_{j=1}^{m} (aX_j - acA_j) \leq az_1, \ \text{for } m < k, \ \sum_{j=1}^{k} (aX_j - acA_j) > az_1 \\ \Leftrightarrow \ & \text{Ruin occurs for Ruin problem (A) at } T_k. \end{aligned} \tag{3.12}$$

In view of Lemmas 3.2, 3.3, we have the following result.

Theorem 3.4. *(i) Assume (A0)-(A5). Then $\psi^{(A)}(az_1) = \psi^{(N)}(z_1), \ z_1 \geq 0$. Consequently*

$$\psi(z_1, z_2) = \psi^{(E)}(az_1 + z_2) \leq \psi^{(N)}(z_1), \ \forall \ z_1 \geq 0, \ z_2 \geq 0. \tag{3.13}$$

(ii) In addition, if (A6) also holds, then there is a unique Lundberg coefficient $r^{(A)} > 0$ for Ruin problem (A); in fact $r^{(A)} = \frac{1}{a} r^{(N)}$. As $0 < a < 1$ it is clear that $r^{(A)} > r^{(N)}$.

Theorem 3.5. *Assume (A0)-(A6). Then there is a unique Lundberg coefficient $r^{(E)} > 0$ for Ruin problem (E). Moreover $r^{(E)} > r^{(A)} = \frac{1}{a} r^{(N)} > r^{(N)} > 0$. Also there are $0 \leq b_- \leq b_+ \leq 1$ such that*

$$b_- \exp(-r^{(E)}(az_1 + z_2)) \leq \psi(z_1, z_2) = \psi^{(E)}(az_1 + z_2) \leq b_+ \exp(-r^{(E)}(az_1 + z_2)). \tag{3.14}$$

Remark 3.6. *The results above are applicable when (2.6) denotes a renewal risk model with light-tailed interarrival time distribution and general phase-type claim size distribution having unique Lundberg coefficient.*

Remark 3.7. *From Lemma 3.3 and Theorem 3.5, note that ruin probability in the two-dimensional model with risk diversifying treaty decays at a faster rate than ruin probability for Company 1 doing business without treaty. Especially if* $az_1 + z_2 > z_1$, *that is, if* $z_2 > (1-a)z_1$, *the advantage for Company 1 is very clear.*

In the case of Cramer-Lundberg model we can say more.

Theorem 3.8. *Let* $A_j,\ j \geq 1$ *be independent identically distributed random variables having exponential distribution.*

(i) Assume (A0)-(A5). Then (3.13) is satisfied.

(ii) Assume (A0)-(A6). Then there exist constants $C^{(N)},\ C^{(E)} > 0$ *such that*

$$\lim_{x\to\infty} \psi^{(N)}(x) \exp[r^{(N)}x] = C^{(N)},$$
$$\lim_{x\to\infty} \psi^{(E)}(x) \exp[r^{(E)}x] = C^{(E)}.$$

Thus conclusions of Lemma 3.3, Theorems 3.4, 3.5 and Remarks 3.6, 3.7 are applicable here.

Remark 3.9. *We have been assuming* $0 < a < 1$. *Let us see what happens if* $a = 1$. *Of course, all the other assumptions are the same as before. Note that the results of Section 2 hold even if* $a = 1$; *in particular, the two dimensional ruin problem can be reduced to the one dimensional setup. In this case it can be seen that* $\psi^{(E)} = \psi^{(N)},\ x \geq 0$; *in particular* $r^{(E)} = r^{(N)}$, *if the Lundberg coefficient exists. However, recall that we need to compare* $\psi^{(N)}(z_1)$ *with* $\psi^{(E)}(az_1 + z_2)$; *that is, compare* $\psi^{(N)}(z_1)$ *with* $\psi^{(N)}(z_1 + z_2)$. *If (A6) also holds, as* $\psi^{(N)}(x) = C\exp(-r^{(N)}x)(1 + o(1)),\ x \to +\infty$, *it follows that*

$$\psi^{(E)}(z_1 + z_2) = \psi^{(N)}(z_1 + z_2) \approx e^{-r^{(N)}z_2}\psi^{(N)}(z_1)$$

for large values of $(z_1 + z_2)$. *Thus the ruin probability is considerably less if the treaty is operative, and* z_2 *is at least moderately large. (The case when* $a = 1$, *with exponentially distributed claim sizes in a Cramer-Lundberg setup has been considered in Example 3.5 of Ramasubramanian (2012).) The setup in this example can be viewed upon as Company 1 operating as a subsidiary* *of Company 2. Suppose Company 2 is willing to invest only an initial capital of z_1 in the insurance business done by Company 1, with premium income shared between the two at respective rates c_1, c_2. However, Company 2 is willing to take care of any deficit that Company 1 might face upto level z_2+c_2t at any time $t > 0$. Ruin probability $\psi^{(E)}$ might be an objective measure helpful in any decision at a later date concerning dissolving Company 1 with minimal loss, or selling Company 1 at a handsome profit.*

Note. *In fact, even when $0 < a < 1$, the setup can be looked upon as Company 1 operating as a subsidiary of Company 2. In such a situation, Company 2, besides providing an initial capital of only z_1, wants the subsidiary to manage $(1-a)-$fraction of any possible deficit as well. With these stipulations in place, Company 2 can decide at a later date if the performance of the subsidiary has been satisfactory.*

Example 3.10. *Consider the Cramer-Lundberg setup with exponential claim size. Suppose X_1 has exponential distribution with parameter $\theta > 0$. Then $E(X_1) = \frac{1}{\theta}$. Since explicit expression for ruin probability is known (see Rolski et al. (1999), Ramasubramanian (2009)), note that*

$$\begin{aligned} \psi^{(N)}(z_1) &= \frac{\lambda E(X_1)}{c} \exp\left\{-\frac{(c-\lambda E(X_1))}{cE(X_1)} z_1\right\},\ z_1 \geq 0; \\ \psi^{(E)}(az_1+z_2) &= \frac{\lambda a E(X_1)}{ac_1+c_2} \exp\left\{-\frac{[(ac_1+c_2)-\lambda a E(X_1)]}{(ac_1+c_2)E(X_1)}(az_1+z_2)\right\}, \\ & \quad z_1 \geq 0\ z_2 \geq 0; \end{aligned}$$

If $0 < a < 1$, and $c = c_1 + c_2$ it is easy to see that

$$\frac{\lambda a E(X_1)}{(ac_1+c_2)} < \frac{\lambda E(X_1)}{c}$$

and

$$\frac{(ac_1+c_2)-\lambda a E(X_1)}{(ac_1+c_2)E(X_1)} > \frac{c-\lambda E(X_1)}{cE(X_1)}.$$

Also, if $a = 1$, then

$$\psi^{(E)}(z_1+z_2) = \psi^{(N)}(z_1) \exp\left\{-\frac{(c-\lambda E(X_1))}{cE(X_1)} z_2\right\},\ z_1 \geq 0,\ z_2 \geq 0.$$

So the advantage of the treaty is clear.

Remark 3.11. *By the proof of Theorem 6.5.4 of Rolski et al. (1999) note that b_- (resp. b_+) in (3.14) is the infimum (resp. supremum) of the same* *functional of the distribution of $(aX_1 - (ac_1 + c_2)A_1)$. The lower bound in (3.14) is useful only if $b_- > 0$. (Under (A0)-(A6), note that claim size distribution function $F_X(\cdot)$ has a probability density function $f_X(\cdot)$, and by Theorem 3.5, a unique Lundberg coefficient $r^{(E)} > 0$ exists. (If the claim size distribution has nondecreasing hazard rate (mortality rate), that is $x \mapsto f_X(x)/(1 - F_X(x))$ is nondecreasing on $[0, \infty)$, then by Corollary 6.5.7 on p.258 of Rolski et al. (1999) it follows that $b_- > 0$.) Suppose $r^{(E)} > 0, b_- > 0$ in the renewal risk model considered in Theorem 3.5. Then by (3.14) it follows that*

$$\log(1-\psi(z_1, z_2)) = \log(1-\psi^{(E)}(az_1+z_2) \sim -r^{(E)}\cdot(az_1+z_2),\ (az_1+z_2) \to \infty.$$

This is a rough large deviations result, in the sense of Embrechts et al. (1997), p.498. In the case of the Cramer-Lundberg model, under the conditions of Theorem 3.8, we get the stronger precise large deviations result

$$\psi(z_1, z_2) = \psi^{(E)}(az_1 + z_2) \sim C \exp(-r^{(E)} \cdot (az_1 + z_2),\ (az_1 + z_2) \to \infty.$$

In addition if the claim sizes are exponential, then $b_- = b_+$ and hence we have equality in the above.

We now consider the Cramer-Lundberg setup with certain heavy tailed claim size distribution; for such distributions moment generating function does not exist in any neighbourhood of 0. For information needed in the rest of this section see Teugels (1975), Rolski et al. (1999), Ramasubramanian (2009), or the very recent Foss et al. (2013). Let $F(\cdot),\ F^{(a)}(\cdot)$ denote, respectively, the distribution functions of $X_1,\ (aX_1)$; similarly let $F_I(\cdot),\ F_I^{(a)}(\cdot)$ denote the respective integrated tail distribution function. It is clear that $F^{(a)}(x) = F(\frac{1}{a}x),\ \overline{F}^{(a)}(x) = 1 - F^{(a)}(x) = 1 - F(\frac{1}{a}x) = \overline{F}(\frac{1}{a}x),\ 0 < x < \infty$. Also, by the definition of integrated tail distribution

$$\begin{aligned} 1 - F_I^{(a)}(x) &= \frac{1}{E(aX_1)} \int_x^\infty (1 - F^{(a)}(t))dt \\ &= \frac{1}{E(X)} \int_{\frac{1}{a}x}^\infty (1 - F(s))ds = 1 - F_I(\frac{1}{a}x). \end{aligned} \tag{3.15}$$

Lemma 3.12. *If F_I is subexponential, then so is $F_I^{(a)}$.*

Proof is straightforward and hence is omitted.

Denote $\rho^{(N)} = \frac{(c_1+c_2)}{\lambda E(X_1)} - 1,\ \rho^{(E)} = \frac{(ac_1+c_2)}{\lambda a E(X_1)} - 1$. The well known Embrechts-Veraverbeke approximation leads now to the following.

Theorem 3.13. *Let $0 < a < 1$. Consider the Cramer-Lundberg model. Assume (A0)-(A5) Let F denote the claim size distribution function, that is, distribution function of X_1. Assume that the corresponding integrated tail distribution F_I is subexponential. Then $\rho^{(E)} > \rho^{(N)} > 0$ and*

$$\psi^{(N)}(z_1) = \frac{1}{\rho^{(N)}}[1 - F_I(z_1)](1 + o(1)), \text{ as } z_1 \to +\infty, \quad (3.16)$$

$$\psi^{(E)}(az_1 + z_2) = \frac{1}{\rho^{(E)}}[1 - F_I(z_1 + \frac{1}{a}z_2)](1 + o(1)), \text{ as } (z_1 + az_2) \to +\infty. \quad (3.17)$$

Since $\frac{1}{\rho^{(E)}} < \frac{1}{\rho^{(N)}}$, the advantage of the treaty for Company 1 is clear, especially if z_2 is large.

Remark 3.14. *We now consider the advantage for Company 2 from the treaty. As Company 2 does not have any insurance business of its own, and $c_2 > 0$, we have already noted in Remark 3.1 that $Z_2 \to +\infty$ with probability 1, thanks to hypothesis (A5). Also probability of ruin for Company 2 is only $\psi^{(E)}(az_1 + z_2)$ if it operates under treaty. Generally any sensible insurance business is designed to make ruin probability as small as possible. Therefore, if Company 2 does not opt for the treaty, no doubt its capital z_2 will not get depleted, but it also loses a very good opportunity of its wealth growing very large. Thus the choice for Company 2 is very similar to an insurance company going into insurance business or not doing business at all !!*

Remark 3.15. *A well known strategy for risk diversification is proportional reinsurance, where the cedent (that is, primary insurer, Company 1 in our case) diverts a portion of his premium income, say, at rate $\tilde{c}_2$, to the reinsurer; in return, the reinsurer takes care of a proportion $\tilde{a}$ of each claim; see Rolski et al. (1999), Asmussen and Albrecher (2010). A necessary condition for the reinsurer even to consider entering into an agreement is that the net profit condition applicable to his situation must hold, that is*

$$\tilde{c}_2 > \lambda \tilde{a} E(X_1); \quad (3.18)$$

else ruin is certain however large the capital might be; see Rolski et al. (1999), Ramasubramanian (2009). Of course, correspondingly for the cedent one should have

$$\tilde{c}_1 \equiv (c - \tilde{c}_2) > \lambda(1 - \tilde{a})E(X_1). \quad (3.19)$$

An important aspect of reinsurance is that it is expensive in the following sense; see Hipp (2004). Rewrite (3.19), (3.18) as $\tilde{c}_1 = (1+\beta_1)\lambda(1-\tilde{a})E(X_1)$,

$\tilde{c}_2 = (1+\beta_2)\lambda\tilde{a}E(X_1)$, *where* $\beta_1 > 0$, $\beta_2 > 0$ *are, respectively, the loading factors for the cedent, the reinsurer. If* $\beta_2 \leq \beta_1$, *then taking* $\tilde{a} = 1$, *the cedent can transfer the entire liability to the reinsurer, with* $\tilde{c}_1 > 0$ *ensuring unlimited growth of his wealth. So, in order to prevent the arbitrage opportunity alluded to in the opening sentence in Section 1, we should have* $\beta_2 > \beta_1$. *This would imply*

$$\frac{\tilde{c}_2}{c} > \tilde{a}. \tag{3.20}$$

Clearly by (3.20), to get a fraction of each claim paid by the reinsurer, the cedent would have to part with a larger fraction of his premium income, even if the initial capital z_2 *of the reinsurer is 0. This is a far cry from the earlier requirement that* $c_2 > 0$. *So the advantage of the risk diversification treaty over proportional reinsurance for Company 1 is clear.*

Next, note that $P(T_n < \infty) = 1$ *for any* n; *that is, claims do occur with probability 1. Hence in the proportional reinsurance scheme, with probability 1, the reinsurer is required to make claim payments. Now, if the basics of the insurance business of Company 1 are quite sound, as observed in the preceding remark, Company 2 in the risk diversifying treaty would be required to take care of part of the deficit only with a very small probability; that is, with high probability wealth of Company 2 can keep increasing. This may be another advantage over proportional reinsurance.*

4 Finite Horizon Ruin Probabilities

Fix $T > 0$. In this section we consider the probability of ruin occurring during $[0, T]$. Analogous to the definitions in the preceding section, define

$$\begin{aligned} \psi((z_1,z_2),T) &= 1-\varphi((z_1,z_2),T) \\ &= P(\tau_0 \leq T | Z_1(0)=z_1, Z_2(0)=z_2),\ (z_1,z_2)\in\mathbb{R}_+^2; \quad (4.1) \\ \psi^{(E)}(y,T) = 1-\varphi^{(E)}(y,T) &= P(\tau^{(E)} \leq T | H(0)=y),\ y\geq 0; \quad (4.2) \\ \psi^{(N)}(y,T) = 1-\varphi^{(N)}(y,T) &= P(\tau^{(N)} \leq T | H^{(N)}(0)=y),\ y\geq 0; \quad (4.3) \\ \psi^{(A)}(y,T) = 1-\varphi^{(A)}(y,T) &= P(\tau^{(A)} \leq T | H^{(A)}(0)=y),\ y\geq 0; \quad (4.4) \end{aligned}$$

here ψ's denote the finite horizon ruin probabilities, while φ's denote the corresponding finite time survival probabilities for the various ruin problems introduced in the preceding sections. The next gives a comparison between finite horizon ruin probabilities with treaty and without treaty.

Theorem 4.1. *Consider a renewal-risk-type model satisfying (A0)–(A5). Let* $T > 0$ *be fixed. Then*

$$\varphi^{(A)}(y,T) \leq \varphi^{(E)}(y,T), \quad y \geq 0. \tag{4.5}$$

Consequently

$$\psi((z_1, z_2), T) \leq \psi^{(N)}(z_1, T), \quad z_1 \geq 0, z_2 \geq 0. \tag{4.6}$$

Proof is given in the Appendix.

5 Numerical Results

5.1. Exponential/Hyperexponential Claims. We consider only the Cramer-Lundberg setup with exponential or hyperexponential claims; note that these are light tailed claims.

We assume that $N(\cdot)$ is a Poisson process with arrival rate $\lambda = 1$; also we assume that claims X_j are i.i.d. random variables having exponential distribution with parameter $\mu = 1$. Numerical values have been computed using Mathematica.

First, using the closed form expressions given in Example 3.10 (with $\theta = \mu$) we give the respective infinite horizon survival probability in infinite time horizon without treaty and with treaty in the following tables.

Table 1 gives numerical values for the infinite horizon survival probability $1 - \psi^{(N)}(\cdot)$ for Ruin problem (N); note that these are values for infinite horizon survival probability corresponding to the situation without treaty with respective initial capitals. Note that the net profit condition holds.

Tables 2 and 3 give numerical values for infinite horizon survival probability $1 - \psi^{(E)}(\cdot)$ for Ruin problem (E) with $a = \frac{1}{4}$, and indicated values for c_1, c_2, z_1, z_2; these correspond to the situation with treaty. (Since $a = \frac{1}{4}$, it is understood that shareholders of Company 1 can provide, through capital injection, $\frac{3}{4}$ of any possible deficit without difficulty.) As $c_1 > 1$, note that hypothesis (A5) is satisfied and $z_2 > (1 - a)z_1$ in all the cases in the two tables.

From the values above, it is clear that there is a very significant increase in survival probability in infinite horizon when treaty is operational.

By (3.13), we know that, operating under the treaty is advantageous when (A0)–(A5) hold, for any $z_1 \geq 0, z_2 \geq 0$. However, from the expressions given in Example 3.10, with a fixed, it is clear that larger the relative size of z_2 to that of z_1, more significant is the advantage of the treaty. This conforms to common sense, and is borne out by the numerical values above.

Table 1: $\lambda = 1$, $\mu = 1$ and $c = 1.1$

z_1=0	z_1=1	z_1=2	z_1=10
0.0909091	0.169908	0.242043	0.633736

Table 2: $\lambda = 1,\ \mu = 1,\ a = \frac{1}{4},\ c_1 = 1.05$ and $c_2 = 0.05$

z_1=0, $z_2 = 10$	z_1=1, $z_2 = 10$	z_1=2, $z_2 = 10$	z_1=10, $z_2 = 10$	z_1=10, $z_2 = 9$
0.999732	0.99978	0.99982	0.999964	0.999919

Interpreted in another way, relatively smaller values of capital z_2 from Company 2 would not make the treaty attractive for Company 1, and hence the treaty would be less likely to be entered into; in turn, chances of Company 2 making significant gains on his capital become less likely.

Many authors have done extensive work on suitable numerical methods for finding finite horizon ruin probability, with quite a few based on Seal's integral formulae; see Asmussen and Albrecher (2010), Rolski et al. (1999) and references therein. However, we will use the explicit expression for the survival probability derived in equation (24) of Garcia (2005),p.119 to give some numerical results concerning finite horizon survival probability without treaty and with treaty. Again we assume that claims have the exponential distribution with mean $\frac{1}{\mu}$.

By equation (24) of Garcia (2005) the finite horizon survival probability for Ruin problem (N) is given by

$$\phi^{(N)}(z_1, T) = \ 1 + \frac{\exp\{-[(\lambda+c\mu)T+\mu z_1]\}}{\mu}\Big[\sum_{k=0}^{\infty} \frac{(z_1+cT)^k(\lambda\mu T)^{k+1}}{k!}\Big(\frac{1}{(k+1)!} - \sum_{j=0}^{\infty}\Big[(\frac{c}{\lambda})^j + (\frac{1}{\mu})^j\Big]\frac{(\lambda\mu T)^j}{(j+k+1)!}\Big)\Big]. \tag{5.1}$$

This corresponds to the probability of survival up to time T without the treaty. Similarly the finite horizon survival probability for Ruin problem (E) is given by

$$\phi^{(E)}(az_1 + z_2, T) = 1 + \frac{a\exp\{-[(\lambda + (ac_1 + c_2)\frac{\mu}{a})T + \frac{\mu}{a}(az_1 + z_2)]\}}{\mu} \Big[\sum_{k=0}^{\infty} \frac{(az_1 + z_2 + (ac_1 + c_2)T)^k(\lambda\frac{\mu}{a}T)^{k+1}}{k!}\Big(\frac{1}{(k+1)!} - \sum_{j=0}^{\infty}\Big[(\frac{(ac_1 + c_2)}{\lambda})^j + (\frac{a}{\mu})^j\Big]\frac{(\lambda\frac{\mu}{a}T)^j}{(j+k+1)!}\Big)\Big], \tag{5.2}$$

Table 3: $\lambda = 1,\ \mu = 1,\ a = \frac{1}{4},\ c_1 = 1.09$ and $c_2 = 0.01$

z_1=0, $z_2 = 10$	z_1=1, $z_2 = 10$	z_1=2, $z_2 = 10$	z_1=10, $z_2 = 10$	z_1=10, $z_2 = 9$
0.99112	0.992085	0.992945	0.99719	0.995547

which corresponds to the probability of survival up to time T while operating under risk diversifying treaty.

In Table 4, using (5.1), we give numerical values of finite horizon survival probability for Ruin problem (N); note that the parameters chosen here are the same as in Table 1 on p.128 of Garcia (2005), and hence our values are in agreement with those in column (1) of that table.

In Tables 5 and 6, using (5.2), we give some numerical values of finite horizon survival probability for Ruin problem (E). Again, note that $z_2 > (1-a)z_1$ in all the cases below.

It is clear from Tables 4–6 that there is significant increase in finite horizon survival probability when the treaty is operational.

As another illustration we next consider the Cramer-Lundberg model with claim sizes having a hyperexponential distribution; note that hyperexponential distribution is a phase-type distribution. We consider only the infinite horizon ruin probability. Explicit expression for infinite horizon ruin probability is given in Theorem 8.3.1 of Rolski et al. (1999). Assume that claims X_j are i.i.d. random variables having distribution function $F(x) = 1 - b\exp(-\alpha x) - (1-b)\exp(-\beta x)$ with $0 < b < 1,\ \alpha > 0,\ \beta > 0$ and $x > 0$. In this case explicit expression for the Ruin problem (N) is given by

$$\psi^{(N)}(z_1) = \frac{(\varrho - r_1)\exp(-r_1 z_1) + (r_2 - \varrho)\exp(-r_2 z_1)}{(\rho^{(N)} + 1)(r_2 - r_1)} \tag{5.3}$$

where

$$r_1 = \frac{\rho^{(N)}(\alpha+\beta) + \varrho - \sqrt{(\rho^{(N)}(\alpha+\beta)+\varrho)^2 - 4\alpha\beta\rho^{(N)}(\rho^{(N)}+1)}}{2(\rho^{(N)}+1)}$$

Table 4: $\lambda = 1,\ \mu = 1$ and $c = 1.1$

T	z_1=0	z_1=1	z_1=2	z_1=10
1	0.536599	0.761944	0.880294	0.999692
2	0.407136	0.645431	0.794328	0.99865
3	0.344789	0.574022	0.731541	0.99677
4	0.306693	0.524715	0.683593	0.994105
5	0.280402	0.488107	0.645581	0.990767
6	0.260881	0.459571	0.614552	0.986885
7	0.245662	0.436536	0.588633	0.98258
8	0.233374	0.417448	0.566579	0.977958
9	0.223189	0.401304	0.54753	0.973106
10	0.214573	0.387424	0.53087	0.968097

Table 5: $\lambda = 1,\ \mu = 1,\ a = \frac{1}{4},\ c_1 = 1.05$ and $c_2 = 0.05$

T	$z_1 = 0,\ z_2 = 1$	$z_1 = z_2$=1	$z_1 = z_2$=2	$z_1 = 10,\ z_2 = 8$
1	0.972998	0.987069	0.999716	1
2	0.942202	0.968861	0.998836	1
3	0.914091	0.950181	0.997362	1
4	0.889541	0.932559	0.995397	1
5	0.868273	0.916426	0.993065	1
6	0.849804	0.901818	0.990475	1
7	0.833673	0.888635	0.987718	1
8	0.819489	0.876735	0.984866	1
9	0.806935	0.865969	0.981971	1
10	0.795751	0.856205	0.979075	1

$$r_2 = \frac{\rho^{(N)}(\alpha+\beta) + \varrho + \sqrt{(\rho^{(N)}(\alpha+\beta)+\varrho)^2 - 4\alpha\beta\rho^{(N)}(\rho^{(N)}+1)}}{2(\rho^{(N)}+1)}$$

and

$$p = \frac{b}{\alpha\left(b\alpha^{-1} + (1-b)\beta^{-1}\right)},\quad \varrho = \alpha(1-p) + \beta p,\quad \rho^{(N)} = \frac{c_1 + c_2}{a\lambda\left(b\alpha^{-1} + (1-b)\beta^{-1}\right)} - 1.$$

Similarly for the Ruin problem (E) we have

$$\psi^{(E)}(az_1 + z_2) = \frac{(\varrho - r_1)\exp(-r_1(az_1+z_2)) + (r_2-\varrho)\exp(-r_2(az_1+z_2))}{(\rho^{(E)}+1)(r_2-r_1)} \tag{5.4}$$

Table 6: $\lambda = 1,\ \mu = 1,\ a = \frac{1}{4},\ c_1 = 1.09$ and $c_2 = .01$

T	$z_1 = 0,\ z_2$=1	$z_1 = z_2$=1	$z_1 = z_2$=2	$z_1 = 10,\ z_2$=8
1	0.971575	0.986346	0.999697	1
2	0.936985	0.965808	0.998690	1
3	0.903925	0.943676	0.996899	1
4	0.874026	0.921984	0.994390	1
5	0.847382	0.901492	0.991282	1
6	0.823684	0.882438	0.987700	1
7	0.802546	0.864835	0.983758	1
8	0.783604	0.848608	0.979555	1
9	0.766544	0.833644	0.975171	1
10	0.751099	0.819828	0.970673	1

where

$$r_1 = \frac{\rho^{(E)}(\alpha+\beta)+\varrho-\sqrt{(\rho^{(E)}(\alpha+\beta)+\varrho)^2-4\alpha\beta\rho^{(E)}(\rho^{(E)}+1)}}{2(\rho^{(E)}+1)}$$

$$r_2 = \frac{\rho^{(E)}(\alpha+\beta)+\varrho+\sqrt{(\rho^{(E)}(\alpha+\beta)+\varrho)^2-4\alpha\beta\rho^{(E)}(\rho^{(E)}+1)}}{2(\rho^{(E)}+1)}$$

and

$$p = \frac{b}{\alpha\left(b\alpha^{-1}+(1-b)\beta^{-1}\right)}, \quad \varrho = \alpha(1-p)+\beta p,$$
$$\rho^{(E)} = \frac{ac_1+c_2}{a\lambda\left(b\alpha^{-1}+(1-b)\beta^{-1}\right)} - 1$$

In Table 7, using (5.3), we give some numerical values of ultimate survival probability for Ruin problem (N).

Tables 8 and 9 give numerical values for infinite horizon survival probability $1-\psi^{(E)}(\cdot)$ for Ruin problem (E) with $a=\frac{1}{4}$, and indicated values for c_1, c_2, z_1, z_2; these correspond to the situation with treaty.

Again the advantage of operating under the risk diversification treaty is clear.

5.2. Pareto Claims. As mentioned in Remark 1.1, ours is a formal model. However, we consider here a framework where the one-dimensional set up representing the basic insurance business of Company 1, in the absence of risk diversifying treaty, is obtained via a real life situation; the two-dimensional set up representing the situation under risk diversifying treaty includes two additional parameters a, c_2 chosen in an adhoc fashion; (see Remark 1.1 for the reasons). Advantage of operating under the treaty is again seen.

It is known that Pareto distribution, which is a heavy tailed distribution, occurs frequently as claim size distribution in real life situations. Let $\alpha > 0, \kappa > 0$. Recall that the right tail of Pareto(α,κ) distribution is given by

$$\overline{F}(x) = 1-F(x) = \frac{\kappa^\alpha}{(\kappa+x)^\alpha}, \ x>0. \tag{5.5}$$

Table 7: $\lambda=1,\ \alpha=1/2,\ \beta=2,\ b=1/3$ and $c=1.1$

z_1=0	z_1=1	z_1=2	z_1=10
0.0909091	0.157451	0.20907	0.508654

Table 8: $\lambda = 1,\ \alpha = 1/2,\ \beta = 2,\ b = 1/3,\ a = \frac{1}{4},\ c_1 = 1.05$ and $c_2 = 0.05$

z_1=0, $z_2 = 10$	z_1=1, $z_2 = 10$	z_1=2, $z_2 = 10$	z_1=10, $z_2 = 10$	z_1=10, $z_2 = 9$
0.786992	0.793657	0.800114	0.844999	0.823981

Note that only moments of order $< \alpha$ are finite. For $\alpha > 1$, the mean is $\mu = \frac{\kappa}{(\alpha-1)}$. Also, in such a case, the integrated tail distribution is Pareto$(\alpha-1,\kappa)$ distribution. Moreover, both are subexponential distributions.

We consider Ruin problem (N) for Cramer-Lundberg model where claim sizes have Pareto(α,κ) distribution with $1 < \alpha < 2$; this is one dimensional ruin problem. In such a case, the 'expected value principle' is the only natural way to choose the premium rate, that is, $c = (1+\rho)\lambda\mu$, where λ is the rate of claim arrivals, μ is as above, and $\rho > 0$ is the so called safety loading factor; see for example Ramasubramanian (2009). It is easy to see that $\rho = \frac{c}{\lambda\mu} - 1 = \rho^{(N)}$, where $\rho^{(N)}$ is as in (3.16). Fixing $\rho^{(N)}$ would fix c as well. While larger value of $\rho^{(N)}$ would imply greater safety for the insurer, on the other hand it could also make the insurance policy less attractive to the end-user. Note that (3.16) in Theorem 3.13 gives a measure of risk for a large initial capital z_1; hence (3.16) can be used to determine a suitable z_1 so that the $\psi^{(N)}(z_1)$ is very small. Since the integrated tail distribution is Pareto$(\alpha-1,\kappa)$, for a preassigned small $\rho^{(N)} > 0,\ \epsilon > 0$, one may choose z_1 such that

$$\frac{1}{\rho^{(N)}}\frac{\kappa^{(\alpha-1)}}{(\kappa+z_1)^{(\alpha-1)}} = \epsilon. \tag{5.6}$$

We now look at a specific illustration of a Cramer-Lundberg model involving theft claims. Data set of theft claim amounts is given in Chapter 2 of Boland (2007); it is shown there that a Pareto distribution fits the data best as appropriate claim size distribution. Accordingly $\lambda = 120,\ \alpha = 1.88047,\ \kappa = 1872.13176$. Clearly $\mu = \frac{\kappa}{(\alpha-1)} = 2126.3$. We choose safety loading factor $\rho^{(N)} = 0.1$; so for Ruin problem (N), the premium rate is $c = (1.1)\lambda\mu = 280670$; (as this is a collective risk model, c represents the claim income from the totality of all policy holders in a unit of time.) Let $\epsilon = 0.1$. Using (5.6) we get

$$z_1 = \kappa\big[(\epsilon\rho^{(N)})^{-\frac{1}{\alpha-1}} - 1\big] = 347950;$$

Table 9: $\lambda = 1,\ \alpha = 1/2,\ \beta = 2,\ b = 1/3,\ a = \frac{1}{4},\ c_1 = 1.09$ and $c_2 = 0.01$

z_1=0, $z_2 = 10$	z_1=1, $z_2 = 10$	z_1=2, $z_2 = 10$	z_1=10, $z_2 = 10$	z_1=10, $z_2 = 9$
0.592211	0.599755	0.607159	0.661666	0.63543

that is, a safety loading factor $\rho^{(N)} = 0.1$, and an initial capital $z_1 = 347950$ would ensure that the probability of ruin to be approximately 0.1, for Company 1 in the absence of risk diversifying treaty.

Next we look at Ruin problem (E) to assess the effect of risk diversifying treaty. Take $a = \frac{1}{4}$, $c_1 = (1.05)\lambda\mu = 267910$, $c_2 = (0.05)\lambda\mu = 12760$; (clearly $c_1 + c_2 = c$.) The premium rate in this problem is $ac_1 + c_2 = (0.3125)\lambda\mu = 79736$. By Theorem 3.13, the safety loading factor for Ruin problem (E) is

$$\rho^{(E)} = \frac{ac_1 + c_2}{\lambda a \mu} = 0.25.$$

Hence by (3.17) in Theorem 3.13 (with z_2 denoting the initial capital of Company 2),

$$\psi^{(E)}(az_1 + z_2) \quad \approx \quad 4\frac{\kappa^{(\alpha-1)}}{(\kappa + z_1 + 4z_2)^{(\alpha-1)}}, \tag{5.7}$$

where $\approx$ denotes approximate equality. By (5.7)

$$\begin{aligned} \psi^{(E)}(az_1 + z_2) \quad &\approx \quad 0.04, \quad \text{if} \quad z_2 = 0, \\ &\approx \quad 0.022, \quad \text{if} \quad z_2 = \frac{1}{4}z_1, \\ &\approx \quad 0.0097, \quad \text{if} \quad z_2 = z_1. \end{aligned}$$

Advantages of operating under the treaty is clear from the above. While the insurance policy is marketed at an attractive safety loading factor $\rho^{(N)} = 0.1$, under the treaty, Company 1 gets the advantage of a safety loading factor $\rho^{(E)} = 0.25$, provided the shareholders of Company 1 can manage to take care of the fraction $1 - a = 3/4$ of any potential deficit. As for as Company 2 is concerned, even when its initial capital $z_2 = 0$, the probability that it may never need to rescue Company 1 is greater than 0.95; if $z_2 = z_1$ then such a probability exceeds 0.99.

6 Concluding Remarks

In our formal risk diversification model, formulated in terms of Skorokhod problem in an orthant, there is a canonical notion of ruin. As multidimensional problems are usually quite difficult, a natural method is to first look at problems that can be reduced to a mathematically tractable one dimensional problem. This paper is one such attempt where a two dimensional problem is studied using a one dimensional reduction, and that too in a limited context. In such a set up we have demonstrated, both analytically as well as in terms of numerical examples, that operating under the risk diversification treaty is considerably more advantageous than without the treaty.

We now indicate some possible directions for future investigations. We do this in the context of general multidimensional insurance models described in terms of Skorokhod problem in a $d-$dimensional orthant, of which the set up considered in this paper is a special case. One may consult Ramasubramanian (2006), Ramasubramanian (2011), Ramasubramanian (2012) also to get a perspective about these models.

1. Perhaps the most important need is to have a good handle on the ruin probability in a full fledged general multidimensional model in which all the companies have insurance business of their own, as well as operating under risk diversification treaty. A single tractable integro-differential equation for ruin probability, or a Pollaczek-Khinchine type formula for ruin probability will be very useful.

2. A result as indicated in (1) above might also be helpful to try out well-directed simulations, approximations and numerical methods. Such an investigation in the two dimensional case might be good place to start with.

3. In the model there are quite a few parameters, a_j, R_{ij}, c_i, z_i, etc. We have been tacitly assuming that these quantities have been negotiated and agreed upon by the companies of the network. An interesting question is how to choose them in some optimal fashion, and what could be the optimality yardsticks. (Recall that we have already remarked in the preceding section that in our two dimensional model larger values of z_2 will be more advantageous.) This may lead to nice control theoretic or game theoretic problems. Even in the two dimensional situation the problems may not be simple.

4. We have indicated towards the end of Remark 2.6 that our method will not be applicable if claims for different companies of the network occur at different arrival times. Suitable methods have to be devised to handle such situations.

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Appendix: Proofs

Proof of Theorem 2.1: As $c_1, c_2 > 0$, by $(A1)$ note that $H_1(\cdot), H_2(\cdot)$ are positive and strictly increasing on (T_n, T_{n+1}) for $n \geq 0$. So by (SP2), (SP3) it follows that for $n \geq 0, i = 1, 2,\ Z_i(\cdot) > 0$ is strictly increasing on $(T_n, T_{n+1}),\ Z_i(t) > 0,\ Y_i(t) = Y_i(T_n),\ t \in (T_n, T_{n+1})$. Thus the first assertion of the theorem follows.

Next let $t_0 > 0$ be such that $Y_2(t_0) - Y_2(t_0-) > 0$; by (SP3) note that $Z_2(t_0) = 0$; (of course, $t_0 = T_n$ for some n). So by (2.2), we get $Z_2(t_0-) < a(Y_1(t_0) - Y_1(t_0-))$; as $a > 0,\ Z_2(t_0-) \geq 0$ we get $Y_1(t_0) - Y_1(t_0-) > 0$. This proves the assertion (ii).

To prove (iii), suppose $Z_2(T_n) = 0$. It is enough to show that $Y_1(T_n) - Y_1(T_n-) > 0$ by (SP3). Suppose $Y_1(T_n) - Y_1(T_n-) = 0$. Then by (2.2) we get

$$\begin{aligned} 0 &= Z_2(T_n-) + Y_2(T_n) - Y_2(T_n-) - a(Y_1(T_n) - Y_1(T_n-)) \\ &= Z_2(T_n-) + Y_2(T_n) - Y_2(T_n-) > 0 \end{aligned}$$

as $Z_2(\cdot)$ is strictly increasing on (T_{n-1}, T_n) and $Y_2(\cdot)$ is nondecreasing, This is a contradiction and hence (iii) is proved.

For (iv), sufficiency is clear by (SP3). To prove necessity, let $Z_1(T_n) = 0$ (we may assume $t_0 = T_n$ for some n by (i)). If possible let $Y_1(T_n) - Y_1(T_n-) = 0$. Then by (2.1) we get

$$\begin{aligned} 0 &= Z_1(T_n-) - X_n + Y_1(T_n) - Y_1(T_n-) \\ &= Z_1(T_n-) - X_n \end{aligned}$$

So $X_n = Z_1(T_n-)$. However note that $Z_1(T_n-)$ is $\sigma\{T_i,\ i \leq n,\ X_l,\ l \leq n-1\}-$measurable. But X_n is independent of $\sigma\{T_i,\ i \leq n,\ X_l,\ l \leq n-1\}$ by (A2), (A3). And as X_n is absolutely continuous (by (A4)) it now follows that $P(X_n = Z_1(T_n-)) = 0$. Thus $Y_1(T_n) - Y_1(T_n-) > 0$ with probability one.

Finally let $Z_2(T_n) = 0$. If possible let $Y_2(T_n) - Y_2(T_n-) = 0$. By (iii) we have $Z_1(T_n) = 0$ and hence $Y_1(T_n) - Y_1(T_n-) = X_n - Z_1(T_n-)$ by (2.1). So by (2.2), we get $0 = Z_2(T_n-) - a(X_n - Z_1(T_n-))$. So $X_n = \frac{1}{a} Z_2(T_n-) + Z_1(T_n-)$. However $P(X_n = \frac{1}{a} Z_2(T_n-) + Z_1(T_n-)) = 0$ as before using (A2), (A3), (A4). Thus $Y_2(T_n) - Y_2(T_n-) > 0$ with probability 1. This completes the proof of the theorem.

Proof of Theorem 2.2: Eliminating $Y_1(\cdot, \omega)$ from (2.1), (2.2) and using (2.6) result in (2.7). Clearly (S2) holds, $Y_2(0) \equiv 0,\ Y_2(\cdot)$ is nondecreasing. So to complete the proof we need to prove

$$[aZ_1(t, \omega) + Z_2(t, \omega)][Y_2(t, \omega) - Y_2(t-, \omega)] = 0. \tag{6.1}$$

As $Z_2(t) \cdot [Y_2(t) - Y_2(t-)] = 0$ by (SP3), we just need to verify $Z_1(t, \omega) \cdot [Y_2(t, \omega) - Y_2(t-, \omega)] = 0$. If $Y_2(t, \omega) - Y_2(t-, \omega) = 0$ the conclusion is clear.

So let $(Y_2(t,\omega) - Y_2(t-,\omega)) > 0$. Then $Z_2(t,\omega) = 0$ and hence $Z_1(t,\omega) = 0$ by Theorem 2.1. So (6.1) holds.

Proof of Theorem 2.5: As $c_1 > 0, c_2 > 0, a > 0, T_1 > 0$ with probability one, note that $H(t) = aZ_1(t) + Z_2(t) > 0$ for $t \in (0, T_1)$ even if $z_1 = z_2 = 0$. By Theorem 2.1 (iii) $\tau_0 = \tau_2$. As (A4) is assumed, by Theorem 2.1(iv) $\tau_0 = \tau_4$ with probability 1. By (A4) note that $(aX_j),\ j \geq 1$ are absolutely continuous; so it is clear that $\tau_1 = \tau_3$ with probability 1. Clearly by (S1)-(S3) we get

$$\tau_3 = \inf\{t > 0 : aZ_1(t) + Z_2(t) = 0\} = \tau_2$$

with the last equality following from Theorem 2.1(iii). Thus $\tau_0 = \tau_1 = \tau_2 = \tau_3 = \tau_4$ with probability one.

Proof of Lemma 3.2: Now $E(X_1 - cA_1) = E(X_1 - c_1A_1) - c_2E(A_1) < 0$ by (3.1) as $c_2 > 0,\ c = c_1 + c_2$, and $A_1 > 0$ with probability 1. Next $E[aX_1 - (ac_1 + c_2)A_1] = aE(X_1 - c_1A_1) - c_2E(A_1) < 0$, again by (3.1) as $a > 0,\ c_2 > 0,\ A_1 > 0$. Since (3.9) holding is equivalent to net profit condition for Ruin problem (A), the last assertion is now obvious.

Proof of Lemma 3.3: As (A6) holds, $A_1 > 0,\ c > 0$ note that $E[e^{h(X_1-cA_1)}] < \infty$ for all h in a neighbourhood $(-h_1, h_0)$ of 0 where $h_1 > 0$ and h_0 is as in (A6). Moreover, as X_1 and A_1 are independent

$$\lim_{h\uparrow h_0} E[e^{h(X_1-cA_1)}] = \lim_{h\uparrow h_0}\{E(e^{hX_1})\cdot E(e^{-hcA_1})\} = E(e^{h_0cA_1})\cdot \lim_{h\uparrow h_0} E(e^{hX_1}) = +\infty,$$

by (3.11), since $h_0 \in (0,\infty)$. As (NPC) holds for Ruin problem (N) by the preceding lemma, the required result now follows by Proposition 5.5 of Ramasubramanian (2009). The second assertion follows by Theore 6.5.4 of Rolski et al. (1999).

Proof of Theorem 3.4: Part (ii) and the first assertion in (i) follow from Lemmas 3.2, 3.3. As the proof of (3.13) is entirely analogous to that of Theorem 4.1 concerning finite horizon ruin probability, it is not given here.

Proof of Theorem 3.5: Denote $g(h) = E[\exp\{h(aX_1 - (ac_1 + c_2)A_1)\}],\ h \in \mathbb{R}$ whenever it makes sense. By (3.10) in Lemma 3.2, net profit condition holds for Ruin problem (E). By (A6), as in the proof of Lemma 3.3, it follows that $g(h) < \infty$ for all $h \in (-h_2, \frac{1}{a}h_0)$ for some $h_2 > 0$, with h_0 given as in (A6). Since $0 < h_0 < \infty$, by (3.11) note that

$$\begin{aligned}\lim_{h\uparrow \frac{1}{a}h_0} g(h) &= \lim_{h\uparrow \frac{1}{a}h_0} E[e^{haX_1}]E[e^{-h(ac_1+c_2)A_1}] \\ &= E[\exp\{-\frac{1}{a}h_0(ac_1 + c_2)A_1\}] \cdot \lim_{h\uparrow h_0} E[e^{hX_1}] = +\infty.\end{aligned}$$

Then by the proof of Proposition 5.5 in Ramasubramanian (2009), there exist unique $h_m,\ r^{(E)}$ with $0 < h_m < r^{(E)} < \frac{1}{a}h_0$ such that $g(\cdot)$ attains unique minimum at $h = h_m$, strictly increasing on $(h_m,\ \frac{1}{a}h_0),\ g(h) = 1,\ h > 0$ only at $h = r^{(E)}$, and $g(h) < 1$ if and only if $0 < h < r^{(E)}$. So $r^{(E)}$ is the Lundberg coefficient for Ruin problem (E).

To show that $r^{(A)} < r^{(E)}$, note that

$$\begin{aligned} g(r^{(A)}) &= E[\exp\{r^{(A)}(aX_1 - acA_1) - r^{(A)}(1-a)c_2A_1\}] \\ &< E[\exp\{r^{(A)}(aX_1 - acA_1)\}] = 1 \end{aligned}$$

because $r^{(A)} > 0$ is the Lundberg coefficient for Ruin problem (A), $(1-a) > 0,\ c_2 > 0,\ A_1 > 0$. So it follows that $r^{(A)} < r^{(E)}$; hence the second assertion now follows by preceding theorem.

Finally (3.14) follows by (3.4) above and Theorem 6.5.4, pp. 255-256 of Rolski et al. (1999).

Proof of Theorem 3.8: First assertion follows by Theorem 3.4, while the second assertion follows by Theorem 5.4.2, p.172 of Rolski et al. (1999) and Lemma 3.3 and Theorem 3.5.

Proof of Theorem 3.13: Clearly $\rho^{(E)} = \rho^{(N)} + \frac{(1-a)c_2}{a\lambda E(X_1)}$, and hence $\rho^{(E)} > \rho^{(N)} > 0$ by Lemma 3.2. Now by Theorem 5.14 of Ramasubramanian (2009) $\frac{\psi^{(N)}(x)}{(1-F_I(x))} \to \frac{1}{\rho^{(N)}}$ as $x \to +\infty$, whence (3.16) follows. By preceding lemma and again Theorem 5.14 of Ramasubramanian (2009) we have $\frac{\psi^{(E)}(x)}{(1-F_I^{(a)}(x))} \to \frac{1}{\rho^{(E)}}$ as $x \to +\infty$, (3.17) now follows by (3.15).

Proof of Theorem 4.1: Recall that for Ruin problem (A), premium rate is $ac = ac_1 + ac_2$, while for Ruin problem (E) it is $ac_1 + c_2$; clearly $ac_1 + c_2 > a(c_1 + c_2)$. Of course, claim sizes are $aX_j, j \geq 1$ in both cases. Let $H^{(A)}(0) = H(0) = y$. As $ac_1 + ac_2 < ac_1 + c_2$, note that

$$\begin{aligned} &\Big\{\text{No ruin for } H^{(A)}(\cdot) \text{ during } [0,T]\Big\} = \Big\{\sum_{j=1}^{N(t)} aX_j \leq y + (ac_1 + ac_2)t,\ \forall t \in [0,T]\Big\} \\ \subseteqq\ &\Big\{\sum_{j=1}^{N(t)} aX_j \leq y + (ac_1 + c_2)t,\ \forall t \in [0,T]\Big\} = \Big\{\text{No ruin for } H(\cdot) \text{ during } [0,T]\Big\}. \end{aligned}$$

Now (4.5) easily follows.

Next, by (2.11) in Theorem 2.5 we know that

$$\psi((z_1, z_2), T) = \psi^{(E)}(az_1 + z_2, T), \tag{6.2}$$

similar to (3.4). Again, analogous to the situation in Section 3, we need to compare $\psi^{(E)}(az_1 + z_2, T)$ and $\psi^{(N)}(z_1, T)$ corresponding to Ruin problem

(E) and Ruin problem (N) respectively. By (3.12),

$$\psi^{(A)}(az_1, T) = \psi^{(N)}(z_1, T),\ z_1 \geq 0, \tag{6.3}$$

as (A0)-(A5) hold. (Similar argument is applicable to Theorem 3.4 as well.) It is easy to see that $y \mapsto \psi^{(A)}(y, T)$ is decreasing. Hence (6.2), (4.5), (6.3) now imply (4.6).

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