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Symmetrizing and Variance Stabilizing Transformations of Sample Coefficient of Variation from Inverse Gaussian Distribution

Yogendra P. Chaubey and Debaraj Sen
Concordia University, Montreal, Canada
Murari Singh
International Center for Agricultural Research in the Dry Areas (ICARDA), Amman, Jordan

Abstract

Coefficient of variation (CV) plays an important role in statistical practice; however, its sampling distribution may not be easy to compute. In this paper, the distributional properties of the sample CV from an inverse Gaussian distribution are investigated through transformations. Specifically, the symmetrizing transformation as outlined in Chaubey and Mudholkar (1983), that requires numerical techniques, is contrasted with the explicitly available variance stabilizing transformation (VST). The symmetrizing transformation scores very high as compared to the VST, especially in a power family. The usefulness of the resulting approximation is illustrated through a numerical example.

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1 Introduction

Let the mean and standard deviation of the random variable X be denoted by μ and σ respectively. Then the ratio of $\tau = \sigma/\mu$ is called the coefficient of variation (CV) of the corresponding population. Inference on CV is of interest in many areas of applied research that has been addressed by many authors in the literature. The reader may be referred to Chapter 15 of Johnson et al. (1994) that gives details of relevant research in this area.

As mentioned by Johnson and Welch (1940), in practice the use of CV is primarily in situations where the observed measurements are positive, yet most of such studies are based on the assumption of a Gaussian distribution. In this case a natural estimate of τ based on a random sample of size n is given by the ratio of the sample standard deviation s and the sample mean $\bar{X}$, namely the sample CV $\hat{\tau} = s/\bar{X}$. This estimate may provide approximate confidence intervals for τ based on approximations to the distribution of $t = \sqrt{n}/\hat{\tau}$, that follows the non-central Student-t distribution with $n-1$ degrees of freedom and non-centrality parameter $\delta = \sqrt{n}/\tau$. Laubscher (1960) has discussed such an approximation to the distribution of $\hat{\tau}$ based on the variance stabilizing transformation (VST) of the non-central t-distribution, where as Singh (1993) obtained such a transformation based on approximate moments; see Banik and Kibria (2011) for a comprehensive review and comparison of various approximations. It is well recognized that the VST may not yield a symmetrizing transformation that motivated the recent paper by Chaubey et al. (2013) investigating approximately normalizing transformation of CV for Gaussian populations.

The situations where the use of CV may be appropriate, i.e. where the observations are positive, the inverse Gaussian (IG) distribution is often more justified compared to lognormal, gamma and Weibull distributions (see Chhikara and Folks 1977, 1989; Kumagai et al. 1996, Tagaki et al. 1997). Koopmans et al. (1964) have shown "that without some *a priori* information about the range of the parameter μ it is, in fact, impossible to obtain confidence intervals for τ which have finite length with probability one for all values of μ, and σ, except by a purely sequential sampling scheme." This motivated the above authors to investigate the confidence intervals for CV based on a log-normal population where the problem of non-existence of fixed width confidence interval does not arise. The inverse Gaussian distribution is shown to be well approximated by the log-normal distribution by Whitmore and Yalovsky (1978) which implies that the IG distribution also does not share the problem of non-existence of fixed width confidence interval with the Gaussian case. Furthermore, it is plausible that the inverse Gaussian model may be appropriate for transformed observations using the transformation $x \mapsto x^{(\lambda)}$ defined as

$$x^{(\lambda)} = \begin{cases} x^{\lambda} & \text{if} \quad \lambda \neq 0, \\ \ln x & \text{if} \quad \lambda = 0 \end{cases} \tag{1.1}$$

while the original observations may not follow an IG distribution. This has been demonstrated in Chaubey et al. (2016), while illustrating an application of the symmetrizing transformation of CV obtained here using data from an agricultural experiment.

Analogues of symmetry, skewness and kurtosis for the IG distribution in contrast to those for the Gaussian distribution have been discussed by Mudholkar and Natarajan (2002). This paper puts the IG distribution at a place in the class of distributions for positive valued data, similar to that enjoyed by the Gaussian distribution on the platform of distributions on the real line. The Likelihood ratio test for CV of an IG population has been investigated by Hsieh (1990) and more recently by Chaubey et al. (2014) demonstrating that this test is "best invariant" under the group of scale transformations.

The purpose of this paper is to investigate properties of variance stabilizing and symmetrizing transformations for CV in the context of the IG population. The organization of the paper is as follows. In Section 2, some basic properties of the IG distribution along with that of the corresponding sample CV are listed. As mentioned before, Chaubey et al. (2013) considered the inverse of the sample CV in the context of the Gaussian distribution, as the sample mean in the denominator may present computational problems. It will be seen that this is not a problem in the IG case as the CV is a function of the product of the mean and inverse of the dispersion parameter and both of these are unbiasedly estimable.

The present study supplements the study in the Gaussian case on one hand that the variance stabilizing transformation may be seriously lacking the symmetry and on the other hand it brings out the fact that a simple transformation in the power family may be quite appealing. Though the numerical algorithm used for the Gaussian case is not applicable here, an alternative algorithm is provided that effectively produces the required transformation. Other important contribution of this paper is the analysis of the nature of the transformation for large and small values of CV (see Section 4) that, in turn, has prompted us to study the power transformation family (see Section 5).

Section 3 presents the general formulae for the variance stabilizing transformation (VST) and that for the symmetrizing transformation (ST) which is conjectured to provide a better approximation as compared to that given by the VST. An analysis of these transformations is also carried out in this paper with the aim of examining a) stability of variance of the symmetrizing transformation, b) symmetry of the distribution of variance stabilizing transformation, c) the probability distribution of the sample CV based on these transformations. Section 4 presents an analysis of the symmetrizing transformation for small and large values of CV that motivates the examination of the power transformation family that has been carried out in Section 5. Section 6 presents a numerical investigation comparing various approximation for computing the distribution of the CV. The final section, Section 7 presents a numerical example illustrating the usefulness of the symmetrizing transformation in the context of hypothesis testing.

2 Inverse Gaussian Distribution and Estimation of CV

The probability density function *(pdf)* of the inverse Gaussian random variable X with mean μ and dispersion parameter λ, to be denoted by $IG(\mu,\lambda)$, is given by

$$f(x|\mu,\lambda)=\left\{\frac{\lambda}{2\pi x^3}\right\}^{1/2}\exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\};\ \ x>0,\ \mu>0,\ \lambda>0. \tag{2.1}$$

This distribution was studied in detail by Tweedie (1957a, 1957b) and was brought to limelight later by a seminal paper by Folks and Chhikara (1978). For a broad review and applications of the IG family and other related results, the reader may refer to the texts by Chhikara and Folks (1989) and Seshadri (1993, 1998). The variance of this density is given by μ^3/λ, hence the corresponding population CV is given by $\tau=\sqrt{\mu/\lambda}$. For our purpose we will consider the parameter $\phi=\tau^2=\mu/\lambda$. Consider a random sample $X_1, X_2, ..., X_n$ from $IG(\mu,\lambda)$, then we have two standard results concerning the distributions of the sample mean $\bar{X}=n^{-1}\sum_{i=1}^{n}X_i$ and $U=(n-1)^{-1}\sum_{i=1}^{n}\left(\frac{1}{X_i}-\frac{1}{\bar{X}}\right)$, namely

$$(i)\ \bar{X}\sim IG(\mu,n\lambda)\quad \text{and}\quad (ii)\ (n-1)\lambda\, U\sim\chi^2_{n-1}. \tag{2.2}$$

Further, the random variables $\bar{X}$ and U are independent, thus we get an unbiased estimator of ϕ given by

$$\hat{\phi}=\hat{\mu}\,\widehat{\left(\frac{1}{\lambda}\right)}=\bar{X}\,U. \tag{2.3}$$

Using the distributional properties of $\bar{X}$ and U we can write

$$\hat{\phi}\overset{\mathcal{D}}{=}\frac{ZY}{\nu} \tag{2.4}$$

where $Z\sim IG(\phi,n)$, $Y\sim\chi^2_\nu$, $Z\overset{\text{ind}}{\sim}Y$ and $\nu=n-1$. Thus using the moments of IG from Chhikara and Folks (1989) and those of the χ^2_ν random variable, the independence of $\bar{X}$ and U provides the following four raw moments of $\hat{\phi}$ that will be useful for later use:

$$E(\hat{\phi})\ =\ \phi, \tag{2.5}$$

$$E(\hat{\phi}^2) = \phi^2\left(1+\frac{\phi}{n}\right)\left(1+\frac{2}{\nu}\right), \tag{2.6}$$

$$E(\hat{\phi}^3) = \phi^3\left(1+\frac{3\phi}{n}+\left(\frac{3\phi}{n}\right)^2\right)\left(1+\frac{2}{\nu}\right)\left(1+\frac{4}{\nu}\right), \tag{2.7}$$

and

$$E(\hat{\phi}^4)=\phi^4\left(1+\frac{6\phi}{n}+\left(\frac{15\phi}{n}\right)^2+\left(\frac{15\phi}{n}\right)^3\right)\left(1+\frac{2}{\nu}\right)\left(1+\frac{4}{\nu}\right)\left(1+\frac{6}{\nu}\right). \tag{2.8}$$

The central moments of $\hat{\phi}$ may therefore be deduced as

$$E(\hat{\phi}-\phi)^2=\mu_2(\phi) = \phi^2\left[\frac{2}{\nu}+\left(1+\frac{2}{\nu}\right)\frac{\phi}{n}\right], \tag{2.9}$$

$$E(\hat{\phi}-\phi)^3=\mu_3(\phi) = \phi^3\left[\frac{8}{\nu^2}+\frac{12}{\nu}\left(1+\frac{2}{\nu}\right)\frac{\phi}{n} +3\left(1+\frac{6}{\nu}+\frac{8}{\nu^2}\right)(\frac{\phi}{n})^2\right], \tag{2.10}$$

and $$E(\hat{\phi}-\phi)^4=\mu_4(\phi) = \phi^4\left[\frac{12}{\nu^2}\left(1+\frac{4}{\nu}\right)+\frac{12}{\nu}\left(1+\frac{14}{\nu}+\frac{24}{\nu^2}\right)\left(\frac{\phi}{n}\right) +3\left(1+\frac{36}{\nu}+\frac{188}{\nu^2}+\frac{240}{\nu^3}\right)\left(\frac{\phi}{n}\right)^2 +15\left(1+\frac{12}{\nu}+\frac{44}{\nu^2}+\frac{48}{\nu^3}\right)\left(\frac{\phi}{n}\right)^3\right]. \tag{2.11}$$

3 Symmetrizing and Variance Stabilizing Transformations

The variance stabilizing transformation as first proposed by Bartlett (1947) is now widely available in standard texts (see e.g. Rao 1973). The general formulation for a symmetrizing transformation following the same approach as of Bartlett (1947) has been put forward in Chaubey and Mudholkar (1983, 1984). Note that there have been attempts in proposing the symmetrizing transformations in a particular class. For example Hinkley (1975) considered symmetrizing transformation in the family of power transformations that has been further elaborated in Hinkley (1977) and Taylor (1985) (see also Hall 1992, and Yeo and Johnson 2000). The power transformations are easier to handle and the general symmetrizing transformation may be fretted upon due to numerical complexity. In the present case, we demonstrate that even though the general symmetrizing transformation is computationally possible, the power transformation provides an excellent approximation.

Let T_n denote a statistic based on a random sample of size n, constructed to estimate a parameter ϕ. Here we can take $T_n = \bar{X}U$, the unbiased estimator of the squared sample CV. Further, assume that $\sqrt{n}(T_n - \phi)$ tends to follow $N(0, \sigma^2(\phi))$ as $n \to \infty$. Denote the j^{th} central moment of T_n by

$$\mu_j(\phi) = E(T_n - \mu(\phi))^j, \; j = 1, 2, ... \tag{3.1}$$

where

$$\mu(\phi) = E(T_n).$$

We denote by $\xi_1(\phi) = \mu(\phi) - \phi$ as the bias of T_n and by $\mu_2(\phi) = \sigma^2(\phi) + \mu^2(\phi)$ as the mean squared error. Then for a smooth function $g(T_n)$, we approximately have for large n, the variance (μ_{2g}) of $g(T_n)$ as (see Chaubey and Mudholkar 1983),

$$\mu_{2g} = (g'(\phi))^2(1 + \xi_1(\phi)R)^2 \left[\mu_2(\phi) + R_1\mu_3(\phi) + \frac{1}{4}R_1^2(\mu_4(\phi) - {\mu_2}^2(\phi))\right] \tag{3.2}$$

where

$$R = \frac{g''(\phi)}{g'(\phi)} \text{ and } R_1 = \frac{R}{1 + \xi_1(\phi)R} \; . \tag{3.3}$$

And the third central moment μ_{3g} of T_n up to order $O(1/n^2)$ is given by

$$\mu_{3g} = (g'(\phi))^3(1 + \xi_1(\phi)R)^3 \left[\mu_3(\phi) + \frac{3}{2}R_1(\mu_4(\phi) - \mu_2^2(\phi))\right], \tag{3.4}$$

where we have omitted terms containing central moments of order higher than 4 (this assumes that the third and fourth central moments are of order $O(1/n^2)$ and the higher order moments are of lower order). In the present case $\xi_1(\phi) = 0$, hence $R_1 \equiv R$.

The *variance stabilizing transformation* (VST) $g_v(\phi)$, may now be obtained using Eq. 3.2, ignoring the last two terms which are of $O(n^{-2})$, as

$$g_v'(\phi) = \frac{C}{\sigma(\phi)} \tag{3.5}$$

where C is a constant. Hence

$$g_v(\phi) = C \int \frac{1}{\sigma(\phi)} d\phi. \tag{3.6}$$

The approximate *symmetrizing transformation* (ST) $g_s(\phi)$, is obtained by equating the third moment of $g(X_n)$ given in Eq. 3.4 to zero, that gives

$$g_s(\phi) = \int \mathrm{e}^{-w(\phi)} d\phi \tag{3.7}$$

where

$$w(\phi) = \frac{2}{3} \int \left\{ \frac{f_1(\phi)}{f_2(\phi)} \right\} d\phi \tag{3.8}$$

with $f_1(.)$ and $f_2(.)$ being defined as

$$f_1(\phi) = \mu_3(\phi), \tag{3.9}$$

$$f_2(\phi) = \mu_4(\phi) - \mu_2^2(\phi). \tag{3.10}$$

In general the integrals in Eqs. 3.6, 3.7 and 3.8 may not be available in explicit forms. We will see that the VST is explicitly available, however, the ST is not. Chaubey et al. (2013) provided R-codes for solving the integral in Eq. 3.7 numerically and that will be adopted here also. However, explicit solutions for large and small values of ϕ are developed that prompts us in considering symmetrizing transformation in the family of power transformations. These are detailed in the next subsections.

3.1. Variance Stabilizing Transformation for $\hat{\phi}$. The variance stabilizing transformation (VST), denoted by say $g_v(\hat{\phi})$ is obtained, from Eq. 3.4 by substituting

$$\sigma^2(\phi) = n\mathrm{Var}(\hat{\phi}) = n(a\phi^2 + b\phi^3), \tag{3.11}$$

where

$$a = \frac{2}{\nu},\ b = (1 + \frac{2}{\nu})/n. \tag{3.12}$$

Thus the (approximate) VST is given by

$$\begin{aligned} g_v(\phi) &= \int \frac{1}{\sqrt{nVar(\hat{\phi})}} d\phi \\ &= \frac{1}{\sqrt{n}} \int \frac{1}{\phi\sqrt{a + b\phi}} d\phi \end{aligned} \tag{3.13}$$

The integral may be obtained explicitly by substitution $\phi = (a/b)\tan^2\theta$ (see also Gradshteyn and Ryzhik 2007, formula 2.266; there is an additional constant of integration in our formula). This gives

$$\int \frac{d\phi}{\phi\sqrt{a + b\phi}} = \frac{2}{\sqrt{a}} \ln\left(\frac{\sqrt{b\phi}}{\sqrt{a} + \sqrt{a + b\phi}} \right). \tag{3.14}$$

Thus we have

$$g_v(\phi) = \sqrt{\frac{2\nu}{n}}\ \ln\left[\frac{\sqrt{(1+2/\nu)\phi}}{\sqrt{(2n/\nu)}+\sqrt{(2n/\nu)+(1+2/\nu)\phi}}\right]. \tag{3.15}$$

It may be verified that the above function is an increasing function taking negative values.

Remark: It may be seen that $g_v(\phi)$ is proportional to

$$\sinh^{-1}\left(\frac{B}{\sqrt{\phi}}\right) = \ln\left[\frac{B}{\sqrt{\phi}}+\sqrt{1+\frac{B^2}{\phi}}\right], \tag{3.16}$$

where

$$B = \sqrt{\frac{2n}{n+1}}. \tag{3.17}$$

The VST as derived in the Gaussian case is very similar to the one obtained here (see Singh 1993) as given by

$$g\left(\frac{1}{\sqrt{\phi}}\right) = \sinh^{-1}\left(\frac{B_G}{\sqrt{\phi}}\right) = \ln\left[\frac{B_G}{\sqrt{\phi}}+\sqrt{1+\frac{B_G^2}{\phi}}\right], \tag{3.18}$$

where $B_G = (1+\frac{3}{4\nu})\sqrt{\frac{n}{2\nu}}$.

3.2. Symmetrizing Transformation of $\hat{\phi}$. In order to obtain symmetrizing transformation of $\hat{\phi}$, we use the expression in Eq. 3.7 where $f_1(\phi)$ and $f_2(\phi)$ are computed using the values of $\mu_3(\phi)$ and $\mu_4(\phi)$ from Eqs. 2.10 and 2.11 respectively. As mentioned earlier, such an approach was used by Chaubey et al. (2013) for the Gaussian case, where the indefinite integral in Eq. 3.7) was obtained numerically using the algorithm

$$\int s(x)dx = S(x) = \int_0^x s(u)du + S(0). \tag{3.19}$$

This algorithm can not be used here, as the integrand involved in Eq. 3.8 diverges for the values of ϕ near zero. However, the formula

$$\int_1^x s(u)dx = S(x) - S(1) \text{ for } x \geq 1 \tag{3.20}$$

$$-\int_x^1 s(u)dx = S(x) - S(1) \text{ for } x < 1 \tag{3.21}$$

that differs from $S(x)$ by an additive constant, may be used instead. This algorithm, as applied to computing the integrals in Eqs. 3.7 and 3.8, generates $g_s^*(\phi) = c_0 g_s(\phi) + c_1$ where c_0 and c_1 are unknown constants, instead of generating $g_s(\phi)$. This does not pose a problem in studying the distribution of $g_s(\hat{\phi})$ as its standardized version has the same distribution as that of $g_s^*(\hat{\phi})$. The computer codes used for applying this algorithm were written in `R` (Ihaka and Gentleman, 1996) to produce the function `fsym.IG` as displayed in Appendix A, that gives the value of $g_s^*(\phi)$ for a given value of the parameter ϕ and a given sample size n. Figure 1 is used to demonstrate the nature of this transformation for various sample sizes. We note that this function is an increasing function of ϕ.

Note that the logarithm involved in the variance stabilizing transformation acts on a value that is necessarily less than 1, hence the values of the transformation are negative as depicted in Fig. 2. However, such is not the case with the symmetrizing transformation, hence the two transformations will be qualitatively quite different.

3.3. Variance Stabilizing and Symmetrizing Behavior of Transformations. In this section, we investigate the following questions.

1. How far the variance stabilizing transformation Eq. 3.6 symmetrizes the distribution?

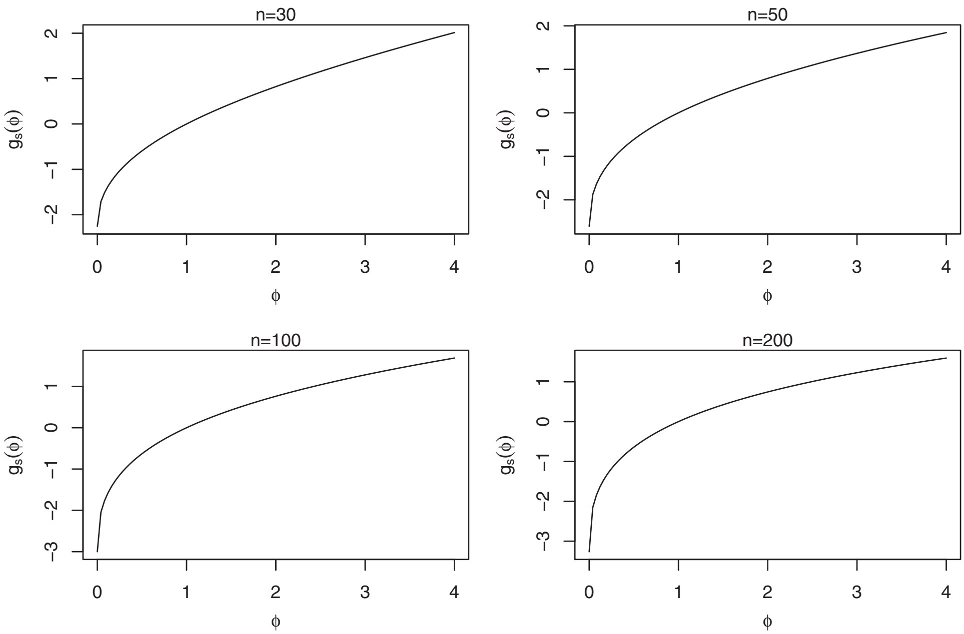

Figure 1: Symmetrizing transformation, $g_s(\phi)$, of CV^2 (ϕ) for varying values of sample size (n)

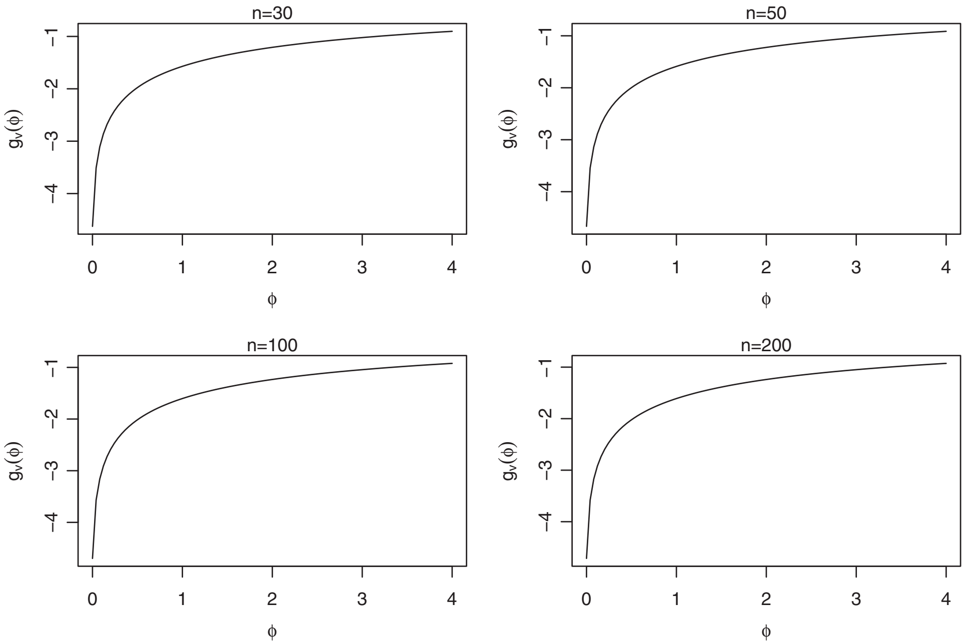

Figure 2: Variance stabilizing transformation, $g_v(\phi)$, of CV^2 (ϕ) for varying values of sample size (n)

2. How far the symmetrizing transformation given in Eq. 3.7 stabilizes the variance?

To assess the degree of symmetry of VST (question 1) and untransformed statistic, we evaluate their skewness β_1 using Eqs. 3.2 and 3.4, that is given by

$$\beta_1 = \frac{\mu_3(\phi) + \frac{3}{2}R(\mu_4(\phi) - {\mu_2}^2(\phi))}{[\mu_2(\phi) + R\mu_3(\phi) + \frac{1}{4}R^2(\mu_4(\phi) - {\mu_2}^2(\phi))]^{3/2}}, \tag{3.22}$$

where

$$R = -\frac{2B^2 + 3\phi}{2\phi(B^2 + \phi)} \tag{3.23}$$

for VST, with B as defined in Eq. 3.17, and it equals zero for untransformed case. Figure 3 gives a plot of the skewness of the VST (denoted by $\beta_1(vst)$) and Fig. 4 presents that (denoted by $\beta_1(ut)$) of the untransformed statistic. These plots show that the skewness of VST is a decreasing function of ϕ, however it increases for the untransformed case. Roughly, the skewness of the untransformed CV and that of the VST are in the same ball park, except for their signs.

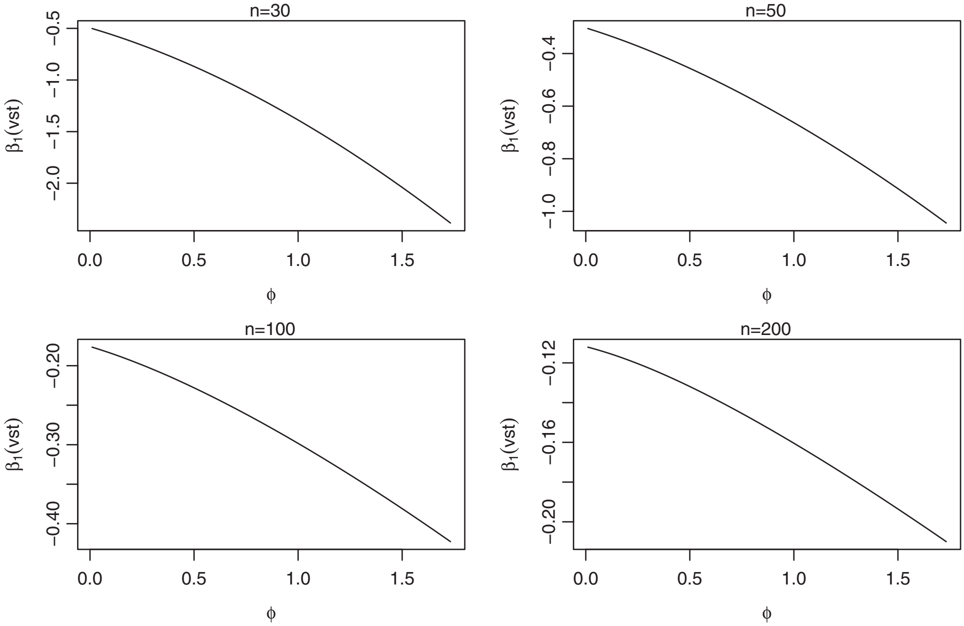

Figure 3: Skewness ($\beta_1(vst)$) of the VST ($g_v(\hat{\phi})$) for varying values of sample size (n)

On the other hand, to see how far the variance stability holds for the symmetrizing transformation (question 2) given in Eq. 3.7, we explore the nature of its variance as a function of n and ϕ. Figure 5 displays these values, as computed from Eq. 3.2, for various sample sizes and CV in the range of $[0, .3]$ which demonstrates that the ST has poor variance stabilizing property.

4 Nature of the Symmetrizing Transformation for Small and Large Values of ϕ

Small values of ϕ. For small values of ϕ we have approximately (for large values of n)

$$\begin{aligned}(\mu_2(\phi))^2 &\approx \frac{4\phi^4}{\nu^2},\\ \mu_3(\phi) &\approx \frac{8\phi^3}{\nu^2},\\ \text{and } \mu_4(\phi) &\approx \frac{12\phi^4}{\nu^2}.\end{aligned}$$

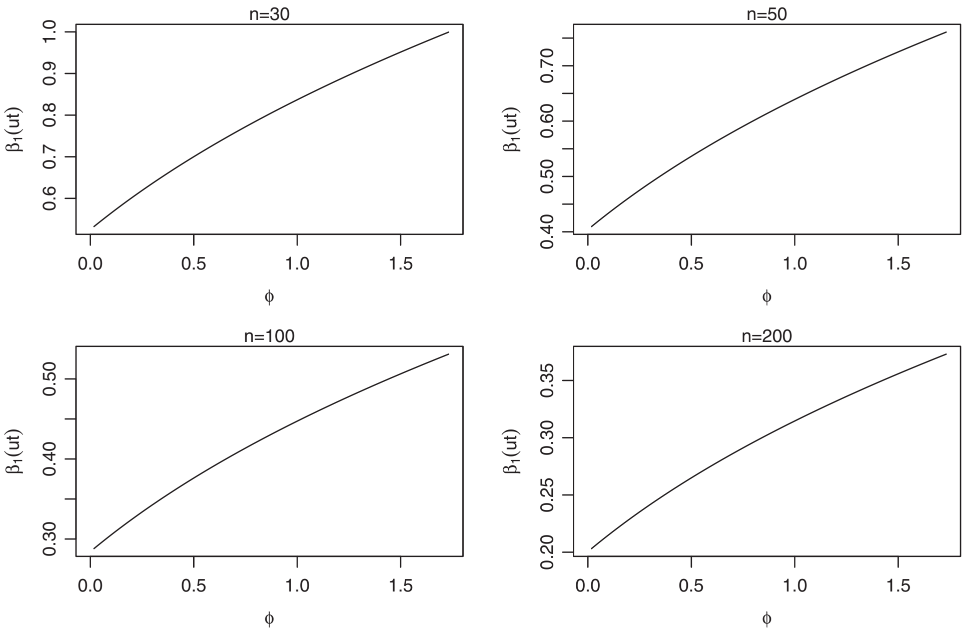

Figure 4: Skewness ($\beta_1(ut)$) of the untransformed CV^2 ($\hat{\phi}$) for varying values of sample size (n)

Hence $f_1(\phi) = 8\phi^3/\nu^3$ and $f_2(\phi) = \mu_4(\phi) - \mu_2^2(\phi) = 8\phi^4/\nu^3$ that gives approximately

$$w(\phi) \approx \frac{2}{3}\int \frac{1}{\phi}d\phi = \frac{2}{3}\ln\phi, \tag{4.1}$$

and the corresponding expression for $g_s(\phi)$ therefore is given by

$$g_s(\phi) = \int \mathrm{e}^{-\frac{2}{3}\ln\phi}d\phi = \frac{1}{3}\phi^{1/3}. \tag{4.2}$$

This transformation heuristically makes sense, as for small values of ϕ, Z is close to μ and therefore $\hat{\phi}$ behaves like a χ^2 random variable for which the Wilson-Hilferty (1931) cube-root transformation is recognized as an excellent normalizing transformation. Now we consider the large values of ϕ.

Large values of ϕ. In this case retaining terms up to order $1/n^2$ and assuming $n \approx \nu$, we can write

$$\begin{aligned} f_1(\phi) &\approx \phi^3[\frac{8}{\nu^2} + \frac{12}{\nu^2}\phi + \frac{3}{\nu^2}\phi^2] \\ &= \frac{\phi^3}{\nu^2}[8 + 12\phi + 3\phi^2]. \end{aligned}$$

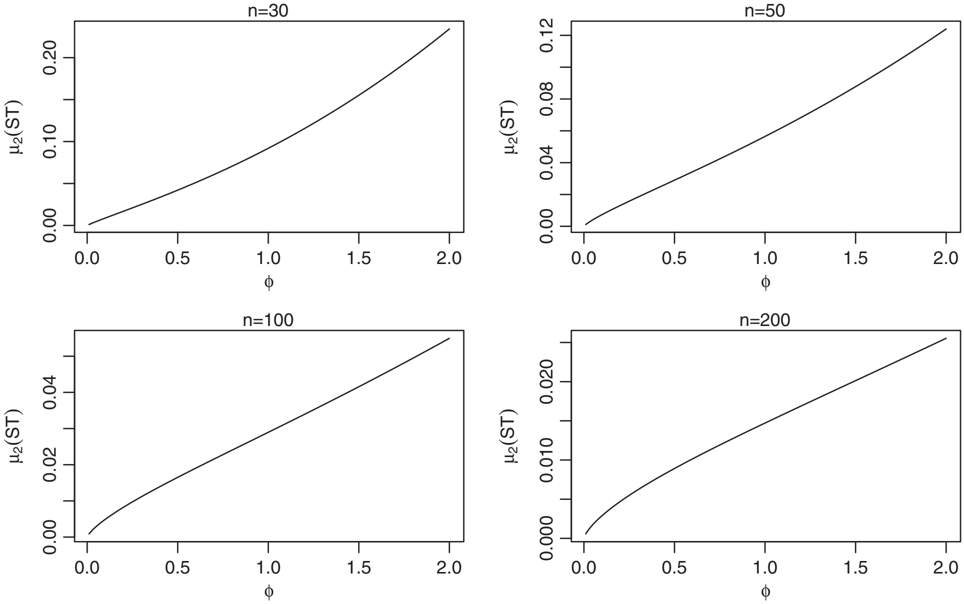

Figure 5: Variance ($\mu_2(ST)$) of symmetrizing transformation ($g_s(\hat{\phi})$) as a function of ϕ for varying values of sample size (n)

and

$$\begin{aligned} f_2(\phi) &\approx \phi^4[\frac{8}{\nu^2} + \frac{8}{\nu^2}\phi + \frac{2}{\nu^2}\phi^2] \\ &= \frac{\phi^4}{\nu^2}[8 + 8\phi + 2\phi^2]. \end{aligned}$$

Thus we approximately have

$$\begin{aligned} \frac{f_1(\phi)}{f_2(\phi)} &= \frac{3\phi^2 + 12\phi + 8}{\phi(8 + 8\phi + 8\phi^2)} \\ &= \frac{3\phi^2 + 12\phi + 8}{2\phi(\phi + 2)^2} \\ &= \frac{1}{\phi} + \frac{1}{2(\phi + 2)} + \frac{1}{(\phi + 2)^2} \end{aligned}$$

and then

$$\begin{aligned} w(\phi) &= \frac{2}{3}\int \frac{f_1(\phi)}{f_2(\phi)} d\phi \\ &= \frac{2}{3}[\ln\phi + \frac{1}{2}\ln(\phi + 2) - \frac{1}{\phi + 2}]. \end{aligned}$$

Using the approximation $-\frac{2}{\phi+2} \approx \ln(1-\frac{2}{\phi+2}) = \ln\frac{\phi}{\phi+2}$ for large ϕ, the above can be simplified to be

$$\begin{aligned} w(\phi) &= \frac{2}{3}[\ln\phi + \frac{1}{2}\ln(\phi+2) + \frac{1}{2}\ln\frac{\phi}{\phi+2}] \\ &= \frac{1}{3}(\ln\phi^2 + \ln\phi) \\ &= \ln\phi. \end{aligned} \tag{4.3}$$

Therefore, for large values of ϕ, we have

$$g_s(\phi) = \int \mathrm{e}^{-\ln\phi}\mathrm{d}\phi = \ln\phi \tag{4.4}$$

and we find that the log-transformation is approximately symmetrizing transformation for large values of ϕ.

The above analysis shows that we might like to search for normalizing transformations in the family of power transformations $\phi \mapsto \phi^\lambda$ where $\lambda = 0$ signifies logarithmic transformation. We can adopt the technique of Jensen and Solomon (1972) that was developed for seeking the best normalizing transformation for a quadratic form. This is explored in the next section.

5 Normalizing Transformation for CV in Power Transformation Family

The technique in Jensen and Solomon (1972) has been adopted to non-negative random variables by Mudholkar and Trivedi (1981) that we outline here. Let $\kappa_r,\ r = 1, 2, ..$ denote the r^{th} cumulant of a non-negative random variable T and assume that $\psi_r = \kappa_r/\kappa_1,\ r = 2, 3, ...$ are bounded as $\kappa_1 \to \infty$. Then, using a Taylor series expansion, we can write the expectation of $(T/\kappa_1)^h$ as

$$\mu'_{1h} = 1 + \frac{h(h-1)\psi_2}{2\kappa_1} + \frac{h(h-1)(h-2)}{24\kappa_1^2}[4\psi_3 + 3(h-3)\psi_2^2] + O(k_1^{-3}). \tag{5.1}$$

The above expression may be used to obtain the r^{th} moment $\mu'_{rh} = E[(T/\kappa_1)^h]^r$ by a simple substitution of h by rh. This provides the following series expansions for the central moments $\mu_r(h)$ of $(T/\kappa_1)^h$, $r = 2, 3, 4$ in terms of the powers of κ_1^{-1} :

$$\mu_{2h} = \frac{h^2\psi_2}{\kappa_1} + \frac{h^2(h-1)}{2\kappa_1^2}[2\psi_3 + (3h-5)\psi_2^2] + O(\kappa_1^{-3}), \tag{5.2}$$

$$\mu_{3h} = \frac{h^3}{\kappa_1^2}[\psi_3 + (3h-1)\psi_2^2] + O(\kappa_1^{-3}), \tag{5.3}$$

$$\mu_{4h} = \frac{3h^4\psi_2^2}{\kappa_1^2} + O(\kappa_1^{-3}). \tag{5.4}$$

If T is asymptotically distributed as $\kappa_1 \to \infty$ then as Mudholkar and Trivedi (1981) argue, so is T^h by Mann-Wald (1943) theorem. In order to accelerate the convergence to normality, we may choose h so that the leading term in $\mu_3(h)$ is zero. The resulting value of h denoted by h_0 thatapproximately symmetrizes $(T/\kappa_1)^h$ is thus given by

$$h_0 = 1 - \frac{\kappa_1\kappa_3}{3\kappa_2^2}. \tag{5.5}$$

In order to use the above formulation for the CV, we take $T = \nu\hat{\phi}/\phi$. The cumulants of T needed for our purpose may be obtained from the central moments of $\hat{\phi}$ given in Eqs. 2.9–2.10 that are given below:

$$\kappa_1 = \nu, \tag{5.6}$$

$$\kappa_2 = \nu[2 + (\nu+2)\frac{\phi}{n}], \tag{5.7}$$

$$\kappa_3 = \nu[8 + 12(\nu+2)\frac{\phi}{n} + 3(\nu^2+6\nu+8)(\frac{\phi}{n})^2]. \tag{5.8}$$

The asymptotic normality of T follows from that of $\hat{\phi}$ and obviously $\psi_r, r = 2, 3, ...$ are bounded as $n \to \infty$. Hence we can approximate the distribution of $(\hat{\phi}/\phi)^{h_0}$ by the normal distribution with mean μ'_{1h_0} and variance $\sigma^2(h_0) = \mu_{2h_0}$ as given in Eqs. 5.1 and 5.2, replacing h by h_0.

We plot the values of the powers h_0 for various sample sizes as a function of ϕ in Fig. 6. It is interesting to note that for small values of ϕ the optimum power is close to 1/3 and for large values of ϕ this is close to zero. Owing to the analysis of the general symmetrizing transformation for small and large ϕ in the previous section, this implies that the power transformation family may be adequate in contrast to the general transformation g_s that can be only computed numerically.

6 A Comparison of the Approximations Using the Transformed Statistics

The approximation afforded by the power transformation family provides an excellent approximation for computing the distribution of the CV as seen from the previous section. It is thus a natural question to ask whether

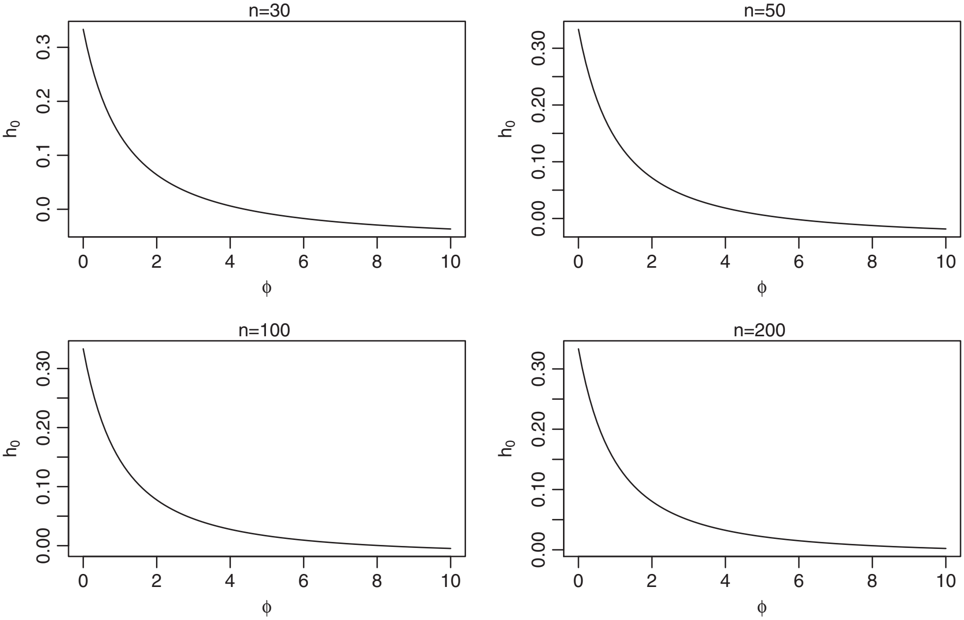

Figure 6: Optimum power (h_0) for symmetrizing power transformation $(\hat{\phi}/\phi)^{h_0}$ for varying values of sample size (n)

the numerical effort is worth in using the general transformation discussed in Section 3.2 over the simplicity of the explicit formulae using the power family of transformations. In order to investigate this issue we compare the two approximations, namely,

(i)

$$(\hat{\phi}/\phi)^{h_0} \sim N(\mu'_{1h_0}, \sigma^2(h_0))$$

with

(ii)

$$g_s(\hat{\phi}) \sim N(\mu_{g_s}, \sigma^2(g_s))$$

where

$$\mu_{g_s} = g_s(\phi) + \frac{1}{2}g''_s(\phi)\mu_2(\phi), \tag{6.1}$$

$$\sigma^2(g_s) = (g'_s(\phi))^2 \left[\mu_2(\phi) + R\mu_3(\phi) + \frac{1}{4}R^2(\mu_4(\phi) - \mu_2^2(\phi))\right] \tag{6.2}$$

with

$$R \equiv R(\phi) = -\frac{2}{3}\frac{\mu_3(\phi)}{\mu_4(\phi) - \mu_2^2(\phi)}$$

and

$$g_s'(\phi) = \mathrm{e}^{-w(\phi)}, \tag{6.3}$$

$w(\phi)$ being as defined in Eq. 3.8. Note that in the above approximation, we require g_s, g_s' and g_s''. These are numerically computed using the algorithm described in the previous section, starting with g_s' as given in Eq. 6.3, the `R-codes` for which are available through function `f1f2.IG` given in the Appendix A; g_s'' is then computed using the formula

$$g_s''(\phi) = g_s'(\phi)R(\phi). \tag{6.4}$$

Additionally we compare these in turn, with $\hat{\phi}$ and $g_v(\hat{\phi})$ approximating their distributions with those of appropriate Gaussian distributions, i.e.

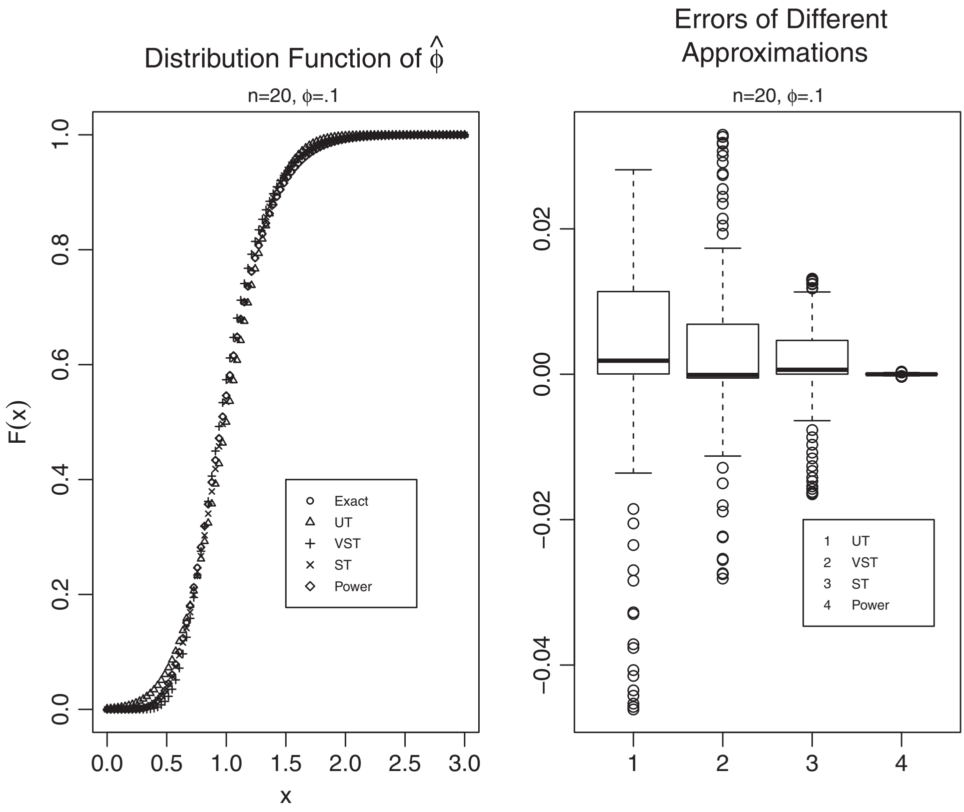

Figure 7: Distribution function, $F(x) = \Pr(\hat{\phi} \leq x)$, of $\hat{\phi}$ under normal approximation for various transformations: $n = 20, \phi = .1$

(iii)

$$\hat{\phi} \sim N(\phi, \mu_2(\phi))$$

and

(iv)

$$g_v(\hat{\phi}) \sim N(\mu_{g_v}, \sigma^2(g_v))$$

where

$$\mu_{g_v} = g_v(\phi) + \frac{1}{2} g_v''(\phi)\mu_2(\phi) \tag{6.5}$$

and

$$\sigma^2(g_v) = g_v'(\phi)^2 \left[\mu_2(\phi) + R_v\mu_3(\phi) + \frac{1}{4}R_v^2(\mu_4(\phi) - {\mu_2}^2(\phi))\right], \tag{6.6}$$

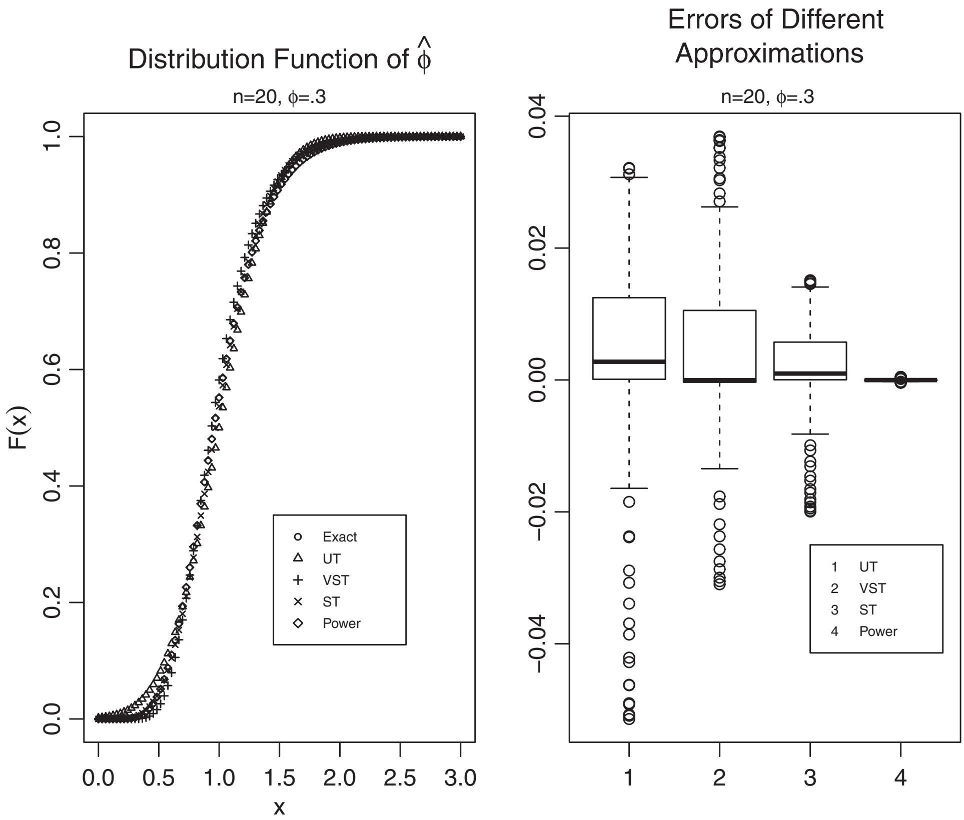

Figure 8: Distribution function, $F(x) = \Pr(\hat{\phi} \leq x)$, of $\hat{\phi}$ under normal approximation for various transformations: $n = 20, \phi = .3$

where

$$
\begin{aligned}
R_v \equiv R_v(\phi) &= -\frac{(2B^2+3\phi)}{2\phi(B^2+\phi)}, \\
g_v'(\phi) &= \frac{1}{\sqrt{n}}\frac{1}{\phi\sqrt{a+b\phi}} \\
g_v''(\phi) &= g_v'(\phi)R_v(\phi),
\end{aligned}
$$

with $a = 2/(n-1), b = (1+a)/n$ and $B = \sqrt{\frac{2(n-1)}{n+1}}$ as given in Eq. 3.15.

Figures 7, 8, 9 and 10 plot the distribution functions of $\hat{\phi}$ along with various approximations, where the exact values are computed using an integral formula outlined in Chaubey et al. (2014). These figures display the qualitative nature of the approximations and convey that the basic nature

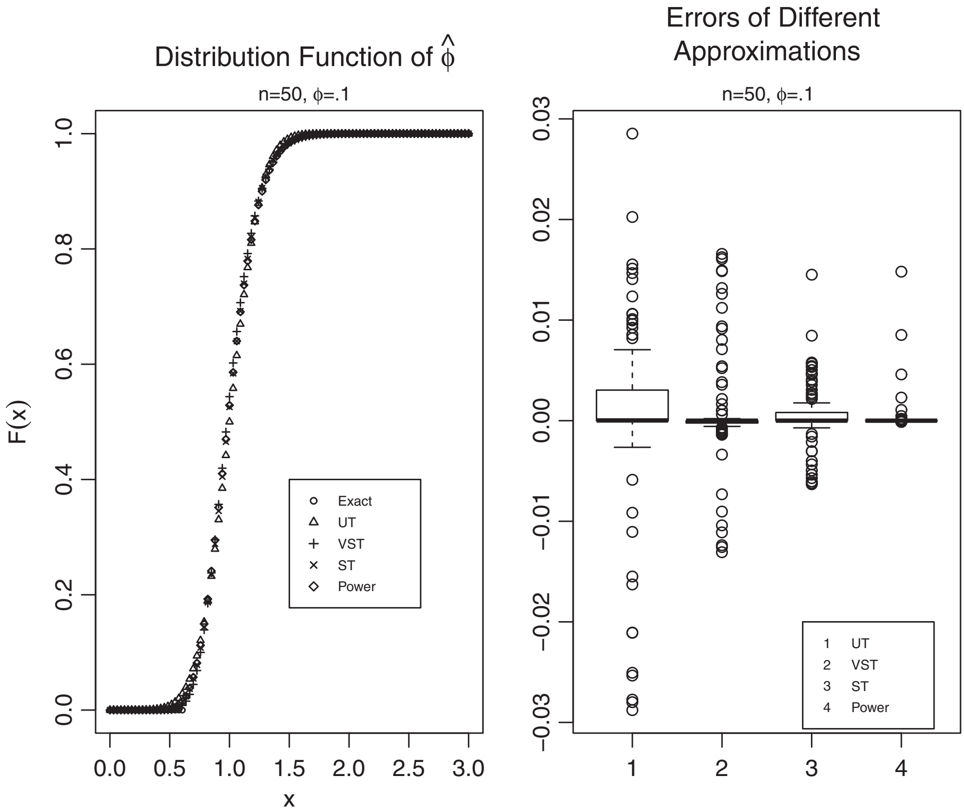

Figure 9: Distribution function, $F(x) = \Pr(\hat{\phi} \leq x)$, of $\hat{\phi}$ under normal approximation for various transformations: $n = 50, \phi = .1$

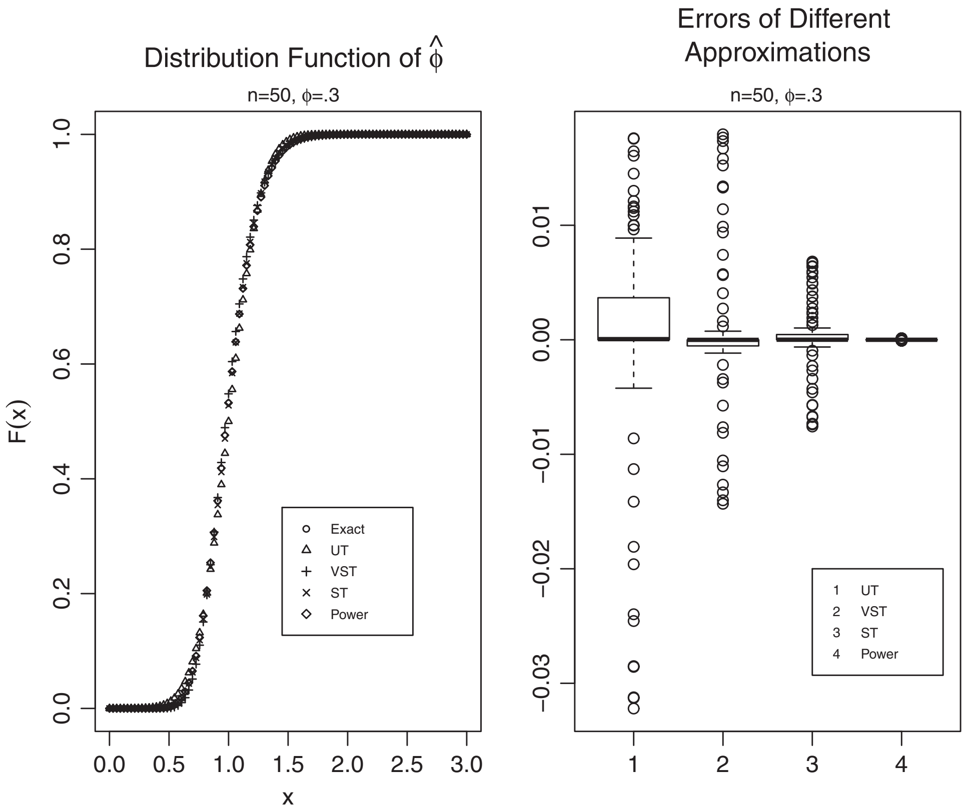

Figure 10: Distribution function, $F(x) = \Pr(\hat{\phi} \le x)$, of $\hat{\phi}$ under normal approximation for various transformations: $n = 50, \phi = .3$

of the approximations are the same for the values of the parameters investigated. Hence we plot the errors as boxplots in the second box that clearly demonstrates that the normal approximations rendered by the untransformed statistic and the VST show the same performance whereas the symmetrizing transformation gives a significant improvement. However the power transformation gives even a better performance.

One is tempted to ask 'why the general symmetrizing transformation derived numerically is poorer than the power transformation?' This may be due to accuracy lost during iterative computations of the integrals and approximation of the derivatives involved. Due to the simple nature of the power transformation and its accuracy in approximating the probabilities, we refrain from investigating the general symmetrizing transformation any further.

7 Effect of Bias on the Transformed Statistics for Small Samples

In earlier sections we have addressed the questions (i) how far does the VST symmetrize the distributions and (ii) how far does the ST stabilize the variance. In the background of these questions is a hidden concern of how do these affect the approximating nature of the transformations. A related concern about the quality of the approximation may be dictated by the bias in finite samples; for the untransformed statistic exact mean is available but such is not the case for the resulting transformations studied earlier. It may be crucial to point out here that the bias correction may not improve variance stabilization or symmetrization. Though we have seen from Figs. 7–10, that standardized statistics based on transformations provide better approximations than that based on the untransformed statistic, there may still be a lingering question whether there exist corrections for the VST and ST that can improve variance stabilization, as has been questioned by a reviewer. In order to answer this question, we would have to use expansion of moments of the transformed statistic $g(\hat{\phi})$ of order higher than $O(1/n^2)$,

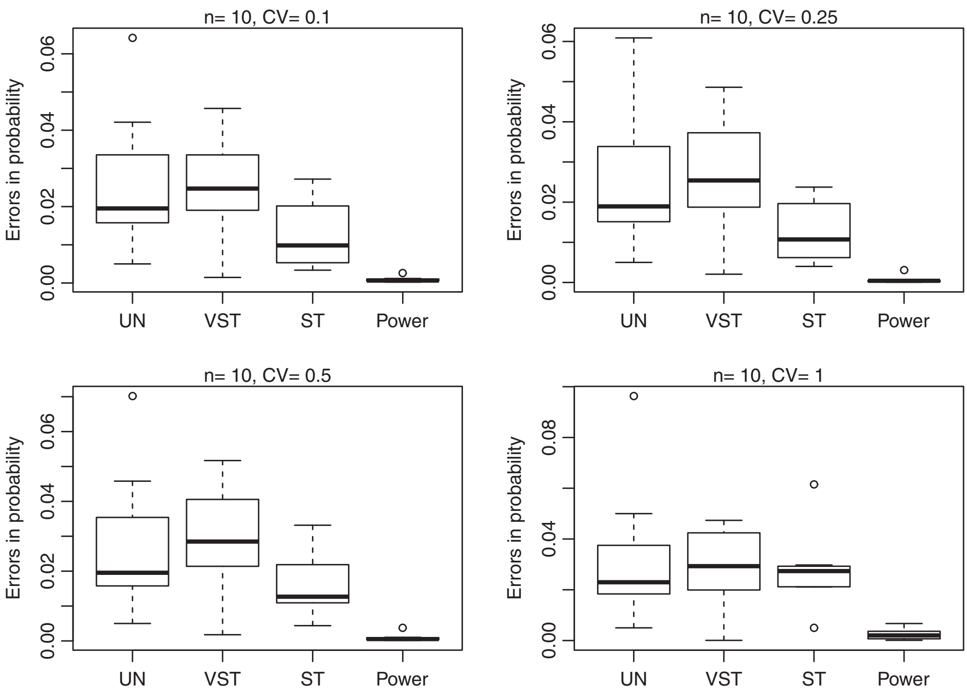

Figure 11: Errors in probabilities using the normal approximation for different transformations based on 50,000 simulations, $n = 10$

because we have already incorporated terms up to order $O(1/n^2)$. We have not found that to be analytically tractable. However, we have compared the approximations for probabilities for small sample sizes using the simulated values of the mean and variance based on 50,000 replications. Figures 11, 12, 13 and 14 display errors of probabilities obtained by simulation and the normal approximation of transformations. This indirectly answers the concern whether any further correction for bias would improve approximating quality of transformations.

It is also natural to ask about the nature of variance of the power transformation as we saw in Section 3, with respect to the symmetrizing transformation, as was put forward by the reviewer. We have shown analytically in Section 5 that for large values of n the symmetrizing transformation is the same as the power transformation for small and large values of ϕ. We have seen (see Fig. 5) that the variance of the symmetrizing transformation

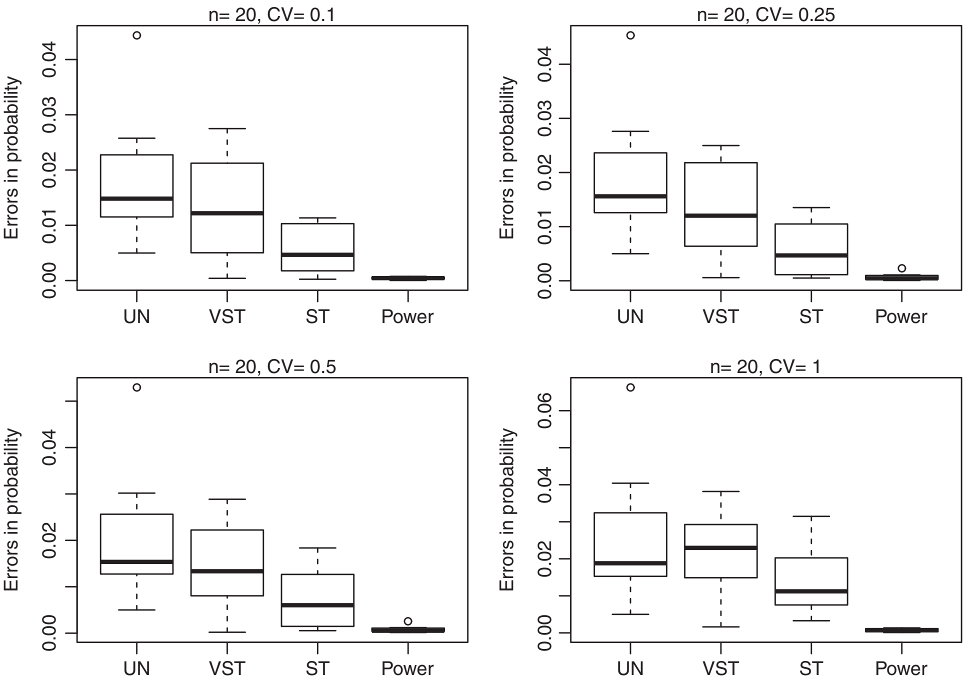

Figure 12: Errors in probabilities using the normal approximation for different transformations based on 50,000 simulations, $n = 20$

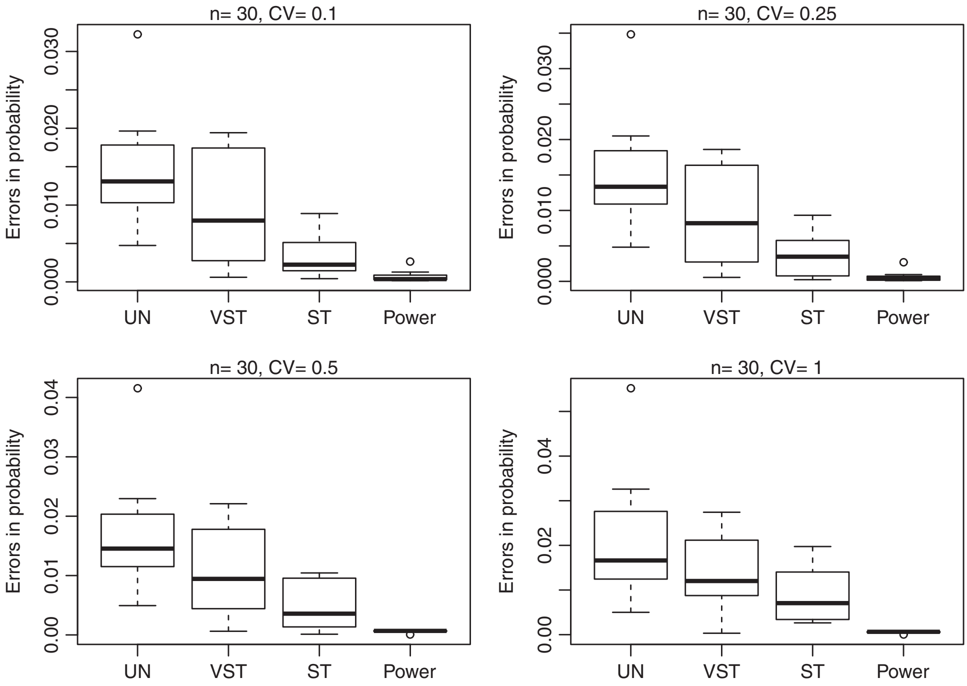

Figure 13: Errors in probabilities using the normal approximation for different transformations based on 50,000 simulations, $n = 30$

increases as a function of ϕ, however, in the case of the power transformation, this pattern holds for the values of ϕ in the range $\phi \in [0, .5)$ (see Fig. 15). Roughly speaking in a small neighborhood of $\phi = 0.5$ the variance of the power transformation may be assumed to be constant but in general it is not.

8 Numerical Examples

The following data set, that show active repair times (in hours) of an airborne communication transceiver, is obtained from Chhikara and Folks (1977). These are used for testing of hypothesis for coefficient of variation assuming that these represent a random sample from an inverse Gaussian population. Figure 16 demonstrates the excellent quality of the fit of the inverse Gaussian distribution with estimated parameters in terms of the probabilities and quantiles. The excellence of the inverse Gaussian distribution fit to these data has also been quantitatively demonstrated in the paper

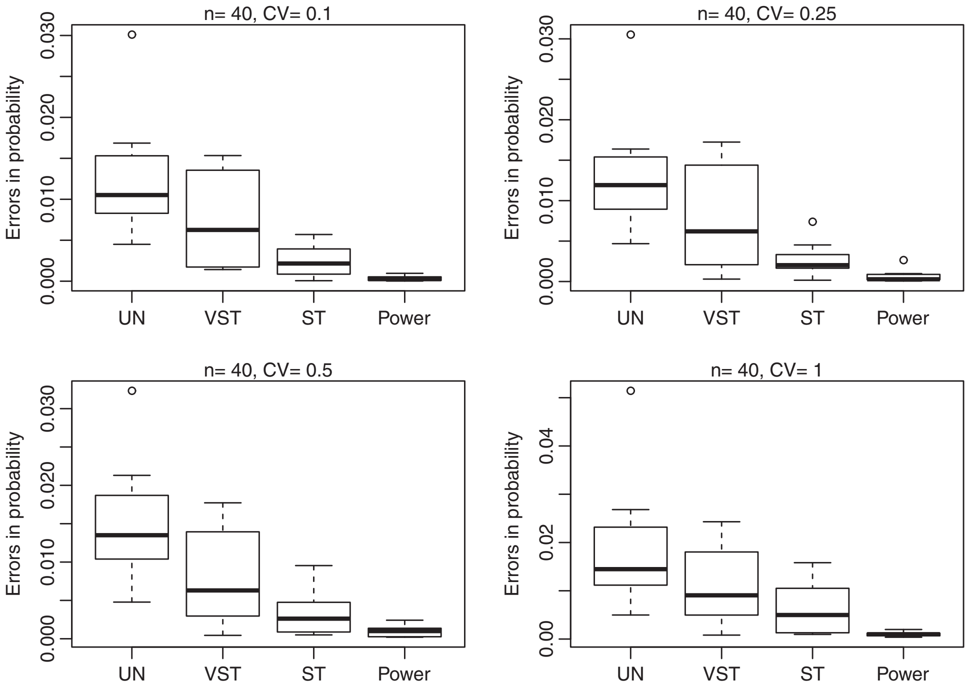

Figure 14: Errors in probabilities using the normal approximation for different transformations based on 50,000 simulations, $n = 40$

by Henze and Klar (2002) where the P-values for various tests of goodness-of-fit have been provided that range from 0.81 to 0.95 (see Table 8 of Henze and Klar 2002).

Such a test is carried out using the exact distribution in Chaubey et al. (2014). Let us test $H_0 : \phi \leq 2$ against $H_1 : \phi > 2$. Here $n = 46$, $\bar{X} = 3.6065$, $\sum_i((1/X_i) - (1/\bar{X})) = 27.73$. Hence, $\hat{\phi} = 3.6065 \times 27.73/45 = 2.2224$. We may use the test statistic based on the symmetrizing power transformation

$$Z = \frac{(\hat{\phi}/\phi_0)^{h_0} - \mu'_{1h_0}}{\sigma(h_0)}.$$

Here $\phi_0 = 2,\ h_0 = 0.0707,\ \mu'_{1h_0} = 0.99715,\ \sigma^2(h_0) = 0.0004347339$, and the observed value of Z is $Z_{obs.} = 0.4955$. As shown in Chaubey et al. (2014), the best invariant test under scale transformations rejects H_0 for larger values of $\hat{\phi}$, the test based on the Z statistic will reject the null hypothesis for $Z > Z_{obs}$ and the corresponding $P-$value $= 1 - 0.68988 = 0.31012$.

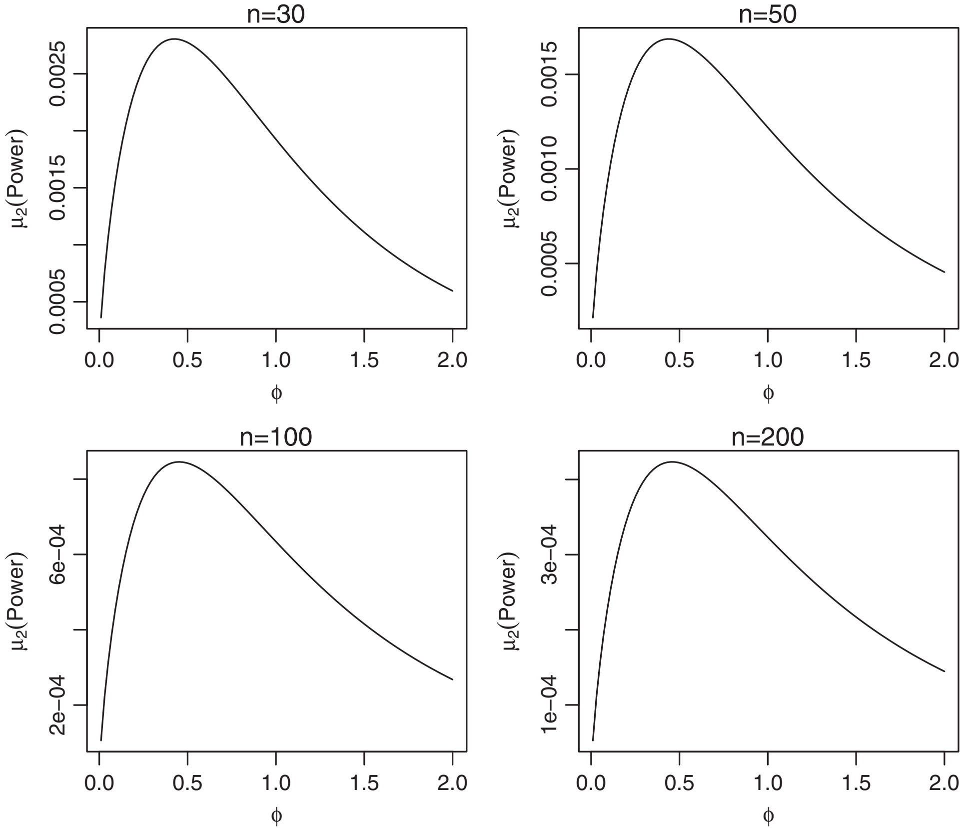

Figure 15: Variance ($\mu_2(Power)$) of symmetrizing power transformation ($\hat{\phi}^{h_0}$) as a function of ϕ for varying values of sample size (n)

The exact P-value given in Chaubey et al. (2014) is 0.31087 that may be noted to be very close to the approximate value obtained by the symmetrizing transformation. This value is quite large for a 1% level of significance

Table 1: Active repair times (in hours) of an airborne communication transceiver

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8	0.8
1.0	1.0	1.0	1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0	2.0	2.2
2.5	2.7	3.0	3.0	3.3	3.3	4.0	4.0	4.5	4.7	5.0	5.4	5.4
7.0	7.5	8.8	9.0	10.3	22.0	24.5						

Source: Chhikara and Folks (1977)

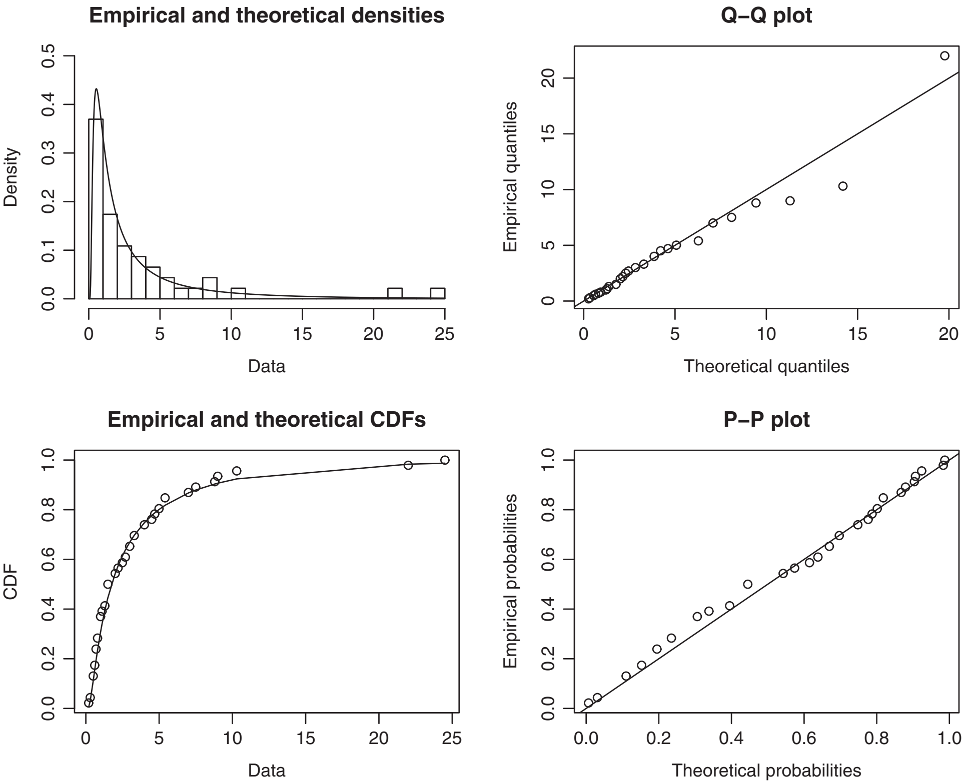

Figure 16: Inverse Gaussian fit for the Repair Times Data-set (Table 1) in terms of probability density function, cumulative distribution function (CDF), quantitle-quantile and probability - probability plots

and therefore a squared CV of less than equal to 2 is accepted. This is not a surprising result for this data as the unbiased estimate of ϕ is $\hat{\phi}$ is just slightly larger than 2. For more applications of the power transformation of the coefficient of variation, the reader may refer to Chaubey et al. (2016).

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Appendix A: R-Codes for the Symmetrizing Function

```
###Computing the symmetrization transformation as a function
of phi
###Symmetrizing function
##Input:
##phi: Value of (mu/lambda)
##ss: Sample size n
##Output: g_s(phi)
###Computing the symmetrization transformation as a function
of phi
################
### Symmetrizing unction
fsym.IG<-function(phi,ss){
if (phi>1) {xl<-1;xu<-phi}
else {xl<-phi;xu<-1}
fval<- integrate(f1f2Int.IG,xl,xu,subdivisions=1000,ss=ss)
$value
if (phi<1) fval<--fval
fval}
######################
hfun.IG<-function(phi,ss=ss){
if (phi>0) {
nu<-ss-1;d<-phi/ss
c21<-(2/nu);c22<-(1+c21)
c31<-8/nu^{2};c32<-12*c22/nu;c33<-3*(1+(6/nu)+(8/nu^{2}))
c41<-12*(1+(4/nu))/nu^{2};c42<-12*(1+(14/nu)+(24/nu^{2}))/nu
c43<-3*(1+(36/nu)+(188/nu^{2})+(240/nu^{3}))
c44<-15*(1+(12/nu)+(44/nu^{2})+(48/nu^{3}))
mu2<-phi^{2}*(c21+c22*d)
mu3<-phi^{3}*(c31+c32*d+c33*d^{2})
mu4<-phi^{4}*(c41+c42*d+c43*d^{2}+c44*d^{3})
result<-mu3/(mu4-mu2^{2})}
else result<-Inf
result}
##Vector version of hfun.IG
hfunInt.IG<-function(x,ss)sapply(x,hfun.IG,ss=ss)

##The following function computes g_{s}'(phi)
f1f2.IG<-function(phi,ss){
if (phi==0) result<-Inf
```

```
else {
if (phi>1){xl<-1;xu<-phi}
else {xl<-phi;xu<-1}
 fval<- integrate(hfunInt.IG,xl,xu,subdivisions=1000,ss=ss)
 $value
if (phi<1) fval<--fval
result<- exp(-(2/3)*fval)}
result}

#Vectorised version of f1f2.IG
f1f2Int.IG<-function(x,ss)sapply(x,f1f2.IG,ss=ss)
```

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Yogendra P. Chaubey
Debaraj Sen
Department of Mathematics and Statistics,
Concordia University,
Montreal, QC H3G 1M8, Canada
E-mail: yogen.chaubey@concordia.ca;
debaraj.sen@concordia.ca

Murari Singh
International Center for Agricultural Research in the Dry Areas,
Amman, Jordan
E-mail: m.singh@cgiar.org

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