Sankhyā : The Indian Journal of Statistics
2017, Volume 79-B, Part 2, pp. 278-291

Optimal Block and Row–Column Designs for CDC Methods

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Abstract

A simple method of construction of three series of optimal block designs for complete diallel cross method (4) has been presented by using nested balanced incomplete block designs of Dey et al. (*Calcutta Statist. Assoc. Bull.* **35**, 161–167 1986). The proposed designs are different from Gupta and Choi (*Communication in Statistics-Theory and Methods* **27(11)**, 2827–2835 1998) and Das et al. (*Statistics and Probability Letters* **36**, 427–436 1998)'s designs in parametric values. From the proposed designs we have also derived three series of optimal row–column designs for complete diallel cross method (4) and three series of A–optimal row–column designs for complete diallel cross method (2), respectively. By using the proposed method, we have also obtained some more designs of the above types from nested balanced incomplete block designs of Morgan et al. (*Discrete Mathematics* **231**, 351–389 2001). Tables of these optimal /A-optimal designs have been provided. Illustration of the layouts of the proposed designs along with their analysis has been provided.

AMS (2000) subject classification. 62 k 05.
Keywords and phrases. Nested balanced incomplete block design, Row–column design, Complete diallel cross, Optimality

1 Introduction

The diallel cross is a type of mating designs used in plant breeding to study the genetic properties of a set of inbred lines. Suppose there are p inbred lines and it is desired to perform two types of diallel cross experiments (i) involving$p(p-1)/2$ crosses, where cross $(i \times j) = (j \times i)$ and (ii) involving $p(p+1)/2$ crosses of the two types $(i \times j) = (j \times i)$ and $(i \times i)$, respectively, for $i, , j = 1, \ldots, p$. The first experiment is the complete diallel cross (CDC) method (4) and second is the complete diallel cross (CDC) method (2) (see, Griffing, 1956) with p inbred lines having $p(p-1)/2$ distinct F_1 crosses and p self and $p(p-1)/2$ distinct F_1 crosses, respectively. The problem of generating optimal mating designs for CDC method (4) has been investigated by several authors (see e.g. Singh et al. 2012). For CDC method (2) model of Griffing (1956) involves the general combining ability (gca) and specific combining ability (sca) effects of lines. Let n_c denote the total number of crosses involved in CDC method (2). It is desired to compare the average effects or gca effects of lines. Generally, the experiments of CDC Method (2) are conducted using either a completely randomized design (CRD) or a randomized complete block (RCB) design involving n_c crosses as treatments. The number of crosses in such mating design increases rapidly with increase in number of lines p. Thus, if p is large adoption of CRD or an RCB design is not appropriate unless the experimental units are extremely homogeneous. It is for this reason that the use of incomplete block design as environment design is needed for CDC method (2).

The available literature restricts to universal optimality and combinatorial aspects for CDC method (4) in the one-way elimination of heterogeneity set up except by Gupta and Choi (1998) who studied complete diallel cross method (4) in row–column (two-way elimination of heterogeneity) designs. Parsad et al. (2005) also constructed optimal row–column designs for CDC method (4) and optimal row–column designs for double cross experiments, respectively, by using nested balanced incomplete block designs for odd value of 'p' parents. However, row–column designs for CDC method (2) have not received any attention so far. This is because it is difficult to obtain connected designs for cross effects in row and column set up.

Aggarwal (1974) gave analysis of confounding diallel experiments II (method (2)) by using triangular partially balanced incomplete block designs. Sharma and Fanta (2011) gave optimal incomplete block designs for CDC method (2) by using mutually orthogonal Latin squares of order p, where p is a prime or power of a prime.

In the present article, we are presenting a simple method of construction of three series of optimal block designs for CDC experiment method (4) by using nested incomplete block designs of Dey et al. (1986) for p parental lines where p is prime or power of a prime. From the above designs, we derive three series of row–column designs for CDC experiment method (4) and for CDC experiment method (2) for 'p' parental lines in which two sources of nuisance variability is to be controlled. We have also investigated nested incomplete block designs of Morgan et al. (2001) and obtained block designs for CDC method (4), row–column designs for CDC method (4) and row–column designs for CDC method (2). Tables of these designs have also been provided. Row–column designs for CDC experimental method (4) is optimal while row–column designs for CDC method (2) is A–optimal. These designs require the minimal number of experimental units for estimating all pairwise comparison among *gca* parameters. These designs are different in parametric values from Gupta and Choi (1998), Das et al. (1998) and Parsad et al. (2005). Illustration of the layouts of the proposed designs along with their analysis has been provided. We have considered the model that includes the gca effects, apart from block effects, but no specific combining ability effects.

2 Universal Optimality of Designs for OneWay Heterogeneity

Let d be a block design for a CDC experiment method (4) involving p inbred lines, b blocks each of size k. This means that there are k crosses in each of the blocks of d. Further, let r_{dt} and s_{di} denote the number of replication of cross t and the number of replications of the line i in different crosses, respectively, in d $[t = 1, 2, \ldots, n_c; \mathrm{i} = 1, 2, \ldots, p]$. Evidently, $\sum r_t = bk, \sum s_i = 2bk = 2n$ and $n = bk$, the total number of observations. We took the following additive model for the observationsobtained from design d

$$\mathbf{y} = \mu \mathbf{1_n} + \Delta_1' \mathbf{g} + \Delta_2' \beta + \mathbf{e} \tag{2.1}$$

where $\mathbf{y}$ is an $n \times 1$ vector of observations, $\mathbf{1_n}$ is the $n \times 1$ vector of ones, Δ_1' is the $n \times p$ design matrix for lines and Δ_2' is an $n \times b$ design matrix for blocks, that is, the $(h\lambda)^{\text{th}}$ element of Δ'_1 (respectively, of Δ'_2) is 1 if the $(h, \ell)^{\text{th}}$ observation pertains to the ℓ^{th} line (respectively, of block) and is zero otherwise. μ is a general mean, $\mathbf{g}$ is a $p \times 1$ vector of line parameters, $\boldsymbol{\beta}$ is a $b \times 1$ vector of block parameters and $\mathbf{e}$ is an $n \times 1$ vector of residuals. It is assumed that vector $\boldsymbol{\beta}$ is fixed and $\mathbf{e}$ is normally distributed with $\mathrm{E}(\mathbf{e}) =, V(\mathbf{e}) = \sigma^2\ \mathbf{I}$ and Cov $(\boldsymbol{\beta}, \mathbf{e}) = 0$, where $\mathbf{I}$ is the identity matrix of conformable order.

The method of least squares for the analysis of proposed design d leads to the following reduced normal equations for estimating the general combining effects of lines under model (2.1).

$$\mathbf{C}_d \hat{\boldsymbol{\tau}} = \mathbf{Q}_d \tag{2.2}$$

where $\mathbf{C}_d = \mathbf{G}_d - \mathbf{N}_d \mathbf{K}_d^{-1} \mathbf{N}_{d'}$ and $\mathbf{Q}_d = \mathbf{T} - \mathbf{N}_d \mathbf{K}_d^{-1} B$

In the above expressions, $\mathrm{G}_d = \Delta_1 \Delta_1' = (g_{\mathrm{dii}'})$, $g_{\mathrm{dii}} = \mathrm{s}_{\mathrm{di}}$ and for $i \neq i'$, g_{dii} is the number of crosses in d in which the lines i and i' appear together. $\mathbf{N}_d = \boldsymbol{\Delta_1 \Delta_2'} = (n_{dij})$, n_{dij} is the number of times the line i occurs in block j of d and $K_d = \Delta_2 \Delta_2'$ is the diagonal matrix of block sizes. $\mathbf{T} = \boldsymbol{\Delta_1'}\mathbf{y}$ and $\mathbf{B} = \boldsymbol{\Delta_2'}\mathbf{y}$ are the vectors of lines totals and block totals of order $p \times 1$ and $b \times 1$, respectively for design d

A design d will be called connected if and only if rank (Cd) $= p - 1$, or equivalently, if and only if all elementary comparison among gca effects are estimable using d. We denote by $\boldsymbol{D}(p,b,k)$, the class of all such connected block design $\{d\}$ with p lines, b blocks each of size k.

In Section 3, we will present Kiefer's (1975) criterion of the universal optimality of $\mathbf{D}(p,b,k)$.

3 Universal Optimality of Designs for TwoWay Heterogeneity Setting

Let d be a row–column design with k rows and b columns for CDC method (4) and CDC method (2) involving p lines and $n = bk$. For the data obtained from d, we postulate the following model.

$$\mathbf{y} = \mu\mathbf{1_n} + \Delta'_{\mathbf{1}}\,\mathbf{g} + \ \boldsymbol{\Delta'_2}\boldsymbol{\beta} + \boldsymbol{\Delta}'_{\mathbf{3}}\gamma + \mathbf{e} \tag{3.1}$$

where $\mathbf{y}$ is an $n \times 1$ vector of observed responses, μ is the general mean, $\mathbf{g}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are column vectors of p gca parameters, k row effects and b column effects, respectively, $\boldsymbol{\Delta'_1}$ $(n \times p), \boldsymbol{\Delta}'_2(n \times k), \boldsymbol{\Delta}'_3(n \times b)$ are the corresponding design matrices, respectively and $\mathbf{e}$ denotes the vector of independent random errors having mean 0 and covariance matrix $\sigma^2\mathbf{I}_n$.

Let $N_{d1} = \boldsymbol{\Delta}_1\boldsymbol{\Delta}'_2$ be the $p \times$ k incidence matrix of lines vs rows and $N_{d2} = \boldsymbol{\Delta}_1\boldsymbol{\Delta}'_3$ be the $p \times b$ incidence matrix of treatments vs columns and $\boldsymbol{\Delta}_2\boldsymbol{\Delta}'_3 = \mathbf{1}_k\mathbf{1}_b$. Let $\boldsymbol{r}_{\mathrm{dl}}$ denote the number of times the $\mathrm{t^{th}}$ cross appears in the design d, $\mathrm{t} = 1, 2, ..., n_c$ and similarly s_{di} denote the number of times the i^{th} line occurs in design d, $i = 1, ...p$. Under (3.1), it can be shown that the reduced normal equations for estimating the gca effects of lines, after eliminating the effect of rows and columns, in design d are

$$\mathbf{C}_d\hat{\mathbf{g}} = \mathbf{Q}_d \tag{3.2}$$

where

$$\mathbf{C}_d = \mathbf{G}_d - \frac{1}{b}\mathbf{N}_{d1}\mathbf{N}'_{d1} - \frac{1}{k}\mathbf{N}_{d2}\mathbf{N}'_{d2} + \frac{\mathbf{s}_{d1}\mathbf{s}'_{d1}}{\mathbf{s}'_{d1}\mathbf{1}}$$

and

$$\mathbf{Q_d} = \mathbf{T} - 1/b\mathbf{N_{d1}}\mathbf{R} - 1/k\mathbf{N_{d2}}\mathbf{C} + (G/bk,)\mathbf{s_{d1}}$$

$\mathbf{C_d}$ is a $p \times p$ information matrix of the treatments and $\mathbf{G}_d = \boldsymbol{\Delta}_1\boldsymbol{\Delta}'_1 = (g_{d\,i\,i'}), \mathbf{N}_{\mathrm{d1}} = (n_{\mathrm{dij.}}).n_{dij}$ is the number of times line i occurs in row j of d, $\mathbf{N}_{\mathrm{d2}} = (\mathrm{n_{di.t}}), \mathrm{n_{i.t}}$ is the number of times the cross i occurs in column t of d. $\boldsymbol{s}_{\mathrm{d1}}$ is the replication vector of lines in design d. $\mathbf{Q}$ is a $p \times 1$ vector of adjusted treatments (crosses) total. $\mathbf{T}$ is a $p \times 1$ vector of treatment (line) totals, $\mathbf{R}$ is a $k \times 1$ vector of rows totals, $\mathbf{C}$ is a $b \times 1$ vector of columns totals, respectively, in design d. G is a grand total of all observations in design d.

Now we state the following theorem of Parsad et al. (2005) without proof.

Theorem 3.1. *Let $d^*\varepsilon\mathbf{D_1}(p,b,k)$ be a row-column design and $d^*\varepsilon\boldsymbol{D}(p,b,k)$ be a block design for diallel crosses satisfying*

(i) *Trace* $(C_{d^*}) = k^{-1}b\,\{2k(k-1-2x)+px(x+1)\}$

(ii) $(C_{d*}) = (p-1)^{-1}k^{-1}b\{2k(k-1-2x)+px(x+1)\}(\mathbf{I}p - p^{-1}\mathbf{1}\mathrm{p}\mathbf{1'}\mathrm{p})$ *is completely symmetric.*

where $x = [2k/\mathrm{p}], \mathbf{I}p$ *is an identity matrix of order* p *and* $\mathbf{1}p\mathbf{1}'p$ *is a* $p \times p$ *matrix of all ones. Furthermore, using* $d^*\varepsilon\mathbf{D}_1(p,b,k)$ *or* $d^*\varepsilon\mathbf{D}(p,b,k)$ *all elementary contrasts among gca effects are estimated with variance*

$$[2\mathbf{b}^{-1}(p-1)k/\{2k(k-1-2x)+px(x+1)\}]\sigma^2.$$

Then according to Kiefer (1975), $d^*\varepsilon\mathbf{D_1}(p,b,k)$ *or* $d^*\varepsilon\boldsymbol{D}(p,b,k)$ *is universally optimal and, and in particular minimizes the average variance of the best linear unbiased estimator of all elementary contrasts among the gca effects.*

Now we will show a connection between CDC designs method (4), row–column CDC designs method (4) and row–column CDC designs method (2) through nested balanced incomplete block designs of Preece (1967).

Definition 3.1. *A nested balanced incomplete block design with parameters* (v, b_1, k_1, $r*$, λ_1, b_2, k_2, λ_2, t) *is a design for* v *treatments, each replicated* $r*$ *times with two systems of blocks such that:*

(a) *the second system is nested within the first, with each block from the first system, called henceforth as 'block' containing exactly t blocks from the second system, called hereafter as 'sub-blocks';*

(b) *ignoring the second system leaves a balanced incomplete block design with parameters* v, b_1, k_1, $r*$, λ_1

(c) *ignoring the first system leaves a balanced incomplete block design with parameters* v, b_2, k_2, $r*$, λ_2*;*

The following parametric relations hold for a nested balanced incomplete block design:

$$_v r* = b_1k_1 = tb_1k_2 = b_2k_2, \lambda_1(v-1) = r*(k_1-1), \lambda_2(v-1)_= r*(k_2-1).$$

Consider now a nested balanced incomplete block design d with parameters $v = p,\ b_1,\ b_2,\ k_1,\ r*,\ k_2 = 2,$ *and t. If we identify the treatments of d as lines of a CDC experiment method (4) and perform crosses among the lines appearing in the same sub block of d and considering these t sub blocks as single block, we get a block design d* for a CDC experiment method (4) involving p lines with p(p− 1)/2 crosses, each replicated* $r = 2b_2/\{p(p-1)\}$ *times in* $b = b_1/t$ *blocks with block size* $k = tk_2$*. Such a design* $d{*}\varepsilon D\ (p, b, k)$*; also, for such a design* $n_{d*ij} = 0$ *or 1 for* $i = 1, 2, ..., p,\ j = 1, 2, ..., b.$ *and*

$$\boldsymbol{C}_{d^*} = (p-1)^{-1}p(p-3)[\boldsymbol{I}_p - p^{-1}\mathbf{1}_p\mathbf{1}'_p] \tag{3.3}$$

where $\boldsymbol{I}_p$ *is an identity matrix of order p and* $\boldsymbol{1}_p$ *is a unit column vector of order p. Clearly* $\boldsymbol{C}_{d^*}$ *given by (3.3) is completely symmetric and tr* $(\boldsymbol{C}_{d^*}) = p(p-3)$ *which equals the equality given in theorem 3.1. Thus the design d* is universally optimal in D (p, b, k) and using d* each elementary contrast among general combining ability effects is estimated with a variance*

$$2(p-1)\sigma^2/p(p-3) \tag{3.4}$$

*From the above design we can derive the row–column design d** for CDC experiment method (4), if we consider the arrangement of* $p(p-1)/2$ *crosses in* $b = b_1/t$ *blocks and* $k = tk_2$ *rows and each cross is replicated* $r = 2b_2/\{p(p-1)\}$ *times in arrangement. Such a design* $d^{**}\varepsilon\boldsymbol{D}_{\mathbf{1}}(p, b, k)$*; also, for such a design* $n_{d*ij} = 0$ *or 1 for* $i = 1, 2, ..., p,\ j = 1, 2, ..., b.$ *and*

$$\boldsymbol{C}_{d^{**}=}(p-1)^{-1}p(p-3)[\mathbf{I}_p - \{(2p^2-5p-1)/p^2(p-3)\}\mathbf{1}_p\mathbf{1}'_p] \tag{3.5}$$

where $\boldsymbol{I}_p$ *is an identity matrix of order p and* $\boldsymbol{1}_p$ *is a unit column vector of order p. Clearly* $\boldsymbol{C}_{d^{**}}$ *given by (3.5) is completely symmetric and tr* $(\boldsymbol{C}_{d^{**}}) =$ *p (p− 3) which equals the equality given in theorem 3.1. Thus the design d* is universally optimal in* $\boldsymbol{D}_{\mathbf{1}}$ *(p, b, k) and using d**, each elementary contrast among gca effects is estimated with a variance*

$$2(p-1)\sigma^2/p(p-3) \tag{3.6}$$

Further, if the nested balanced incomplete block design with parameters $v = p$, $b_1,\ b_2 = t\ b,\ k_1,\ k_2 = 2$ *is such that* $\lambda_2 = 1$ *or equivalently if* $b_1k_1 = p(p-1)$, *then the optimal design d* and d** derived from d*, has each cross replicated just once and hence use the minimal number of experimental units.*

Since $n_{d*ij=}$ *0 or 1 for* $i = 1, 2, ..., p,\ j = 1, 2, ..., b$*, a CDC design method (2) can be derived from* d^{**} *by attaching the cross of the type* $(i \times i)$ *in* j^{th} *block when it does not appear in* j^{th} *block, where* $i = 1, 2, ..., p$; $j = 1, 2, ..., b.$

Hence we get a row–column design d^{***} for a CDC method (2) experiment involving p lines with $p_(p+1)/2$ crosses arranged in $b = b_1/t$ blocks each of size $k = k_1 + 1$ and each replicated $r = 2b\ (k_1+1)/\{p(p+1)\}$. Such a design $d***\varepsilon\mathbf{D_1}(p, b, k)$; and, for such a design $n_{d*ij=}x$ or $x+1$ for $i = 1, 2, ..., p$, $j = 1, 2, ..., b$. and

$$\mathbf{C}_{d^{***}} = p[\mathbf{I}_p - \frac{2p^2+5p+1}{p^2(p+1)}\mathbf{1}_p\mathbf{1}_p'] \tag{3.7}$$

The $\boldsymbol{C}_{d^{***}}$ given by (3.7) is completely symmetric matrix. Hence

$$\text{trace}(\mathbf{C}_{d^{***}}) = p(p-1) \tag{3.8}$$

and using d^{***} each elementary contrast among gca effects is estimated with a variance

$$\frac{2}{p}\sigma^2 \tag{3.9}$$

and the generalized inverse of $\boldsymbol{C}_{\mathrm{d***}}$ is given below

$$\boldsymbol{C}^{-}_{d^{***}=}\frac{1}{p}[\boldsymbol{I}_p - \frac{2p^2+5p+1}{(p^3-9p^2-25p-5)}\mathbf{1_p}\mathbf{1}_p'] \tag{3.10}$$

$$\text{trace}(\boldsymbol{C}^{-}_{d^{***}}) = (p-1)/p \tag{3.11}$$

To prove A-optimality of row–column design d^{***} for CDC method (2) we state the following definition.

Definition 3.2. *According to Hedayat et al. (1988), an A-optimal design for two-way elimination of heterogeneity is one which minimizes trace $\boldsymbol{P}\ \boldsymbol{C}_{d^{***}}\boldsymbol{P}'$, where $\boldsymbol{P}$ is a $p \times (p-1)$ matrix having normalized rows orthogonal to each other and to the vector $(1, ..., 1)'$, where $\boldsymbol{P}$ is as given below.*

$$\mathbf{P} = \begin{pmatrix} 1 & -1 & 0 & . & . & . & 0 \\ 1 & 0 & -1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & 0 & 0 & . & . & . & -1 \end{pmatrix},$$

Now

$$\boldsymbol{P}\boldsymbol{C}_{\mathbf{d}^{***}}\boldsymbol{P}'_{=}\frac{1}{p}[\boldsymbol{I}_{p-\mathbf{1}}] \tag{3.12}$$

Hence trace $(\boldsymbol{P}\ \boldsymbol{C}_{\mathbf{d}^{***}}\boldsymbol{P}') = (p-1)/p$ which in equal to the value given in (3.11).

Now we state the following theorems.

Theorem 3.2. *The existence of a nested balanced incomplete block design d with parameters $v = p$, b_1, $b_2 = tb$, k_1, $k_2 = 2$ and t implies* ***(i)*** *universally optimal block design d^* for CDC experiment method (4) with parameters $v = p$, $n_c = p(p-1)/2$, $b = b_1/t$, $k = k_1$ and $r = 2b_2/\{p(p-1)\}$* ***(ii)*** *universally optimal row–column design d^{**} for CDC experiment method (4) with parameters $v = p$, $n_c = p(p-1)/2$, $b = b_1/t$, $k = k_1$ and $r = 2b_2/\{p(p-1)\}$ and* ***(iii)*** *A-optimal row–column design d^{***} for CDC experiment method (2) with parameters $v = p$, $n_c = p(p+1)/2$, $b = b_1/t$, $k = k_1 + 1$ and $r = 2b(k_1 + 1)/p(p+1)$.*

Theorem 3.3. *If the row–column design for CDC method (4) is obtained through universally optimal block design for CDC method (4) constructed through a nested balanced incomplete block design then the row–column design for CDC method (4) will always be universally optimal.*

4 Method of Construction

Series 1: Let $p = 4m + 1, m \geq 1$ be a prime or a prime power and x be a primitive element of the GF (p). Consider the following m initial blocks.

$$\{(x^i, x^{i+2m}), (x^{i+m}, x^{i+3m})\}, i = 0, 1, 2, ...m - 1.$$

As shown by Dey et al. (1986), these initial blocks, when developed in the sense of Bose (1939), give rise to a nested balanced incomplete block design with parameters $v = p = 4m+1, k_1 = 4$, $b_1 = m(4m+1)$, $k_2 = 2$. Arranging the following m initial blocks into single block and developing mod (p), will yield an optimal CDC design for method (4) with parameters $p = 4m+1, b = 4m + 1, k = m$, and $r = 1$.

$$\left[\begin{matrix} (x^i,\ x^{i+2m}) \\ x^{i+m},\ x^{i+3m}) \end{matrix}\ i = 0, 1, 2, ..., m - 1\right]$$

From the above design we may obtain the following designs.

(i) Optimal row–column design for CDC experimental method (4) with parameters $p = 4m + 1, k = m, b = 4m + 1$ and $r = 1$.

(i) Row–column design for CDC experimental method (2) with parameters $p = 4m + 1, k = 2m + 1, b = 4m + 1$ and $r = 1$ by supplementing the cross of the type $(i \times i)$ in the j^{th} block when it does not appear in the j^{th} block, where $i, j = 1, 2, ..., 4m + 1$.

Procedure to obtain the above designs has been explained below by illustrative examples.

Example 1. *Let $m = 2$. We get the following two blocks.*

$$\begin{bmatrix} (1,\ 2) & (x,\ 2x) \\ (2x+1,\ x+2) & (2x+2, x+1) \end{bmatrix}$$

Now we convert the both columns in single block as given below.

$$\begin{bmatrix} (1,\ 2) \\ (2x+1,\ x+2) \\ (x,\ 2x) \\ (2x+2, x+1) \end{bmatrix}$$

where x is a primitive element of GF(3^2) and the elements of GF(3^2) are $0, 1, 2, x, x+1, x+2, 2x, 2x+1, 2x+2$. Adding successively the non-zero elements of GF (3^2) to the contents of the single column, the CDC design method (4) is obtained with parameters $p = 9, b = 9, k = 4$, and $r = 1$.

Block design for CDC method (4)

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
2×3	1×3	1×2	5×6	4×6	4×5	8×9	7×9	7×8
6×8	4×9	5×7	2×9	3×7	1×8	3×5	1×6	2×4
4×7	5×8	6×9	1×7	2×8	3×9	1×4	2×5	3×6
5×9	6×7	4×8	3×8	1×9	2×7	2×6	3×4	1×5

The above design can yield optimal row–column design for CDC method (4) with parameters $p = 9, b = 9, k = 4$ and $r = 1$, when we consider rows as another factor of heterogeneity.

The above design can be converted into CDC design for method (2) with parameters $p = 9, b = 9, k = 5$, by attaching cross of the type $(i \times i)$ with the crosses in j^{th} block in which it does not appear, where $i,\ j = 1, 2, ..., 9$. The row–column design for CDC method (2) is exhibited below, where the lines have been relabelled 1-9, using the correspondence $0 \to 1, 1 \to 2, 2 \to 3, x \to 4, x+1 \to 5, x+2 \to 6, 2x \to 7, 2x+1 \to 8, 2x+2 \to 9$:

Row–column design for CDC method (2)

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
R_1	2×3	1×3	1×2	5×6	4×6	4×5	8×9	7×9	7×8
R_2	6×8	4×9	5×7	2×9	3×7	1×8	3×5	1×6	2×4
R_3	4×7	5×8	6×9	1×7	2×8	3×9	1×4	2×5	3×6
R_4	5×9	6×7	4×8	3×8	1×9	2×7	2×6	3×4	1×5
R_5	1×1	2×2	3×3	4×4	5×5	6×6	7×7	8×8	9×9

Series 2: Let $p = 6m + 1, m \geq 1$ be a prime or a prime power and x be a primitive element of the Galois field of order p, GF (p). Consider the initial blocks

$$\{(x^i,\ x^{i+3m}),\ (x^{i+m}\ ,\ x^{i+4m}), (x^{i+2m},\ x^{i+5m})\}, i = 0, 1, 2, ...m - 1.$$

Dey et al. (1986) showed that these initial columns, when developed give a solution of a nested incomplete block design with parameters $v = p = 6m + 1, b_1 = m(6m + 1), k_1 = 6, k_2 = 2, \lambda_2 = 1$.

By arranging the above initial blocks into a single block as given below and developing the single block over mod (p),

$$\begin{bmatrix} (x^i,\ x^{i+3m)}) & \\ (x^{i+m}, x^{i+4m}) & , i = 0, 1, \ldots, m-1 \\ (x^{i+2m},\ x^{i+5m)} & \end{bmatrix}$$

we obtain an optimal block design as well as row–column design for CDC method (4) with minimal number of experimental units with parameters $p = 6m+1, b = 6m+1, k = 3m$, and $r = 1$. This design can be converted into optimal row–column CDC design for method (2) with parameters $p = 6m+1$, $b = 6m+1, k = 3m+1$, by attaching cross of the type $(i \times i)$ with the crosses in the j^{th} block in which cross of the type $(i \times i)$ does not appear at all, where $i, j = 1, 2, ..., 6m + 1$.

Example 2. *Letting $m = 2$, we get the following two initial blocks.*

$$\begin{bmatrix} (1, 12) & (2, 11) \\ (4, 9) & (8, 5) \\ (3, 10) & (6, 7) \end{bmatrix}$$

Now we arrange these two blocks in a single block as given below.

$$\begin{bmatrix} (1,\ 12) \\ (2,\ 11) \\ (4,\ 9) \\ (8,\ 5) \\ (3,\ 10) \\ (6,\ 7) \end{bmatrix}$$

Now developing the above block mod (13), we obtain optimal CDC design for method (4) as well as row–column design CDC for method (4) with parameters p = 13, k = 6, b = 13. Now attaching cross of the type $(i \times i)$ with the crosses in j^{th} block in which cross of the type $(i \times i)$ does not appear at all, where i, j $= 1, 2, ..., 13$, we obtain row–column design for CDC method (2) with parameters $p = 13, k = 13, b = 7$.

Series 3: Let $p = 2m + 1, m \geq 2$ be a prime or a prime power, then cyclically developing the following m blocks

$$(0, 2m), (1, 2m-1), (2, 2m-2), ..., (m-1, m+1) \text{mod}(2m+1)$$

yields an optimal block design as well as row–column design for CDC method (4) with parameters $p = 2m+1$, $k = m$, $b = 2m+1$. A optimal row–column design for CDC method (2) with parameters $p = 2m + 1$, $b = 2m + 1$, and $k = m + 1$ can be obtained by the procedure described in method 1.

Example 3. *Let $m = 3$. Then $p = 7$ and developing the following block mod (7)*

$$\begin{bmatrix} (0,\ 6) \\ (1, 5) \\ (2,\ 4) \end{bmatrix}$$

we obtain optimal block (row–column) design for CDC method (4) with parameters $p = 7$, $k = 3$, $b = 7$. Now attaching cross of the type $(i \times i)$ with the crosses in j^{th} block in which cross of the type $(i \times i)$ does not appear at all, where $i,\ j = 1, 2, ..., 7$. We can obtain row–column design for CDC method (2) with parameters $p = 7$, $k = 4$, $b = 7$.

Note: The m blocks given in series 3 form a nested balanced incomplete block design with parameters $v = p = 2m + 1, b_1 = 2m + 1, k_1 = 2m, b_2 = m(2m + 1), k_2 = 2, \lambda_2 = 1$ *and* $t = m$.

5 Illustration

We show the layouts of CDC experiment method (4), CDC row–column design method (4) and CDC experiment method (2) using their proposed designs, i.e. d^*, d^{**} and d^{***}, respectively. For this purpose, we took data from an experiment on the number of tillers per plant in pearl millet, reported by Sharma (1998) on page 220. The author used a randomized complete block design with $v = 36$ as he considered p^2 possible crosses including selfing and reciprocal crosses, among $p = 6$ inbred lines. For the purpose of illustration we take the data of relevant crosses from this experiment. The layout and observations in parentheses are given below. There are 15 crosses and each cross is replicated once.

Designs d^*, d^{**} and d^{***}

	B_1	B_2	B_3	B_4	B_5
R_1	$1 \times 4(5)$	$2 \times 0(7)$	$3 \times 1(3)$	$4 \times 2(7)$	$0 \times 3(3)$
R_2	$2 \times 3(8)$	$3 \times 4(6)$	$4 \times 0(8)$	$0 \times 1(6)$	$1 \times 2(10)$
R_3	$0 \times 0(4)$	$1 \times 1(8)$	$2 \times 2(9)$	$3 \times 3(3)$	$4 \times 4(9)$

Table 1: Analysis of variance of the data on number of tillers using design d^*, d^{**} and d^{***}, respectively

Source	d^*			d^{**}			d^{***}		
	DF	SS	MS	DF	SS	MS	DF	SS	MS
gca	4	25.6	6.40	4	25.6	6.40	4	62.35	15.58
Error	5	18.5	4.62	4	18.5	4.62	10	15.25	1.52
Total	9	44.1		9	44.1		14	77.6	

Since the proposed designs use the minimum number of experimental units, sums of squares of crosses are equal to total sums of squares. The error sums of squares contain sums of squares due to sca$_+$ error sums squares. The estimate of yield of (cross)$_{ij}$ $= g_i + g_j +$ general mean and estimate of sca can be obtained by the difference observed yield of cross-estimate yield of cross

The following exhibit the vectors of block totals **B,** adjusted treatment totals **Q** and treatment total, respectively, of $d^*(d^{**})$ and d^{***}(Table 1).

$$
\begin{aligned}
&B_{d^*}(B_{d^{**}}) = (13, 13, 11, 13, 13)' \\
&B_{d^{***}} = (17, 21, 20, 16, 22) \\
&Q_d^* = (-1, -1, 6, -5, 1) \\
&Q_d^{**} = (-1, -1, 6, -5, 1) \\
&Q_d^{***} = (-5.66, 1, 11.33, -11.33, 4.66) \\
&\mathbf{T}^* = (24, 32, 20, 26, 24)' \\
&\mathbf{T}^{**} = (24, 32, 20, 26, 24) \\
&\mathbf{T}^{***} = (32, 40, 50, 26, 44)
\end{aligned}
$$

Universally optimal block designs, row–column designs for CDC method (4) and A-optimal row–column designs for CDC method (2), respectively, with $p \leq 16$, $s \leq 30$ obtained by the above method from NBIB designs of Morgan et al. (2001), are listed in Tables 2, 3, 4, and 5. These are the designs other than the designs catalogued byDey and Midha (1996), Das et al. (1998) and Parsad et al. (2005).

Table 2: Estimates of the general combining ability effects and their estimated error on the number of tillers using d^*, d^{**} and d^{***}, respectively

Line	gca*	S.E.	gca**	S.E.	gca***	S.E.
0	−0.4	±0.63	−0.4	±0.63	−1.12	±0.44
1	−0.4	±0.63	−0.4	±0.63	0.19	±0.44
2	2.4	±0.63	2.4	±0.63	2.26	±0.44
3	−2	±0.63	−2	±0.63	−2.26	±0.44
4	0.4	±0.63	0.4	±0.63	0.93	±0.44

Table 3: Universally optimal block design for complete diallel crosses with $p \leq 16$, $s \leq 30$ generated by using NBIB designs of Morgan et al. (2001)

S. no	p	b	k	Source
1.	8	7	4	MPR4w
2.	9	9	4	MPR8
3.	6	5	6	MPR13
4.	11	11	5	MPR14
5.	13	13	6	MPR20
6.	15	15	7	MPR31
7.	16	15	8	MPR33w
8.	11	11	10	MPR49
9.	8	28	3	MPR50c
10.	14	13	14	MPR57
11.	15	15	14	MPR59
12	12	11	6	MPR16
13	12	11	6	MPR17
14	16	15	8	MPR37
15	12	11	12	MPR53

Table 4: Universally optimal row–column design for complete diallel crosses with $p \leq 16$, $s \leq 30$ generated by using NBIB designs of Morgan et al. (2001)

S. no	p	b	k	Source
1.	9	9	4	MPR8
2.	11	11	5	MPR14
3.	13	13	6	MPR20
4.	15	15	7	MPR31
5.	11	11	10	MPR49
6.	15	15	14	MPR59
7.	12	11	12	MPR53

Table 5: A-optimal block design for complete diallel cross (method (2) with $p \leq 16$, $s \leq 30$ generated by using NBIB designs of Morgan et al. (2001)

S. no.	p	b	k	Source
1.	8	8	3	MPR 4w
2.	13	13	7	MPR20w
3.	13	13	7	MPR 21
4.	13	13	7	MPR 23
5.	15	15	8	MPR31
6.	16	16	9	MPR33w
7	9	9	3	MPR 5w
8	9	9	3	MPR8

Acknowledgments. We are grateful to the anonymous referee for his critical comments.

References

AGGARWAL, K.R. (1974). Analysis of Li(s) and triangular designs. *Journal of the Indian Society for Agricultural Statistics* **26**, 3–13.

BOSE, R. (1939). On the construction of balanced incomplete block designs. *Ann Eugen* **9**, 353–399.

DAS, A., DEY, A. and DEAN, A.M. (1998). Optimal designs for diallel cross experiments. *Statistics and Probability Letters* **36**, 427–436.

DEY, A., DAS, U.S. and BANERJEE, A.K. (1986). Construction of nested balanced incomplete block designs. *Calcutta Statist. Assoc. Bull.* **35**, 161–167.

DEY, A. and MIDHA, C.K. (1996). Optimal designs for diallel crosses. *Biometrika* **83**, 2, 484–489.

GRIFFING, B. (1956). Concepts of general and specific combining ability in relation to diallel crossing system. *Aus. J. Biol. Sci.* **9**, 463–493.

GUPTA, S. and CHOI, K. (1998). Optimal row-column designs for complete diallel crosses. *Communication in Statistics-Theory and Methods* **27**, 11, 2827–2835.

HEDAYAT, A.S., JACROUX, M. and MAJUMDAR, D. (1988). Optimal Designs for Comparing Test Treatments with Controls. *Statistical Science* **3**, 4, 462–491.

KIEFER, J. (1975). *Construction and optimality of generalized Youden designs.* North Hollond, Amsterdam, SRIVASTAVA, J.N. (ed.), p. 333–353.

MORGAN, J.P., PREECE, D.A. and REES, D.H (2001). Nested balanced incomplete block designs. *Discrete Mathematics* **231**, 351–389.

PARSAD, R., GUPTA, V.K. and GUPTA, S. (2005). Optimal Designs for Diallel and Double Cross Experiments. *Utilitas Mathematica* **68**, 11–32.

PREECE, D.A. (1967). Nested balanced incomplete block design. *Biometrika* **54**, 479–486.

SHARMA, M. K. and FANTA, S. (2011). Optimal Block Designs For CDC experiments **LXIX no. 3**, 2, 297–307.

SHARMA, R.J. (1998). *Statistical and Biometrical Techniques in Plant Breeding, New Age International Publication (p) Limited.* Publishers, New Delhi.

SINGH, M., GUPTA, S. and PARSAD, R. (2012). *Geneticcrosses experiments.* Wiley, New York, HINKELMANN, K (ed.), p. 1–71.

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Paper received: 2 January 2016.