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Bayesian Estimation of the Change Point Using CUSUM Control Chart

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Abstract

The process personnel always seek the opportunity to improve the processes. One of the essential steps for process improvement is to quickly recognize the starting time or the change point of a process disturbance. The proposed approach combines the CUSUM control chart with the Bayesian estimation technique. We show that the control chart has some information about the change point and this information can be used to make an informative prior. Two Bayes estimators corresponding to the informative and a non informative prior along with MLE, frequentist approach, are considered. Their mean square error of estimators, are compared through a series of simulations. The results show that the Bayes estimator with the informative prior is more accurate and more precise for almost all values of the shift in the process mean compared to Bayes estimator with a non-informative prior and MLE.

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1 Introduction

The quality of products is always the key factor for a successful business. In order to have a high quality of products, the maintenance of a high quality process is essential. The typical technique for maintaining or monitoring a process is the use of statistical process control (SPC) charts. In this article, we will consider the cumulative sum (CUSUM) control chart, which may be used when we are interested in detecting small shifts in the process mean.

When the control chart signals that the process is out-of-control, the process personnel must initiate a search for the special cause of the process disturbance. It is important to distinguish the difference between the change point and the out-of-control signal time which is triggered by the SPC charts. The change point is the time that the disturbances affect the process and the SPC signal time is the time that the out-of-control signal will be triggered.

Actually, the former is followed by the latter. The purpose of knowing the change point time is to simplify the search for the special cause. If the change point time can be determined, the special cause can be identified more quickly, and appropriate actions needed to improve quality can be implemented sooner.

In their paper, Samuel et al. (1998) addressed the issue of estimating the change point of a normal process. Pignatiello and Samuel (2001) used the EWMA and CUSUM charts and the MLE to estimate the change point of a process. Shao and Hou (2004) provided some statistical properties for the change point estimators. In addition, Shao and Hou (2006) derived the change point estimators under the case where the S chart and MLE are used in a gamma process. Later Shao et al. (2006) used an X-bar control chart and MLE to estimate the change point of a gamma process. From Bayesian point of view, Dehghan Monfared and Lak (2013) used an X-bar control chart for estimating the change point of a normal process. The control chart has some information about the change point and this information can be used to make an informative prior, so, one can improve the estimation of parameters. Therefore, in this study, we use the Bayesian approach to the CUSUM chart for estimating the change point of a normal process.

It is important, however, that when one uses ML approach he/she is not able to use his/her prior information about the process to estimate the related parameters. For example, it is known that when the probability distribution of the quality characteristic of the product considerably changes it is expected that, this change will immediately be detected by SPC chart. On the other hand, the mild changes may go unnoticed for a moment. Thus, one can use these information to estimate the change point more accurately and also to obtain a good prior density.

The rest of this article is organized as follows. Section 2 introduces the model. In Section 3, the maximum likelihood estimator of the change point is derived. In Section 4, two Bayes estimators of the change point are computed. Section 5 compares the efficiency of three estimators and conclusions are made in Section 6.

2 The Model

This study assumes the process is initially in control, and the sample observations come from a normal distribution with a known mean, μ_0 and a known standard deviation, σ_0 (which need to be known or at least a good approximation of them exist to construct a CUSUM control chart). For the ith subsample, $i = 1, 2, 3, ...,$with size n_i, the CUSUM control chart plots the statistics $C_i^+ = \max\left\{0, z_i - k + C_{i-1}^+\right\}$ and $C_i^- = \max\left\{0, -z_i - k + C_{i-1}^-\right\}$ where $C_0^+ = C_0^- = 0$, $z_i = \frac{(\bar{X}_i - \mu_0)}{\sigma_0/\sqrt{n_i}}$ is the standardized subgroup average, σ_0 is the standard deviation of the process when it is in-control and k is the reference value (almost always taken as 0.5). If the test statistic $C_i^+(C_i^-)$ exceeds its decision interval, $h^+(h^-)$, one concludes that the process mean has increased (decreased) and that the process is therefore out of control. In this paper, we consider a CUSUM chart with reference value $k = 0.5$ and decision interval $h^+ = h^- = 4.77$. The CUSUM chart with these parameters has an in-control Average Run Length (ARL) of 370, the same as a 3 sigma Shewhart $\bar{X}$ control chart. To see a full discussion of how to specify the reference value k and decision interval $h^+(h^-)$ for a CUSUM control chart, see Montgomery (2009). It is assumed that after an unknown point in time τ (known as the process change point), the process mean changes from μ_0 to μ_1. It is also assumed that once the parameter μ_0 changed it remains at the new level of μ_1 until the root causes of the disturbance have been identified and removed. Let X_{ij} denote the jth observation in subgroup i with normal distribution $N(\cdot, \cdot)$. That is,

$$
\begin{aligned}
X_{ij} \overset{iid}{\sim} N(\mu_0, \sigma_0^2), i &= 1, 2, \ldots, \tau, \\
j &= 1, 2, \ldots, n_i,
\end{aligned}
$$

and

$$
\begin{aligned}
X_{ij} \overset{iid}{\sim} N(\mu_1, \sigma_0^2), i &= \tau + 1, \ldots, T, \\
j &= 1, 2, \ldots, n_i,
\end{aligned}
$$

where n_i is i th subgroup size, T is the signal time, actuated by cumulative sum (CUSUM) control chart, and is not a false alarm. The parameters μ_0, σ_0 and μ_1 are the process parameters, and $\overset{iid}{\sim}$ stands for independent and identically distributed. For convenience, one can set $n_i = n$ for $i = 1, \ldots, T$. In addition, it is assumed that $\delta = \frac{\mu_1 - \mu_0}{\sigma_0/\sqrt{n}}$ (or equivalently $\mu_1 = \mu_0 + \delta\sigma_0/\sqrt{n}$), is the unknown magnitude of the change in the process mean in the scale of the standard deviation of the ith subgroup sample mean. To obtain the estimates of the unknown parameters, especially change point, one needs the likelihood of the parameters which is the joint density of observation given the unknown parameters. Therefore, for fixed T and $\mathbf{x_i} = (x_{i1}, \ldots, x_{in}), i = 1, \ldots, T$, the joint density of the observations is,

$$\begin{aligned} f(\mathbf{x_1}, \ldots, x_T|\mu_1, \tau) &= f(\mathbf{x}|\mu_1, \tau) \\ &= A(\tau)\exp\left\{-\frac{1}{2\sigma_\tau^2}(\mu_1 - \bar{\bar{x}}_\tau)^2\right\}, \end{aligned} \tag{2.1}$$

where in the last expression $\bar{\bar{x}}_\tau = \frac{1}{T-\tau}\sum\limits_{i=\tau+1}^{T} \bar{x}_i$, $\sigma_\tau^2 = \frac{\sigma_0^2}{n(T-\tau)}$ and

$$A(\tau) = (2\pi\sigma_0^2)^{-\frac{nT}{2}} \exp\left\{-\frac{n}{2\sigma_0^2}\left(\sum_{i=1}^{T}\bar{x}_i^2 - 2\mu_0\sum_{i=1}^{\tau}\bar{x}_i + \tau\mu_0^2 - (T-\tau)\bar{\bar{x}}_\tau^2\right)\right\}.$$

For details see Appendix. Throughout the paper, if $\tau = 0$ the values of the product $\prod\limits_{i=1}^{\tau}$ and the sum $\sum\limits_{i=1}^{\tau}$ are defined as 1 and 0, respectively.

3 MLE of the Change Point

To find the MLE of the parameters τ and μ_1 it suffices to maximize the logarithm of the likelihood function with respect to these parameters. The logarithm of the likelihood function is

$$\log L(\mu_1, \tau|\mathbf{x}) = \log f(\mathbf{x}|\mu_1, \tau) = \log A(\tau) - \frac{n(T-\tau)}{2\sigma_0^2}\left(\mu_1 - \bar{\bar{x}}_\tau\right)^2. \tag{3.1}$$

We note that there are two unknowns in the log-likelihood function: τ and μ_1. If the change point τ were known, the MLE of μ_1 would be $\hat{\mu}_1 = \bar{\bar{X}}_\tau = (T-\tau)^{-1}\sum_{i=\tau+1}^{T}\bar{X}_i$, the average of the $T-\tau$ most recent subgroup averages. Substituting $\hat{\mu}_1$ into equation (1), we get

$$\log L(\tau|\mathbf{x}) = \frac{-nT}{2}\log(2\pi\sigma_0^2) - \frac{n}{2\sigma_0^2}\left\{\sum_{i=1}^{T}\bar{x}_i^2 - 2\mu_0\sum_{i=1}^{\tau}\bar{x}_i + \tau\mu_0^2 - (T-\tau)\bar{\bar{x}}_\tau^2\right\}.$$

It then follows that the value of τ that maximizes the log-likelihood function is,

$$\hat{\tau} = arg\max_\tau\left\{(T-\tau)(\bar{\bar{x}}_\tau - \mu_0)^2\right\}.$$

That is, $\hat{\tau}$ is the value of τ in the range $\tau = 0, 1, \ldots, T-1$ which maximizes $(T-\tau)(\bar{\bar{x}}_\tau - \mu_0)^2$. Due to complicated nature of this function, it has to be maximized in τ numerically, as we have done in Section 5.

4 Bayesian Estimation of the Change Point

To estimate the parameters by the Bayesian approach, first of all we need a prior distribution for parameters. This prior should reflect our prior information about the parameters. In absence of any relevant prior knowledge, one can use a noninformative prior. In our problem it is interesting to note that when μ_1 is known, then given μ_1, the structure of the CUSUM control chart leads us to a prior distribution for the change point, τ, based on the following argument.

Let $g(\cdot|\mu_1)$ be the conditional probability distribution of $\Delta^\star = T - \tau$ given μ_1. This probability distribution plays the main role in specifying an informative prior for τ. Although specifying the conditional probability distribution of $\Delta^\star$ given μ_1 maybe difficult in general, using Monte Carlo simulation methods, for each μ_1, it can be done with arbitrary accuracy (see Appendix for details).

The following theorem says, if the process shift is too large, the conditional probability distribution of $\Delta^\star$ given μ_1 is approximately a degenerate distribution at 1. This theorem helps to construct a prior density, which we describe it after the theorem.

Theorem 1. *Let $\Delta^\star = T - \tau$ be the difference between the signal time T and the change point τ, then as $\Delta\mu = \mu_1 - \mu_0$ approaches plus or minus infinity, the conditional probability distribution of $\Delta^\star$ given μ_1 approaches a degenerate distribution at 1, with probability 1.*

Proof. See Appendix.

Using this theorem, there exist a large enough positive real number M so that, $g(1|\mu_1 = \mu_0 + M\sigma_0) = g(1|\Delta\mu = M\sigma_0) \approx 1$,where $g(\cdot|\mu_1)$ is conditional density of $\Delta^\star$. Therefore, when $\mu_1 > \mu_0 + M\sigma_0$ (or equivalently, when $\Delta\mu > M\sigma_0$), $g(\cdot|\mu_1) \approx I_{\{1\}}(\cdot)$. The same result holds when $\mu_1 < \mu_0 - M\sigma_0$. Thus, $g(\cdot|\mu_1) \approx 1$ for $|\mu_1 - \mu_0| > M\sigma_0$.

From Bayesian point of view the observed value T, is fixed and the parameter τ is a random variable, one can use $P(\Delta^\star = \delta^\star|\mu_1) = g(\delta^\star|\mu_1)$ as a prior distribution for $\Delta^\star$, although he/she may constraint its support to the set $\{1, 2, \cdots, T\}$ and use a truncated distribution

$$\tilde{g}(\delta^\star|\mu_1) = \frac{g(\delta^\star|\mu_1)}{\sum_{k=1}^{T} g(k|\mu_1)}, \delta^\star = 1, 2, \cdots, T,$$

for $\Delta^\star$. Then given μ_1 one can write

$$\begin{aligned} P(\tau = u|\mu_1) &= P(T-\tau = T-u|\mu_1) \\ &= P(\Delta^\star = T-u|\mu_1) = \tilde{g}(T-u|\mu_1), u = 0,1,2,\cdots,T-1. \end{aligned}$$

Using this conditional probability, which can be approximated well for each μ_1 , a conditional prior for τ can be specified as follows. First note that $\tilde{g}(\cdot|\mu_1) = g(\cdot|\mu_1) \approx I_{\{1\}}(\cdot)$ for $|\mu_1 - \mu_0| > M\sigma_0$. To approximate $\tilde{g}(\cdot|\mu_1)$ when $|\mu_1 - \mu_0| < M\sigma_0$, we split the interval $(\mu_0, \mu_0 + M\sigma_0)$ into congruent subintervals $(\mu_1^{(i-1)}, \mu_1^{(i)})$, $i = 1, 2, \cdots, N$ with length $\frac{M\sigma_0}{N}$, $\mu_1^{(0)} = \mu_0$ and $\mu_1^{(N)} = \mu_0 + M\sigma_0$. Assume $p_{i\tau} = \tilde{g}(T-\tau|\mu_1^{(i)})$ is specified for $i = 0, 1, \cdots, N$. Then, for each $\mu_1^{(i-1)} < \mu_1 < \mu_1^{(i)}$, we approximate $\tilde{g}(T-\tau|\mu_1)$, the conditional distribution of τ given μ_1, by a line that passes through two points $(\mu_1^{(i-1)}, p_{i-1\tau})$ and $(\mu_1^{(i)}, p_{i\tau})$. Thus, given μ_1, using the symmetry of the normal distribution, the following conditional prior can be suggested for τ.

$$\tilde{\pi}_{M,N}(\tau|\mu_1) = \begin{cases} p_{i-1\tau} + m_{i\tau}(\mu_1 - \mu_1^{(i-1)}), & \text{if} \mu_1^{(i-1)} < \mu_1 < \mu_1^{(i)} \\ & \text{for} i = 1, 2, \ldots, N \\ \tilde{\pi}(\tau|2\mu_0 - \mu_1), & \mu_0 - M\sigma_0 \le \mu_1 < \mu_0 \\ I_{\{T-1\}}(\tau), & |\mu_1 - \mu_0| \ge M\sigma_0 \end{cases} \tag{4.1}$$

where,

$$m_{i\tau} = \frac{p_{i\tau} - p_{i-1\tau}}{\mu_1^{(i)} - \mu_1^{(i-1)}} = \frac{N(p_{i\tau} - p_{i-1\tau})}{M\sigma_0}, \qquad I_{\{T-1\}}(\tau) = \begin{cases} 1, & \text{if} \tau = T-1 \\ 0, & \text{o.w.} \end{cases}$$

for $\tau = 0, 1, \cdots, T-1$.

Therefore, the posterior distribution for τ is

$$\pi(\tau|\mathbf{x}) = \frac{\int_{-\infty}^{\infty} f(\mathbf{x}|\tau, \mu_1)\tilde{\pi}_{M,N}(\tau|\mu_1)\pi(\mu_1)d\mu_1}{m(\mathbf{x})},$$

where,

$$m(\mathbf{x}) = \sum_{\tau=0}^{T-1} \int_{-\infty}^{\infty} f(\mathbf{x}|\tau, \mu_1)\tilde{\pi}_{M,N}(\tau|\mu_1)\pi(\mu_1)d\mu_1, \tau = 0, 1, \ldots, T-1.$$

Using this posterior, under squared error loss, it is known that when the parameter space is R, the Bayes estimate of τ is $a(\mathbf{x}) = E(\tau|\mathbf{x})$ which minimizes $E((\tau - a(\mathbf{x}))^2|\mathbf{x})$. So the Bayes estimate is an integer value which minimizes $E((\tau - a(\mathbf{x}))^2|\mathbf{x})$. It can be shown that this integer value is the nearest integer number to $E(\tau|\mathbf{x})$.

4.1. Specifying Hyper Parameters. The hyper parameter M, can be set to be 10 because simulation studies show that when shift in mean is greater than $10\sigma_0$, the change instantly is detected by CUSUM control chart, and $T-1=\tau$ with probability 1 . In addition, for a sufficiently small $\epsilon>0$, N can be chosen such that

$$\sum_{k=0}^{\infty}|\tilde{\pi}_{n_1}(k)-\tilde{\pi}_{n_2}(k)|<\epsilon, \forall n_1, n_2>N,$$

where

$$\tilde{\pi}_{n_j}(\tau)=\int_{-\infty}^{+\infty}\tilde{\pi}_{M,n_j}(\tau|\mu_1)\pi(\mu_1)d\mu_1, j=1,2.$$

Simulation studies show that $N=100$ works quite well. For convenience throughout the paper the prior in (3) is shown as $\tilde{\pi}(\tau|\mu_1)$.

4.2. Bayes Estimator of the Change Point. Suppose $\pi(\mu_1)$, the prior distribution for μ_1, is identified, and let given μ_1, $\pi(\cdot|\mu_1)$ be a prior for τ. Under the squared error loss, the Bayes estimator is the nearest integer to the posterior mean,

$$E(\tau|\mathbf{x})=\frac{\sum_{\tau=0}^{T-1}\int_{-\infty}^{+\infty}\tau f(\mathbf{x}|\mu_1,\tau)\pi(\tau|\mu_1)\pi(\mu_1)d\mu_1}{\sum_{\tau=0}^{T-1}\int_{-\infty}^{+\infty}f(\mathbf{x}|\mu_1,\tau)\pi(\tau|\mu_1)\pi(\mu_1)d\mu_1}. \tag{4.2}$$

If we use the noninformative priors, $\pi(\mu_1)=1$ and $\pi(\tau|\mu_1)=T^{-1}, \tau=0,1,\dots,T-1$, the integrals in the above expression is computable and it is easy to show that

$$E(\tau|\mathbf{x})=\frac{\sum_{\tau=0}^{T-1}\tau(T-\tau)^{-\frac{1}{2}}L^{\star}(\tau|\mathbf{x})}{\sum_{\tau=0}^{T-1}(T-\tau)^{-\frac{1}{2}}L^{\star}(\tau|\mathbf{x})}, \tag{4.3}$$

where,

$$L^{\star}(\tau|\mathbf{x})\propto\exp\left\{-\frac{n}{2\sigma_0^2}\left(-2\mu_0\sum_{i=1}^{\tau}\bar{x}_i+\tau\mu_0^2-(T-\tau)\bar{\bar{x}}_{\tau}{}^2\right)\right\}$$

In the other hand, when $\pi(\mu_1)=1$ and $\pi(\tau|\mu_1)=\tilde{\pi}(\tau|\mu_1)$, the conditional informative prior in (3), computing $E(\tau|\mathbf{x})$ is not as straightforward as before.

Nothing the fact that,

$$\pi(\tau|\mathbf{x})\propto\int_{-\infty}^{\infty}f(\mathbf{x}|\tau,\mu_1)\tilde{\pi}(\tau|\mu_1)\pi(\mu_1)d\mu_1,$$

we have

$$\begin{aligned}\int_{-\infty}^{+\infty} f(\mathbf{x}|\mu_1,\tau)\pi(\tau|\mu_1)\pi(\mu_1)d\mu_1 &= \int_{-\infty}^{\mu_0-M\sigma_0} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)d\mu_1 \\ &+ \int_{\mu_0-M\sigma_0}^{\mu_0} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)d\mu_1 \\ &+ \int_{\mu_0}^{\mu_0+M\sigma_0} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)d\mu_1 \\ &+ \int_{\mu_0+M\sigma_0}^{+\infty} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)d\mu_1.\end{aligned} \tag{4.4}$$

Now, we compute expressions on the right side of the "=" in (6). Define

$$\begin{aligned}D_{i\tau}(x) &= \Phi\left(\frac{\mu_1^{(i)}-x}{\sigma_\tau}\right) - \Phi\left(\frac{\mu_1^{(i-1)}-x}{\sigma_\tau}\right) \\ d_{i\tau}(x) &= \phi\left(\frac{\mu_1^{(i)}-x}{\sigma_\tau}\right) - \phi\left(\frac{\mu_1^{(i-1)}-x}{\sigma_\tau}\right),\end{aligned}$$

where, $\Phi(\cdot)$ and $\phi(\cdot)$ are probability density function and probability distribution function of the standard normal distribution, respectively. For the first expression we have

$$\begin{aligned}\int_{-\infty}^{\mu_0-M\sigma_0} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)d\mu_1 &= \int_{-\infty}^{\mu_0-M\sigma_0} A(\tau)\exp\left\{-\frac{1}{2\sigma_\tau^2}(\mu_1-\bar{\bar{x}}_\tau)^2\right\} \\ &\quad \times I_{\{T-1\}}(\tau)d\mu_1 \\ &= A(\tau)I_{\{T-1\}}(\tau)\sqrt{2\pi}\sigma_\tau\Phi\left(\frac{\mu_0-M\sigma_0-\bar{\bar{x}}_\tau}{\sigma_\tau}\right).\end{aligned}$$

The second one is,

$$\begin{aligned}\int_{\mu_0-M\sigma_0}^{\mu_0} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)d\mu_1 &= A(\tau)\sqrt{2\pi}\sigma_\tau\sum_{i=1}^{N}\Bigg\{(p_{i-1\tau}-m_{i\tau}\mu_1^{(i-1)}) \\ &\quad \times D_{i\tau}(2\mu_0-\bar{\bar{x}}_\tau)m_{i\tau}\Big[\sigma_\tau d_{i\tau}(2\mu_0-\bar{\bar{x}}_\tau) \\ &\quad +(\bar{\bar{x}}_\tau-2\mu_0)D_{i\tau}(2\mu_0-\bar{\bar{x}}_\tau)\Big]\Bigg\},\end{aligned} \tag{4.5}$$

for details see Appendix and for the third part,

$$\int_{\mu_0}^{M\sigma_0+\mu_0} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)d\mu_1 = A(\tau)\sqrt{2\pi}\sigma_\tau \sum_{i=1}^{N}\Bigg\{\Big[p_{i-1\tau} - m_{i\tau}\mu_1^{(i-1)}\Big]D_{i\tau}(\bar{\bar{x}}_\tau) + m_{i\tau}\Big[-\sigma_\tau d_{i\tau}(\bar{\bar{x}}_\tau) + \bar{\bar{x}}_\tau D_{i\tau}(\bar{\bar{x}}_\tau)\Big]\Bigg\}, \quad (4.6)$$

for details see Appendix and for the forth term,

$$\int_{\mu_0+M\sigma_0}^{+\infty} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)\pi(\mu_1)d\mu_1 = A(\tau)I_{\{T-1\}}(\tau)\sqrt{2\pi}\sigma_\tau\left[\Phi\left(\frac{\bar{\bar{x}}_\tau-\mu_0-M\sigma_0}{\sigma_\tau}\right)\right],$$

For details see Appendix. Therefore, the posterior probability distribution is obtained,

$$\begin{aligned}\pi(\tau|\mathbf{x}) \propto B(\tau) &= \int_{-\infty}^{+\infty} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)\pi(\mu_1)d\mu_1 \\ &= A(\tau)\sqrt{2\pi}\sigma_\tau\{I_{\{T-1\}}(\tau)\left[\Phi\left(\frac{\mu_0-M\sigma_0+\bar{\bar{x}}_\tau}{\sigma_\tau}\right)+\Phi\left(\frac{\mu_0-M\sigma_0-\bar{\bar{x}}_\tau}{\sigma_\tau}\right)\right] \\ &+\sum_{i=1}^{N}\Bigg\{\Big[p_{i-1\tau}-m_{i\tau}\mu_1^{(i-1)}\Big]\Big[D_{i\tau}(\bar{\bar{x}}_\tau)+D_{i\tau}(2\mu_0-\bar{\bar{x}})\Big] \\ &-m_{i\tau}\sigma_\tau\Big[d_{i\tau}(\bar{\bar{x}}_\tau)+d_{i\tau}(2\mu_0-\bar{\bar{x}})\Big] \\ &+m_{i\tau}\Big[\bar{\bar{x}}_\tau D_{i\tau}(\bar{\bar{x}}_\tau)+(2\mu_0-\bar{\bar{x}}_\tau)D_{i\tau}(2\mu_0-\bar{\bar{x}}_\tau)\Big]\Bigg\}.\end{aligned}$$

Then, $\pi(\tau|\mathbf{x}) = \frac{B(\tau)}{\sum_{\tau=0}^{T-1} B(\tau)}$ and $E(\tau|\mathbf{x}) = \sum_{\tau=0}^{T-1} \tau\pi(\tau|x)$.

5 Comparison of Three Estimators

In this section, the efficiency of three estimators, the Bayes estimator with a non informative prior, the Bayes estimator with the aforementioned informative prior, with $M = 10, N = 100$, $\sigma_0 = 1$, and the MLE are evaluated through a series of simulations. The criteria of efficiency is mean square error, i.e. $MSE = E(\hat{\theta}-\theta)^2$,where θ is the parameter and $\hat{\theta}$ is its estimation. Of course, in simulation the MSE is estimated with $\frac{1}{s}\sum_{i=1}^{s}(\hat{\theta}_i-\theta)^2$ where s is the number of iteration. From the Bayesian point of view, we assume that μ_1 has the probability density $\pi(\mu_1) = 1, -\infty < \mu_1 < +\infty$. In addition, we assume that our estimators are used with the CUSUM control chart. When the CUSUM control chart signals that the mean of the process

Table 1: The Change point estimates and their standard deviations (in parenthesis) relative to an CUSUM control chart for Bayesian approaches. (True change point $\tau = 100$ and subgroup size $n = 1$)

δ	0.5	1	1.5	2	2.5	3
$\bar{T}$	135.81	109.97	105.98	103.76	102.96	102.44
	(30.32)	(5.41)	(2.67)	(1.2)	(0.89)	(0.7)
$\bar{\hat{\tau}}^{(IB)}$	116.86	101.95	100.29	100.02	99.92	99.88
	(19.74)	(5.71)	(3.76)	(1.92)	(1.07)	(0.71)
$\bar{\hat{\tau}}^{(NB)}$	98.93	95.52	96.06	98.10	98.94	99.44
	(17.5)	(8.72)	(7.66)	(4.07)	(2.03)	(1.77)
$\bar{\hat{\tau}}^{(ML)}$	101.37	100.23	98.61	99.37	99.67	99.73
	(25.84)	(6.69)	(8.62)	(4.16)	(1.8)	(1.46)

has changed, each of three estimators are then applied to the data to estimate the time of the change or the change point. A Monte Carlo simulation study was conducted to study the performance of these three estimators. Suppose $n = 1$, sample observations are randomly generated from standard Normal distribution for subgroups $1, 2, \ldots, \tau$. Then, starting with subgroup $\tau + 1$, observations were randomly generated from $N(\delta, 1)$ until the CUSUM control chart triggers a signal. For each of the values of $\tau = 100, 200, 400$ and $\delta = 0.5, 1, 1.5, 2, 2.5, 3$, this procedure is repeated a total of 1000 times. For each simulation run, all three estimators, namely Bayes estimator with a non informative prior denoted by $\hat{\tau}^{(NB)}$, Bayes estimator corresponding to our proposed prior denoted by $\hat{\tau}^{(IB)}$ and MLE denoted by $\hat{\tau}^{(ML)}$ are computed by using the aforementioned methods. The average values of $\hat{\tau}^{(NB)}$,

Table 2: The Change point estimates and their standard deviations (in parenthesis) relative to an CUSUM control chart for Bayesian approaches. (True change point $\tau = 200$ and subgroup size $n = 1$)

δ	0.5	1	1.5	2	2.5	3
$\bar{T}$	231.51	209.53	205.34	203.6	202.9	202.37
	(25.85)	(5.02)	(2.37)	(1.17)	(0.89)	(0.6)
$\bar{\hat{\tau}}^{(IB)}$	214.29	201.76	200.03	200.04	199.96	199.90
	(17.14)	(5.37)	(3.93)	(1.13)	(0.8)	(1.34)
$\bar{\hat{\tau}}^{(NB)}$	192.61	191.81	192.29	196.33	197.24	199.13
	(22.16)	(11.64)	(13.94)	(6.79)	(7.33)	(2.45)
$\bar{\hat{\tau}}^{(ML)}$	202.7	200.19	197.56	199.19	199.16	199.76
	(33.22)	(9.56)	(17.28)	(5.29)	(8.28)	(2.19)

Table 3: The Change point estimates and their standard deviations (in parenthesis) relative to an CUSUM control chart for Bayesian approaches. (True change point $\tau = 400$ and subgroup size $n = 1$)

δ	0.5	1	1.5	2	2.5	3
$\bar{T}$	435.26	410.24	405.44	403.71	402.93	402.46
	(27.47)	(5.7)	(2.07)	(1.45)	(0.79)	(0.68)
$\bar{\hat{\tau}}^{(IB)}$	416.21	402.00	400.21	400.13	399.88	399.88
	(22.78)	(5.54)	(2.33)	(1.33)	(1.1)	(0.63)
$\bar{\hat{\tau}}^{(NB)}$	382.98	376.81	385.19	392.35	394.79	398.15
	(34.51)	(32.62)	(24.04)	(13.86)	(10.16)	(4.96)
$\bar{\hat{\tau}}^{(ML)}$	402.25	396.42	399.51	399.10	399.24	399.87
	(39.24)	(30.68)	(4.83)	(6.58)	(6.75)	(0.89)

$\hat{\tau}^{(IB)}$ and $\hat{\tau}^{(ML)}$ denoted by $\bar{\hat{\tau}}^{(NB)}$, $\bar{\hat{\tau}}^{(IB)}$ and $\bar{\hat{\tau}}^{(ML)}$, respectively along with their standard deviations are given in Tables 1, 2 and 3.

We consider that for each fixed δ, for $\tau = 100, 200, 400$, the bias and standard deviation of $\hat{\tau}^{(IB)}$ is almost constant, while $\hat{\tau}^{(ML)}$ has constant bias, its standard deviation in most cases increases as τ increases. At last, we note that $\hat{\tau}^{(NB)}$ has non constant bias, its standard deviation increasing with τ in most cases.

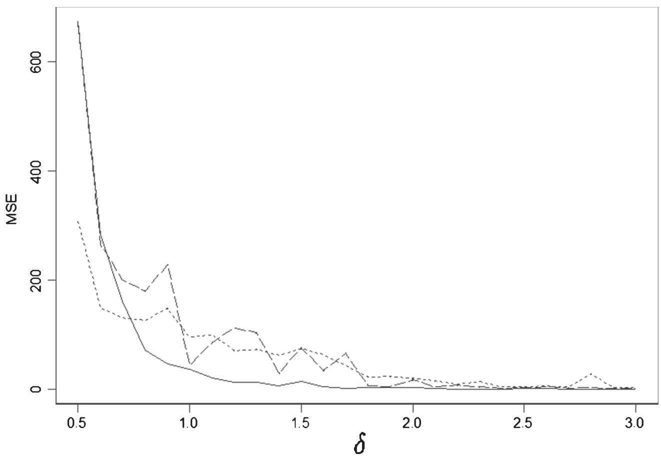

Figure 1: The MSE's of Bayes estimators with informative prior (*solid curve*), non informative prior (*dotted curve*) and MLE (*broken curve*). (True change point $\tau = 100$ and subgroup size $n = 1$)

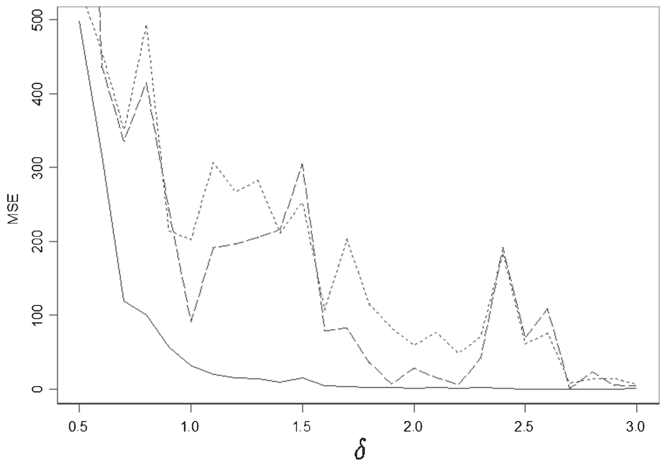

Figure 2: The MSE's of Bayes estimators with informative prior (*solid curve*), non informative prior (*dotted curve*) and MLE (*broken curve*). (True change point $\tau = 200$ and subgroup size $n = 1$)

To draw the figures for three estimators using their MSE, the horizon axis is δ and it changes from 0.5 to 3 by step 0.1. Based on MSE (Figure 1, 2, 3) criterion:

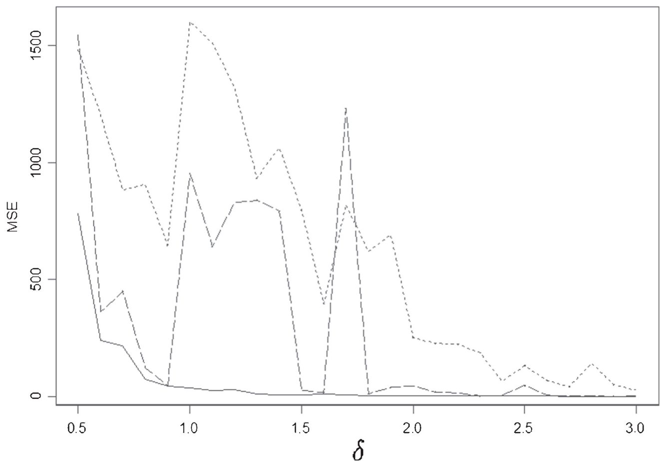

Figure 3: The MSE's of Bayes estimators with informative prior (*solid curve*) and non informative prior (*dotted curve*) and MLE (*broken curve*). (True change point $\tau = 400$ and subgroup size $n = 1$)

For $\tau = 100$, $\hat{\tau}^{(IB)}$ outperforms $\hat{\tau}^{(ML)}$ for $\delta = 0.7$ up to 3 and outperforms $\hat{\tau}^{(NB)}$ for $\delta = 0.8$ up to 3 but neither $\hat{\tau}^{(ML)}$ nor $\hat{\tau}^{(NB)}$ has a clear advantage over the other.

For $\tau = 200$, $\hat{\tau}^{(IB)}$ outperforms both $\hat{\tau}^{(ML)}$ and $\hat{\tau}^{(NB)}$ for for all values of δ and $\hat{\tau}^{(ML)}$ in most cases outperforms $\hat{\tau}^{(NB)}$.

For $\tau = 400$, $\hat{\tau}^{(IB)}$ outperforms $\hat{\tau}^{(ML)}$ and $\hat{\tau}^{(NB)}$ for all values of δ and $\hat{\tau}^{(ML)}$ outperforms $\hat{\tau}^{(NB)}$ for almost all values of δ.

For $\tau = 100, 200, 400$ we note that the efficiency of $\hat{\tau}^{(IB)}$ relative to $\hat{\tau}^{(NB)}$ and $\hat{\tau}^{(ML)}$ increases as τ increases.

6 Conclusion

In this study, we proposed a Bayesian approach for estimation of the change point when implementing an CUSUM control chart. First, it is shown that how it is possible to use the information in a CUSUM control chart and construct an informative prior for the change point. Then we used this informative prior to estimate the change point. In addition, to show the extent of effect of ignoring the information in the CUSUM control chart on the Bayes estimator, a non informative prior was also used. The efficiency of the resulting Bayes estimators and MLE were compared, through a series of simulations. This simulations show that based on the MSE criterion, unless the difference between means of the process before and after the change point time is too small, the CUSUM control chart has some information about the change point and using our proposed informative prior to estimate the change point is better than a non informative prior. In addition, when we use this informative prior, unless the shift in process mean is too small, the Bayes estimator, based on the MSE criterion, outperforms MLE while if the shift in process mean is not too small and if this information is ignored, the Bayes estimator is dominated by MLE. It was also shown that the efficiency of the Bayes estimator to two other estimators increases as τ increases.

The method we used here to construct an informative prior from a CUSUM control chart can be used with other control charts. In addition it can be applied to the cases where there exist more than one change point. In this paper, we used the CUSUM control chart to monitor a normal process. There are other types of control charts (for example, S control chart) which could be considered and our proposed method applied.

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Appendix

Approximating the conditional distribution of $T-\tau$

Because of the complexity of the structure of $g(\cdot|\mu_1)$, we use the following Monte Carlo simulation method to approximate it. Given a random sample $\Delta_1^\star, \cdots, \Delta_m^\star$ from $g(\cdot|\mu_1)$ then $\hat{g}(\cdot|\mu_1) = \frac{1}{m}\sum_{i=1}^m I_{\{\Delta_i\}}(\cdot)$ is an approximation for $g(\cdot|\mu_1)$. To draw $\Delta_i^\star$, the i th observation, note that if $\bar{X}_{\tau+1}^{(i)}, \cdots, \bar{X}_{T_i}^{(i)}, i = 1, \cdots, m$ be independent observations from $N(\mu_1, \frac{\sigma_0^2}{n})$. T_i's are the times at which the control chart trigger a signal, and $(C_{\tau+1}^-, C_{\tau+1}), \cdots, (C_{T_i}^-, C_{T_i}^+)$ are their control bounds respectively, then the random variables $\Delta_1^\star = T_1 - \tau, \cdots, \Delta_m^\star = T_m - \tau$ is a random sample of size m from $g(\cdot|\mu_1)$. Note that to obtain the control bounds we need to know C_τ^- and C_τ^+ which are unknown. But at time point τ, the process is under the control, thus it is assumed that $C_\tau^- = C_\tau^+ = 0$. By doing this the aforementioned method of drawing a random sample from $g(\cdot|\mu_1)$ does not depend on the parameter τ, thus for simplicity it is assumed that $\tau = 0$. Then we propose the following algorithm to derive a sample from $g(\cdot|\mu_1)$ and then approximate it by $\hat{g}(\cdot|\mu_1)$.

1. In step i, the sequence $\overline{X}_t^{(i)}, t = 1, \cdots, T_i$ are drawn from $N\left(\mu_1, \frac{\sigma_0^2}{n}\right)$, until at time T_i, the control chart triggers a signal.

2. Set $\Delta_i^\star = T_i$.

3. If $i \leq m$ go to step 1. In this paper whenever it is necessary, a Monte Carlo simulation method with $m = 5000$ replication is used to approximate $g(\cdot|\mu_1)$, where μ_1 is a known constant.

Likelihood function

$$\begin{aligned} f(\mathbf{x_1}, \ldots, \mathbf{x_T}|\mu_1, \tau) &= f(\mathbf{x}|\mu_1, \tau) \\ &= \prod_{i=1}^{\tau}\prod_{j=1}^{n} f(x_{ij}) \prod_{i=\tau+1}^{T}\prod_{j=1}^{n} f(x_{ij}|\mu_1) \\ &= \prod_{i=1}^{\tau}\prod_{j=1}^{n} (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_0^2}(x_{ij}-\mu_0)^2\right\} \\ &\quad \prod_{i=\tau+1}^{T}\prod_{j=1}^{n} (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_0^2}(x_{ij}-\mu_1)^2\right\} \end{aligned}$$

$$
\begin{aligned}
&= (2\pi\sigma_0^2)^{-\frac{nT}{2}} \exp\Bigg\{-\frac{1}{2\sigma_0^2}\Bigg(\sum_{i=1}^{\tau}\sum_{j=1}^{n}(x_{ij}-\mu_0)^2 \\
&\quad + \sum_{i=\tau+1}^{T}\sum_{j=1}^{n}(x_{ij}-\mu_1)^2\Bigg)\Bigg\} \\
&= (2\pi\sigma_0^2)^{-\frac{nT}{2}} \exp\Bigg\{-\frac{n}{2\sigma_0^2}\Bigg(\sum_{i=1}^{T}\bar{x}_i^2 - 2\mu_0\sum_{i=1}^{\tau}\bar{x}_i + \tau\mu_0^2 \\
&\quad -2\mu_1\sum_{i=\tau+1}^{T}\bar{x}_i + (T-\tau)\mu_1^2\Bigg)\Bigg\} \\
&= (2\pi\sigma_0^2)^{-\frac{nT}{2}} \exp\Bigg\{-\frac{n}{2\sigma_0^2}\Bigg(\sum_{i=1}^{T}\bar{x}_i^2 - 2\mu_0\sum_{i=1}^{\tau}\bar{x}_i + \tau\mu_0^2 \\
&\quad -(T-\tau)\bar{\bar{x}}_\tau^2 + (T-\tau)\times(\mu_1-\bar{\bar{x}}_\tau)^2\Bigg)\Bigg\} \\
&= A(\tau)\exp\Bigg\{-\frac{n(T-\tau)}{2\sigma_0^2}\Bigg(\mu_1-\bar{\bar{x}}_\tau\Bigg)^2\Bigg\} \\
&= A(\tau)\exp\Bigg\{-\frac{1}{2\sigma_\tau^2}(\mu_1-\bar{\bar{x}}_\tau)^2\Bigg\},
\end{aligned}
$$

Proof of theorem

$$
\begin{aligned}
\lim_{\Delta\mu\to\pm\infty} g(1|\mu_1) &= \lim_{\Delta\mu\to\pm\infty} P_{\mu_1}(\Delta^\star=1) = \lim_{\Delta\mu\to\pm\infty} P_{\mu_1}(T-\tau=1) \\
&= \lim_{\Delta\mu\to\pm\infty} P_{\mu_1}(T=\tau+1) \\
&= \lim_{\Delta\mu\to\pm\infty} P_{\mu_1}(C_{\tau+1}^+ > h^+ \text{or} C_{\tau+1}^- > h^-) \\
&= \lim_{\Delta\mu\to\pm\infty} P_{\mu_1}(C_{\tau+1}^+ > h^+) + \lim_{\Delta\mu\to\pm\infty} P_{\mu_1}(C_{\tau+1}^- > h^-).
\end{aligned}
$$

But, we have

$$
\begin{aligned}
\lim_{\Delta\mu\to+\infty} P_{\mu_1}(C_{\tau+1}^+ > h^+) &= 1-\lim_{\Delta\mu\to+\infty} P_{\mu_1}(C_{\tau+1}^+ < h^+) \\
&= 1-\lim_{\Delta\mu\to+\infty} P_{\mu_1}(Z_{\tau+1}-k+C_\tau^+ < h^+) \\
&= 1-\lim_{\Delta\mu\to+\infty} P\Bigg(\frac{\bar{X}_{\tau+1}-\mu_1}{\sigma_0/\sqrt{n}} + \frac{\mu_1-\mu_0}{\sigma_0/\sqrt{n}} \\
&\quad -k+C_\tau^+ < h^+\Bigg) \\
&\geq 1-\lim_{\Delta\mu\to+\infty} P\Bigg(\frac{\bar{X}_{\tau+1}-\mu_1}{\sigma_0/\sqrt{n}} + \frac{\Delta\mu}{\sigma_0/\sqrt{n}} - k < h^+\Bigg) \\
&= 1-\lim_{\Delta\mu\to+\infty} \Phi\Bigg(-\frac{\Delta\mu\sqrt{n}}{\sigma_0} + h^+ + k\Bigg) = 1.
\end{aligned}
$$

Then

$$\lim_{\Delta\mu\to+\infty} g(1|\mu_1) \geq \lim_{\Delta\mu\to+\infty} P_{\mu_1}(C^+_{\tau+1} > h^+) \geq 1.$$

Thus,

$$\lim_{\Delta\mu\to+\infty} g(1|\mu_1) = 1.$$

Similarly, it is easy to show that $\lim_{\Delta\mu\to-\infty} P_{\mu_1}(C^-_{\tau+1} > h^-) = 1$. Thus $\lim_{\Delta\mu\to\pm\infty} g(1|\mu_1) = 1$.

The second part of equation(6)

$$\begin{aligned}
\int_{\mu_0-M\sigma_0}^{\mu_0} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)d\mu_1 &= \sum_{i=1}^{N}\int_{2\mu_0-\mu_1^{(i)}}^{2\mu_0-\mu_1^{(i-1)}} A(\tau)\exp\left\{-\frac{1}{2\sigma_\tau^2}(\mu_1-\bar{\bar{x}}_\tau)^2\right\} \\
&\quad \times\left\{p_{i-1\tau}-m_{i\tau}(\mu_1+\mu_1^{(i-1)}-2\mu_0)\right\}d\mu_1 \\
&= \sum_{i=1}^{N}\int_{-\mu_1^{(i)}}^{-\mu_1^{(i-1)}} A(\tau)\exp\left\{-\frac{1}{2\sigma_\tau^2}(\mu_1-(\bar{\bar{x}}_\tau-2\mu_0))^2\right\} \\
&\quad \times\left[p_{i-1\tau}-m_{i\tau}(\mu_1+\mu_1^{(i-1)})\right]d\mu_1 \\
&= A(\tau)\sqrt{2\pi}\sigma_\tau\sum_{i=1}^{N}\left\{\left[p_{i-1\tau}-m_{i\tau}\mu_1^{(i-1)}\right]\right. \\
&\quad \times\left[\Phi\left(\frac{2\mu_0-\mu_1^{(i-1)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\right)-\Phi\left(\frac{2\mu_0-\mu_1^{(i)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\right)\right] \\
&\quad -m_{i\tau}\left[\sigma_\tau\left(\phi\left(\frac{2\mu_0-\mu_1^{(i)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\right)-\phi\left(\frac{2\mu_0-\mu_1^{(i-1)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\right)\right)\right. \\
&\quad \left.\left.+(\bar{\bar{x}}_\tau-2\mu_0)\left(\Phi\left(\frac{2\mu_0-\mu_1^{(i-1)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\right)-\Phi\left(\frac{2\mu_0-\mu_1^{(i)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\right)\right)\right]\right\} \\
&= A(\tau)\sqrt{2\pi}\sigma_\tau\sum_{i=1}^{N}\left\{\left[p_{i-1\tau}-m_{i\tau}\mu_1^{(i-1)}\right]\right. \\
&\quad \times\left[\Phi\left(\frac{\mu_1^{(i)}-(2\mu_0-\bar{\bar{x}}_\tau)}{\sigma_\tau}\right)-\Phi\left(\frac{\mu_1^{(i-1)}-(2\mu_0-\bar{\bar{x}}_\tau)}{\sigma_\tau}\right)\right] \\
&\quad -m_{i\tau}\left[\sigma_\tau\left(\phi\left(\frac{\mu_1^{(i)}-(2\mu_0-\bar{\bar{x}}_\tau)}{\sigma_\tau}\right)-\phi\left(\frac{\mu_1^{(i-1)}-(2\mu_0-\bar{\bar{x}}_\tau)}{\sigma_\tau}\right)\right)\right. \\
&\quad +(\bar{\bar{x}}_\tau-2\mu_0)\left(\Phi\left(\frac{\mu_1^{(i)}-(2\mu_0-\bar{\bar{x}}_\tau)}{\sigma_\tau}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& -\Phi\Big(\frac{\mu_1^{(i-1)}-(2\mu_0-\bar{\bar{x}}_\tau)}{\sigma_\tau}\Big)\Big)\Big]\Big\} \\
= \; & A(\tau)\sqrt{2\pi}\sigma_\tau\sum_{i=1}^{N}\Big\{(p_{i-1\tau}-m_{i\tau}\mu_1^{(i-1)}) \\
& \times D_{i\tau}(2\mu_0-\bar{\bar{x}}_\tau)m_{i\tau}\Big[\sigma_\tau d_{i\tau}(2\mu_0-\bar{\bar{x}}_\tau) \\
& +(\bar{\bar{x}}_\tau-2\mu_0)D_{i\tau}(2\mu_0-\bar{\bar{x}}_\tau)\Big]\Big\},
\end{aligned}
$$

The third part of equation(6)

$$
\begin{aligned}
\int_{\mu_0}^{M\sigma_0+\mu_0} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)d\mu_1 &= \sum_{i=1}^{N}\int_{\mu_1^{(i-1)}}^{\mu_1^{(i)}} A(\tau)\exp\Big\{-\frac{1}{2\sigma_\tau^2}(\mu_1-\bar{\bar{x}}_\tau)^2\Big\} \\
& \quad \times\Big\{p_{i-1\tau}+m_{i\tau}(\mu_1-\mu_1^{(i-1)})\Big\}d\mu_1 \\
&= A(\tau)\sqrt{2\pi}\sigma_\tau\Big\{\Big[p_{i-1\tau}-m_{i\tau}\mu_1^{(i-1)}\Big] \\
& \quad \times\Big[\Phi\Big(\frac{\mu_1^{(i)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\Big)-\Phi\Big(\frac{\mu_1^{(i-1)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\Big)\Big] \\
& \quad +m_{i\tau}\Big[\sigma_\tau\Big(\phi\Big(\frac{\mu_1^{(i-1)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\Big)-\phi\Big(\frac{\mu_1^{(i)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\Big)\Big) \\
& \quad +\bar{\bar{x}}_\tau\Big(\Phi\Big(\frac{\mu_1^{(i)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\Big)-\Phi\Big(\frac{\mu_1^{(i-1)}-\bar{\bar{x}}_\tau}{\sigma_\tau}\Big)\Big]\Big\} \\
&= A(\tau)\sqrt{2\pi}\sigma_\tau\sum_{i=1}^{N}\Big\{\Big[p_{i-1\tau}-m_{i\tau}\mu_1^{(i-1)}\Big]D_{i\tau}(\bar{\bar{x}}_\tau) \\
&+ m_{i\tau}\Big[-\sigma_\tau d_{i\tau}(\bar{\bar{x}}_\tau)+\bar{\bar{x}}_\tau D_{i\tau}(\bar{\bar{x}}_\tau)\Big]\Big\},
\end{aligned}
$$

The forth part of equation(6)

$$
\begin{aligned}
\int_{\mu_0+M\sigma_0}^{+\infty} f(\mathbf{x}|\mu_1,\tau)\tilde{\pi}(\tau|\mu_1)\pi(\mu_1)d\mu_1 &= \int_{\mu_0+M\sigma_0}^{+\infty} A(\tau)\exp\Big\{-\frac{1}{2\sigma_\tau^2}(\mu_1-\bar{\bar{x}}_\tau)^2\Big\}I_{\{T-1\}}(\tau)d\mu_1 \\
&= A(\tau)I_{\{T-1\}}(\tau)\sqrt{2\pi}\sigma_\tau\Big[1-\Phi\Big(\frac{\mu_0+M\sigma_0-\bar{\bar{x}}_\tau}{\sigma_\tau}\Big)\Big] \\
&= A(\tau)I_{\{T-1\}}(\tau)\sqrt{2\pi}\sigma_\tau\Big[\Phi\Big(\frac{\bar{\bar{x}}_\tau-\mu_0-M\sigma_0}{\sigma_\tau}\Big)\Big],
\end{aligned}
$$

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