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Adjusted Empirical Likelihood for Time Series Models

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Abstract

Empirical likelihood method has been applied to dependent observations by Monti (*Biometrika*, **84**, 395–405 1997) through the Whittle's estimation method. Similar asymptotic distribution of the empirical likelihood ratio statistic for stationary time series has been derived to construct the confidence regions for the parameters. However, Monti's approach is valid only when the error terms follow a Gaussian distribution. Nordman and Lahiri (*Ann. Statist.*, **34**, 3019–50 2006) derived estimating functions and empirical likelihood ratio statistic using frequency domain empirical likelihood approach for non-Gaussian error term distributions. Nonetheless, the required numerical problem of computing profile empirical likelihood function which involves constrained maximization has no solution sometimes, which leads to the drawbacks of using the original version of the empirical likelihood ratio. In this paper, we propose an adjusted empirical likelihood ratio statistic to modify the one proposed by Nordman and Lahiri so that it guarantees the existence of the solution of the required maximization problem, while maintaining the similar asymptotic properties as Nordman and Lahiri obtained. Simulations have been conducted to illustrate the coverage probabilities obtained by the adjusted version for different time series models which are competitive to the ones based on Nordman and Lahiri's version, especially for small sample sizes.

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1 Introduction

Empirical likelihood (EL) method introduced by Owen (1988) has become a widely applicable tool for constructing confidence regions in nonparametric problems due to its appealing asymptotic distribution of the likelihood-ratio-type statistic which is same as the one under the parametric settings. However, the established methods of EL are mainly for independent observations and therefore are difficult to be apply to dependent observations such as time series data. Progress in applying the EL method to different time series models has been made by different researchers. For example, Mykland (1995) built up the connection between the dual likelihood and the empirical likelihood through the martingale estimating equations and applied to time series model. Kitamura (1997) proposed the empirical likelihood of the blocks of observations to study the weakly dependent processes. Chan and Ling (2006) derived the empirical likelihood method for GARCH models. Chan and Liu (2010) considered the Bartlett corrections of empirical likelihood for short-memory time series. Monti (1997) extended the EL method to stationary time series by using the Whittle's (1953) estimation method to obtain an M-estimator of the periodogram ordinates of time series models which are asymptotically independent. This method reduced the dependent data problem to an independent data problem. Then the original EL method is applied and similar asymptotic results have been obtained. Yau (2012) extended Monit's results to long-memory time series models. However, both Ogata (2005) and Nordman and Lahiri (2006) argued that Monti's (1997) results rely on the Gaussian assumption of the error term. Nordman and Lahiri (2006) further proved that Monti's (1997) results require the model class to be correctly specified and developed frequency domain empirical likelihood based on the spectral distribution through the fourier transformation to study short- and long-range dependence. Thus, Ogata (2005) and Nordman and Lahiri (2006) independently derived estimating functions and EL ratio statistic for the Whittle's estimation. This result can be used for short- and long-memory time series models. Further, Nordman and Lahiri (2014) summarized advances in empirical likelihood (EL) for time series data and presented the frequency domain EL methods based on the periodogram for short- and long-range dependence.

However, as Chen et al. (2008) pointed out, computing profile empirical likelihood function which involves constrained maximization requires that the convex hull of the estimating equation must have the zero vector as an interior point. When the required computational problem has no solution, Owen (2001) suggested assigning $-\infty$ to the log-EL statistic. Chen et al. (2008) mentioned that there are two drawbacks in doing so. To remedy the drawback of EL method, Chen et al. (2008) proposed an adjusted empirical likelihood (AEL) for independent observations by adding an artificial term to guarantee the zero vector to be within the convex hull, therefore, the solution always exists. In their work, they have showed the asymptotic results of the AEL are same as that of the EL. Furthermore, this method can achieve the improved coverage probabilities without making Bartlett correction or bootstrap calibration. By adopting their method, Piyadi Gamage et al. (2017) proposed an AEL-based method to study the long-memory time series models, and developed corresponding asymptotic results.

In this paper, we propose an adjusted empirical likelihood method to extend Nordman and Lahiri's EL method by adopting Chen's idea for dependent observations, specifically short-memory time series models. The rest of the article is organized as follows. In Section 2, the EL for time series models is discussed with a review on M-estimators based on the periodogram. The AEL for a stationary time series model is proposed and the asymptotic distribution of the AEL statistic is established in Section 3. Simulations in Section 4 illustrate the coverage probabilities of the proposed AEL method for invertible moving average (MA) time series model, stationary autoregressive (AR) model, and ARMA model with various sample sizes, values of parameters and distributions of the noise term. Comparisons to the EL method, Bartlett-correction ones have also been made to indicate the advantage of the proposed AEL method. The applications to ARMA(1,1) model and AR(2) model are considered in Section 5. Some discussion is provided in Section 6.

2 Empirical Likelihood for Short-Memory Time Series Models

The empirical likelihood (EL) method was introduced by Owen (1988, 1990, 1991) which combines the reliability from non-parametric methods and flexibility of parametric methods. The most appealing property of the EL method is that the null distribution of the EL ratio statistic follows the standard chi-square distribution under mild conditions similar to the one obtained under the parametric settings. Since then, the EL method has been extensively used in different areas in statistics. See Owen (2001) for more details of the EL method. However, the EL method is designed to deal with the independent observations and has difficulties in applying to dependent observations such as time series. Based on the facts that the periodogram of time series data are asymptotically independent and the parameter of a stationary time series estimated by the Whittle's (1953) method can be viewed as an M-estimator involving periodogram ordinates, Monti (1997) proposed the empirical likelihood version based on M-estimators to reduce a dependent data to an independent data; therefore, Owen's EL method can be applied. Let $x_1, x_2, ..., x_n$ be a set of independent and identically distributed observations from an unknown distribution F_0 and $X \sim F_\beta$ where $\beta \in \mathbb{B}$ is a k^*- vector. β is estimated by an M-estimator β_n, which is the solution of

$$\sum_{j=1}^{n} \psi_j(x_j, \beta) = 0.$$

With the definition of $p_j = P(X = x_j)$, the empirical likelihood ratio for any value $\beta \in \mathbb{B}$ is defined by

$$\hat{\lambda}(\beta) = \sup \prod_{j=1}^{n} p_j \Big/ \prod_{j=1}^{n} \frac{1}{n} = \sup \prod_{j=1}^{n} np_j,$$

subject to the constraints: (i) $\sum_{j=1}^{n} \psi(x_j, \beta)p_j = 0$, (ii) $\sum_{j=1}^{n} p_j = 1$, and (iii) $p_j \geq 0$, $(j = 1, 2, ...n)$. The maximization under the Lagrange multiplier method gives

$$p_j = [n\{1 + \xi(\beta)'\psi(x_j, \beta)\}]^{-1} \qquad (j = 1, 2, ..., n),$$

where $\xi(\beta)$ is the Lagrangian multiplier satisfying constraint (i). The empirical likelihood ratio statistic is thus defined

$$\hat{W}(\beta) = -2 \ln \hat{\lambda}(\beta) = 2 \sum_{j=1}^{n} \ln\{1 + \xi(\beta)'\psi(x_j, \beta)\} \tag{2.1}$$

Owen (1988) showed that $\hat{W}(\beta)$ is asymptotically distributed as $\chi^2_{k^*}$. Consequently, an asymptotic 1-α confidence region is given by $\{\beta \in \mathbb{B} : \hat{W}(\beta) \leq \chi^2_{k^*,1-\alpha}\}$.

Let

$$Z_t = \sum_{s=0}^{\infty} \gamma_s a_{t-s}, \qquad (t = ..., -1, 0, 1, ...), \tag{2.2}$$

be a linear process where $\gamma_0 = 1, \sum_{s=0}^{\infty} \gamma_s^2 < \infty$, and a_t is a sequence of independent and identically distributed random variables with $E(a_t) = 0, E(a_t^2) = \sigma^2 > 0, E(a_t^4) < \infty$. The spectral density of Z_t is given by

$$g(\omega) = \frac{\sigma^2}{2\pi} \left| \sum_{s=0}^{\infty} \gamma_s \exp(-i\omega s) \right|^2, \qquad \omega \in [-\pi, \pi].$$

Let $z_1, z_2, ..., z_T$ be T observations from the process (2.2) with sample mean $\bar{z}$. An approximate log-likelihood function is given by Whittle (1953)

$$\ln\{L(\beta)\} = -\sum_{j=1}^{n} \ln\{g_j(\beta)\} - \sum_{j=1}^{n} \frac{I(\omega_j)}{g_j(\beta)}, \tag{2.3}$$

where $g_j(\beta)$ is the spectral density defined above and

$$I(\omega_j) = \frac{1}{2\pi T} \left[\left\{ \sum_{t=1}^{T} (z_t - \bar{z}) \sin(\omega_j t) \right\}^2 + \left\{ \sum_{t=1}^{T} (z_t - \bar{z}) \cos(\omega_j t) \right\}^2 \right]$$

is the periodogram ordinate evaluated at Fourier frequency $\omega_j = 2\pi j/T$, $j = 1, 2, ..., T-1$. The Whittle's estimator $\hat{\beta}$ maximizes (2.3) over $\mathbb{B}$. Hence, in terms of the ψ-functions

$$\psi_j\{I(\omega_j), \beta\} = \left\{\frac{I(\omega_j)}{g_j(\beta)} - 1\right\}\frac{\partial \ln\{g_j(\beta)\}}{\partial\beta},$$

the estimator has the interpretation of an M-estimator from asymptotically independent periodogram ordinates, as shown by Monti (1997). Thus, a dependent data problem reduces to an independent data problem. However, Nordman and Lahiri (2006) pointed out that Monti's (1997) frequency domain empirical likelihood (FDEL) corresponds estimating functions that involves a non-zero spectral mean due to the nuisance parameter, σ^2. Consequently, the null asymptotic distribution of Monti's EL does not follow the standard chi-square distribution unless the error terms are normally distributed. Thus, Nordman and Lahiri (2006) profiled σ^2 out and proved that Eq. 2.1 has an asymptotic χ^2 distribution when $\kappa_{4,\epsilon} = 0$ (This holds when the error terms follow a Gaussian distribution) in addition to the assumption that the density class being correctly specified (Theorem 1 (ii)–(iii) in Nordman and Lahiri (2006)). Therefore, we profile out the nuisance parameter σ^2, first and then construct the adjusted EL based on the estimating equations derived by Nordman and Lahiri (2006).

Let $\beta = (\beta_{(1)}, \sigma^2)$, where $\beta_{(1)}$ is the parameter of interest. Monti (1997) showed that by maximizing (2.3) with respect to σ^2, the spectral log-likelihood function becomes

$$\ln\{\hat{L}(\beta_{(1)})\} = -n\ln\left\{n^{-1}\sum_{j=1}^{n}\frac{I(\omega_j)}{g_j^1(\beta_{(1)})}\right\} - \sum_{j=1}^{n}\ln\{g_j^1(\beta_{(1)})\} - n \tag{2.4}$$

where

$$g_j^1(\beta_{(1)}) = \frac{1}{2\pi}\frac{|\Theta(e^{-i\omega})|^2}{|\Phi(e^{-i\omega})|^2}.$$

Maximizing (2.4) over $\mathbb{B}$ gives the ψ-function,

$$\tilde{\psi}\{I(\omega_j), \beta_{(1)}\} = \frac{I(\omega_j)}{g_j^1(\beta_{(1)})}\left[\frac{\partial\ln\{g_j^1(\beta_{(1)})\}}{\partial\beta_{(1)}} - n^{-1}\sum_{j=1}^{n}\frac{\partial\ln\{g_j^1(\beta_{(1)})\}}{\partial\beta_{(1)}}\right.$$

Now,

$$\sum_{j=1}^{n}\frac{\partial\frac{1}{g_j^1(\beta_{(1)})}}{\partial\beta_{(1)}}g_j^1(\beta_{(1)}) = \sum_{j=1}^{n}\frac{\partial e^{-\ln g_j^1(\beta_{(1)})}}{\partial\beta_{(1)}}g_j^1(\beta_{(1)})$$

$$= -\sum_{j=1}^{n} e^{-\ln g_j^1(\beta_{(1)})} \frac{\partial \ln\{g_j^1(\beta_{(1)})\}}{\partial \beta_{(1)}} g_j^1(\beta_{(1)})$$
$$= \frac{\partial \ln\{g_j^1(\beta_{(1)})\}}{\partial \beta_{(1)}}.$$

Under certain conditions Ogata (2005) showed that $\sum_{j=1}^{n} \frac{\partial \frac{1}{g_j^1(\beta_{(1)})}}{\partial \beta_{(1)}} g_j^1(\beta_{(1)}) = 0$. Because it is known that $E[I(\omega_j)], \omega_j \in [-\pi, \pi]$ converges to $g_j^1(\beta_{(1)})$, we can see that $E[(\partial/\partial\beta_{(1)})I(\omega_j)/g_j^1(\beta_{(1)})] \to 0$. Thus, Ogata (2005) defined the log empirical likelihood ratio to be $-2\ln\hat{\lambda}(\beta_{(1)})$. In Nordman and Lahiri (2006), they consider the equivalent test statistic as Ogata (2005), except they consider $\omega \in [0, \pi]$. Therefore, in their test statistic, they have "-4" instead of "-2" (Nordman and Lahiri 2014). Here, $\omega \in [0, \pi]$. Hence, the profile empirical likelihood ratio is given by

$$\hat{W}^*(\beta_{(1)}) = -4\ln\hat{\lambda}(\beta_{(1)}) = 4\sum_{j=1}^{n} \ln\left[1 + \xi(\beta_{(1)})'\tilde{\psi}_j(I(\omega_j), \beta_{(1)})\right], \quad (2.5)$$

where the estimator of $\beta_{(1)}$ is the M-estimator corresponding to the ψ-function

$$\tilde{\psi}\{I(\omega_j), \beta_{(1)}\} = \frac{I(\omega_j)}{g_j^1(\beta_{(1)})} \frac{\partial \ln\{g_j^1(\beta_{(1)})\}}{\partial \beta_{(1)}}, \quad (2.6)$$

and $\xi(\beta_{(1)})$ satisfies,

$$\sum_{j=1}^{n} [1 + \xi(\beta_{(1)})\tilde{\psi}_j\{I(\omega_j), \beta_{(1)}]^{-1} \tilde{\psi}_j\{I(\omega_j), \beta_{(1)}\} = 0.$$

Nordman and Lahiri (2006) showed that $\hat{W}^*(\beta_{(1)})$ is still asymptotically distributed as χ_k^2 where $k = \dim \beta_{(1)}$.

3 Adjusted Empirical Likelihood for Time Series Models

The definition of $W^*(\beta_{(1)})$ in Eq. 2.5 depends on obtaining positive $p_j s$ such that

$$\sum_{j=1}^{n} \psi_j(I(\omega_j), \beta_{(1)})p_j = 0,$$

for each $\beta_{(1)}$. Under some moment conditions on $\psi_j(x_j, \beta)$ (Owen 2001), the convex hull $\{\psi_j(x_j, \beta), i = 1, 2, ..., n\}$ contains 0 as its interior point with probability 1 as $n \to \infty$. When the parameter β is not close to β_n, or when n is small, there is a good chance that the solution to constraint doesn't exist which raises some computational issues as mentioned by Chen et al. (2008). To overcome this difficulty, Chen et al. (2008) proposed an adjusted empirical likelihood (AEL) ratio function by adding ψ_{n+1}-th term to guarantee the zero to be an interior point of the convex hull, therefore, the required numerical maximization always has the solution. By doing so, they modified Owen's method and applied to independent observations with the establishment of the Wilks' theorem for AEL statistic same as Owen obtained for EL statistic. As follows, we adopt their idea to modify Monti's method for dependent observations. Denote $\psi_j = \psi_j(\beta) = \psi(x_j, \beta)$ and $\bar{\psi}_n = \bar{\psi}_n(\beta) = \frac{1}{n}\sum_{j=1}^{n} \psi_j$. For some positive constant a_n, define

$$\psi_{n+1} = \psi_{n+1}(\beta) = -\frac{a_n}{n}\sum_{j=1}^{n} \psi_j = -a_n\bar{\psi}_n.$$

Here, we choose $a_n = \max(1, log(n)/2)$ coupled with a trimmed version of $\bar{\psi}_n$ when appropriate suggested by Chen et al. (2008). Hence the empirical likelihood ratio for any value $\beta \in \mathbb{B}$ is adjusted to be,

$$\hat{\lambda}(\beta) = \sup \prod_{j=1}^{n+1} p_j \Big/ \prod_{j=1}^{n+1} \frac{1}{n+1} = \sup \prod_{j=1}^{n+1} (n+1)p_j,$$

where the maximization is subject to: (i) $\sum_{j=1}^{n+1} \psi(x_j, \beta)p_j = 0$, (ii) $\sum_{j=1}^{n+1} p_j = 1$, and (iii) $p_j \geq 0$. Similarly, by Lagrange multiplier method we obtain

$$p_j = [(n+1)\{1 + \xi(\beta)'\psi(x_j, \beta)\}]^{-1} \qquad j = 1, 2, ..., n+1,$$

where $\xi(\beta)$ is the Lagrangian multiplier satisfying,

$$\sum_{j=1}^{n+1} \frac{\psi(x_j, \beta)}{1 + \xi(\beta)'\psi(x_j, \beta)} = 0. \tag{3.1}$$

Thus the adjusted empirical likelihood ratio (AEL) statistic is defined by

$$W^*(\beta_{(1)}) = 4\sum_{j=1}^{n+1} \ln\left[1 + \xi(\beta_{(1)})'\tilde{\psi}_j(I(\omega_j), \beta_{(1)})\right], \tag{3.2}$$

where ψ-function defined as in Eq. 2.6 and $\xi(\beta_{(1)})$ satisfies,

$$\sum_{j=1}^{n+1} \left[1 + \xi(\beta_{(1)})'^{\tilde{\psi}}_j\{I(\omega_j), \beta_{(1)}\right]^{-1} \tilde{\psi}_j\{I(\omega_j), \beta\} = 0.$$

Using a similar argument to Nordman and Lahiri (2006), we can show that the the adjusted profile EL ratio statistic, $\hat{W}^*(\beta_{(1)})$, defined in Eq. 3.2 has an asymptotic χ^2 as given in the following theorem.

THEOREM 1. *Suppose the assumptions A1-A4* (Nordman and Lahiri 2006) *are satisfied. Then,* $W^*(\beta_{(1)})$ *in Eq. 3.2 follows an asymptotic* χ^2 *distribution with* k *degrees of freedom where* $k = \dim \beta_{(1)}$.

The proof is provided in Appendix.

4 Simulations: Comparison of Coverage Errors of Confidence Sets

In this section, we conduct simulations AR, MA, and ARMA models to prove the efficiency of the proposed method. The coverage probabilities are compared along with other existing approaches.

4.1. MA(1) MODEL. A Monte Carlo experiment is conducted to explore the accuracy of the adjusted empirical likelihood confidence regions for MA model. To make a fair comparison with the empirical likelihood confidence regions based on Nordman and Lahiri's method, we choose MA(1), $Z_t = a_t - \theta a_{t-1}$ as Monti used. The simulations are carried out under different distributions for the white noise process a_t: $N(0,1)$ and χ^2_5 distribution centered around zero. The simulations are conducted for different values of $\theta = (0.25, 0.5, 0.7, 0.75, 0.8, 0.85, 0.9)$. In each case, 1000 series of size 20, 30, 40, 70 and 100, are drawn and the coverage probabilities are computed. We choose $a_n = \log(n)/2$ as in the definition of ψ_{n+1}. The adjusted empirical likelihood coverage probabilities are compared with the unadjusted empirical likelihood coverage probabilities. Further, the coverage probabilities of intervals based on theoretical Bartlett correction and estimated Bartlett correction (Diciccio et al. 1991) are used for comparison. Table 1 provides the results for the nominal level of 90%. The values of θ in (0,1) are considered as the behavior of the coverage probabilities is almost symmetric in θ. Compared to the other method, the coverage probabilities of the AEL are closer to the nominal value of 0.90 in most scenarios. Especially when the sample size is small, the AEL method gives more accurate results as compared to other methods. We also find that EL with theoretical or estimated Bartlett correction doesn't seem to give better results than AEL.

4.2. AR(1) MODEL. We simulate the coverage probabilities for the parameters of AR(1), $Z_t = \phi_1 Z_{t-1} + a_t$. We consider $N(0,1)$ and χ^2_5 distribution

Table 1: Coverage probabilities for the parameter of MA(1) models

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.7$	$\theta = 0.75$	$\theta = 0.8$	$\theta = 0.85$	$\theta = 0.9$
Model: $a_t \sim N(0,1)$								
n=20	EL	0.834	0.823	0.789	0.767	0.725	0.637	0.489
	EB	0.842	0.830	0.804	0.782	0.740	0.655	0.519
	TB	0.851	0.833	0.811	0.787	0.751	0.668	0.535
	AEL	0.869	0.869	0.860	0.836	0.828	0.831	0.881
n=30	EL	0.873	0.861	0.819	0.789	0.750	0.691	0.573
	EB	0.881	0.869	0.830	0.804	0.760	0.703	0.588
	TB	0.888	0.872	0.835	0.810	0.768	0.713	0.599
	AEL	0.894	0.887	0.855	0.830	0.796	0.766	0.707
n=40	EL	0.893	0.876	0.834	0.813	0.779	0.723	0.595
	EB	0.902	0.878	0.844	0.823	0.792	0.737	0.610
	TB	0.906	0.885	0.848	0.828	0.796	0.742	0.613
	AEL	0.911	0.892	0.862	0.845	0.815	0.762	0.666
n=70	EL	0.890	0.877	0.840	0.828	0.775	0.741	0.651
	EB	0.901	0.883	0.843	0.834	0.789	0.748	0.658
	TB	0.904	0.885	0.850	0.836	0.796	0.752	0.661
	AEL	0.907	0.891	0.855	0.842	0.798	0.758	0.675
n=100	EL	0.897	0.888	0.877	0.871	0.833	0.782	0.698
	EB	0.902	0.891	0.879	0.878	0.837	0.786	0.682
	TB	0.903	0.895	0.879	0.879	0.839	0.789	0.682
	AEL	0.904	0.900	0.885	0.881	0.844	0.798	0.691
Model: $a_t \sim \chi^2_5 - 5$								
n=20	EL	0.839	0.808	0.770	0.752	0.722	0.653	0.510
	EB	0.853	0.823	0.784	0.765	0.743	0.679	0.533
	TB	0.864	0.844	0.800	0.783	0.764	0.706	0.567
	AEL	0.874	0.855	0.825	0.821	0.823	0.821	0.871
n=30	EL	0.857	0.843	0.804	0.781	0.753	0.709	0.600
	EB	0.863	0.853	0.812	0.796	0.761	0.724	0.619
	TB	0.869	0.859	0.819	0.803	0.780	0.737	0.649
	AEL	0.878	0.872	0.837	0.816	0.800	0.781	0.737
n=40	EL	0.856	0.851	0.815	0.791	0.751	0.685	0.584
	EB	0.864	0.860	0.826	0.799	0.765	0.693	0.594
	TB	0.868	0.866	0.830	0.804	0.769	0.705	0.617
	AEL	0.880	0.878	0.835	0.814	0.782	0.723	0.660

Table 1: (continued)

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.7$	$\theta = 0.75$	$\theta = 0.8$	$\theta = 0.85$	$\theta = 0.9$
n=70	EL	0.893	0.881	0.847	0.827	0.787	0.744	0.637
	EB	0.897	0.887	0.850	0.833	0.791	0.749	0.644
	TB	0.898	0.889	0.856	0.836	0.796	0.754	0.651
	AEL	0.902	0.895	0.854	0.841	0.798	0.758	0.663
n=100	EL	0.896	0.871	0.883	0.800	0.765	0.735	0.658
	EB	0.899	0.882	0.840	0.805	0.772	0.741	0.664
	TB	0.901	0.882	0.842	0.807	0.776	0.744	0.668
	AEL	0.904	0.884	0.846	0.812	0.780	0.744	0.672

EL empirical likelihood, *TB=EL* with theoretical Bartlett correction,
EB=EL with estimated Bartlett correction, *AEL* adjusted EL

centered around zero for the distribution of the white noise process a_t. We conduct 1000 simulations under the sample sizes of 20, 30, 40, 70, 100 and ϕ= 0.25, 0.5, 0.7, 0.9. The coverage probabilities are recorded in Table 2. The coverage probabilities are substantially improved with the AEL method as compared to the EL method and EL with (estimated and theoretical) Bartlett correction methods. The simulation results show that the AEL method gives comparable results than the other methods, especially with small sample sizes.

4.3. AR(2) MODEL. Simulation for the coverage probabilities for the parameters of AR(2), $Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$. We consider $N(0,1)$ and χ^2_5 distribution centered around zero for the distribution of the white noise process a_t. We conduct 1000 simulations under the sample sizes of 20, 30, 40, 70, 100 and $(\phi_1, \phi_2) = (0.1, 0.2), (0.1, 0.7), (0.1, -0.1), (-0.2, -0.1), (-0.7, -0.1)$. The coverage probabilities are recorded in Table 3. The coverage probabilities are substantially improved with the AEL method as compared to the EL method and EL with (estimated and theoretical) Bartlett correction methods. The simulation results show that the AEL method gives competitive results than the other methods, especially with small sample sizes.

4.4. ARMA(1,1) MODEL. We conduct simulations with various values of ϕ and θ for ARMA(1,1) model to illustrate improvement of the AEL over the EL on constructing confidence regions. ARMA model with $\phi = 0.7$ and $\theta = 0.5$ is used here. $\hat{W}(\beta)$ is calculated at different points over the parameter space $(\phi, \theta) \in \{(0,1) \times (0,1)\}$ and contour plots are produced for each of the three sample sizes using threshold value $\chi^2_{2,0.9}$ to construct 90% confidence region. The empirical likelihood confidence regions are also constructed. Figure 1 shows that the 90% adjusted empirical likelihood confidence regions

Table 2: Coverage probabilities for the parameter of AR(1) models

n	Method	$\phi = 0.25$	$\phi = 0.5$	$\phi = 0.7$	$\phi = 0.8$	$\phi = 0.9$
Model: $a_t \sim N(0, 1)$						
n=20	EL	0.860	0.799	0.797	0.705	0.597
	EB	0.869	0.813	0.781	0.724	0.605
	TB	0.875	0.822	0.798	0.747	0.661
	AEL	0.892	0.831	0.808	0.741	0.613
n=30	EL	0.870	0.833	0.800	0.757	0.672
	EB	0.878	0.844	0.807	0.769	0.688
	TB	0.890	0.853	0.816	0.790	0.717
	AEL	0.898	0.860	0.818	0.778	0.687
n=40	EL	0.856	0.859	0.819	0.768	0.697
	EB	0.867	0.867	0.831	0.782	0.719
	TB	0.871	0.871	0.843	0.804	0.755
	AEL	0.878	0.874	0.838	0.793	0.716
n=70	EL	0.860	0.850	0.849	0.807	0.752
	EB	0.867	0.862	0.859	0.819	0.763
	TB	0.870	0.868	0.868	0.825	0.789
	AEL	0.876	0.870	0.864	0.821	0.763
n=100	EL	0.800	0.872	0.873	0.844	0.784
	EB	0.885	0.876	0.882	0.848	0.797
	TB	0.887	0.881	0.887	0.858	0.814
	AEL	0.891	0.881	0.884	0.851	0.793
Model: $a_t \sim \chi^2_5 - 5$						
n=20	EL	0.834	0.788	0.743	0.699	0.622
	EB	0.845	0.802	0.757	0.716	0.630
	TB	0.861	0.821	0.783	0.742	0.668
	AEL	0.877	0.836	0.781	0.732	0.644
n=30	EL	0.840	0.833	0.791	0.752	0.655
	EB	0.853	0.843	0.802	0.773	0.673
	TB	0.862	0.853	0.821	0.798	0.703
	AEL	0.871	0.856	0.812	0.788	0.678
n=40	EL	0.868	0.837	0.821	0.777	0.708
	EB	0.871	0.848	0.826	0.787	0.729
	TB	0.879	0.857	0.835	0.813	0.764
	AEL	0.883	0.857	0.831	0.798	0.732

Table 2: (continued)

n	Method	$\phi = 0.25$	$\phi = 0.5$	$\phi = 0.7$	$\phi = 0.8$	$\phi = 0.9$
n=70	EL	0.870	0.879	0.826	0.819	0.746
	EB	0.880	0.886	0.838	0.826	0.765
	TB	0.881	0.890	0.849	0.838	0.786
	AEL	0.889	0.890	0.844	0.827	0.766
n=100	EL	0.896	0.873	0.862	0.831	0.786
	EB	0.901	0.878	0.872	0.836	0.798
	TB	0.905	0.881	0.881	0.844	0.813
	AEL	0.906	0.883	0.875	0.834	0.796

EL empirical likelihood, *TB=EL* with theoretical Bartlett correction, *EB=EL* with estimated Bartlett correction, *AEL* adjusted EL

for sample sizes 40, 70 and 100 under standard normal error distribution. In each case the mean is subtracted from the white noise processes in order to have mean zero for the error terms. Similarly, with the same nominal level, the confidence contours contain the ones based on the unadjusted EL, especially for the small sample sizes.

Table 4 provides the results for the coverage probabilities of the parameters of ARMA(1,1) model with a nominal level of 90%. We conduct 1000 simulations under the sample sizes of 20, 30, 40, 70, 100. It is clear that the AEL method provides competitive coverage probabilities comparing to the other method, especially with the small sample sizes.

As suggested by one of the reviewers, a comparison between the coverage probabilities under Monti's method and Nordman and Lahiri's method is also been done. Table 5 illustrates the coverage probabilities for the parameter of MA(1) model under the two methods. Nordman and Lahiri (2006) and Ogata (2005) showed that Monti's (1997) method can only be used with time series models having Gaussian error distribution. It can be seen from Table 5 that the coverage probabilities for EL and AEL methods under Nordman and Lahiri's method is shown to be better than that of under Monti's method under both Gaussian and non-Gaussian error distributions.

5 Confidence Regions for the Parameters of ARMA Models

Let Z_t be an ARMA(p, q) process,

$$\Phi(B)Z_t = \Theta(B)a_t,$$

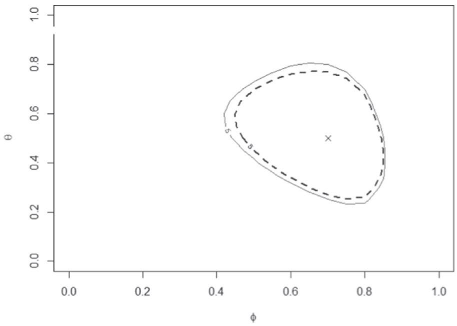
Table 3: Coverage probabilities for the parameter of AR(2) models

n	Method	$\phi = 0.1$ $\phi = 0.2$	$\phi = 0.1$ $\phi = 0.7$	$\phi = 0.1$ $\phi = -0.1$	$\phi = -0.2$ $\phi = -0.1$	$\phi = -0.7$ $\phi = -0.1$
Model: $a_t \sim N(0,1)$						
n=20	EL	0.749	0.512	0.770	0.780	0.708
	EB	0.770	0.536	0.796	0.800	0.730
	TB	0.792	0.661	0.809	0.813	0.758
	AEL	0.825	0.563	0.859	0.865	0.772
n=30	EL	0.798	0.590	0.812	0.815	0.784
	EB	0.818	0.618	0.830	0.834	0.803
	TB	0.827	0.679	0.840	0.841	0.821
	AEL	0.842	0.634	0.859	0.871	0.834
n=40	EL	0.813	0.640	0.842	0.831	0.805
	EB	0.828	0.655	0.855	0.847	0.812
	TB	0.844	0.723	0.860	0.856	0.825
	AEL	0.849	0.662	0.873	0.871	0.827
n=70	EL	0.850	0.712	0.865	0.856	0.843
	EB	0.858	0.727	0.870	0.866	0.850
	TB	0.861	0.758	0.876	0.869	0.856
	AEL	0.867	0.725	0.882	0.877	0.856
n=100	EL	0.848	0.745	0.862	0.865	0.840
	EB	0.853	0.760	0.871	0.873	0.843
	TB	0.857	0.791	0.873	0.875	0.855
	AEL	0.856	0.765	0.884	0.882	0.848
Model: $a_t \sim \chi_5^2 - 5$						
n=20	EL	0.768	0.520	0.793	0.782	0.757
	EB	0.792	0.539	0.807	0.802	0.785
	TB	0.814	0.641	0.821	0.819	0.808
	AEL	0.843	0.570	0.857	0.869	0.832
n=30	EL	0.801	0.589	0.823	0.834	0.795
	EB	0.812	0.613	0.839	0.843	0.813
	TB	0.830	0.681	0.846	0.854	0.836
	AEL	0.841	0.624	0.868	0.880	0.846
n=40	EL	0.834	0.640	0.839	0.845	0.805
	EB	0.841	0.660	0.847	0.855	0.818
	TB	0.851	0.723	0.853	0.860	0.842
	AEL	0.860	0.671	0.868	0.875	0.835

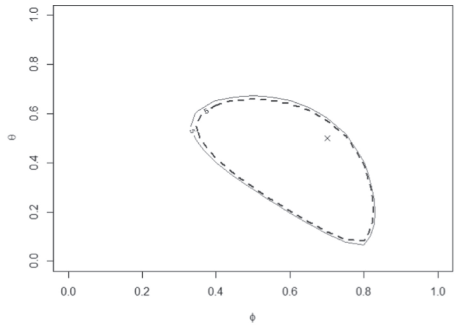
Table 3: (continued)

n	Method	$\phi = 0.1$ $\phi = 0.2$	$\phi = 0.1$ $\phi = 0.7$	$\phi = 0.1$ $\phi = -0.1$	$\phi = -0.2$ $\phi = -0.1$	$\phi = -0.7$ $\phi = -0.1$
n=70	EL	0.843	0.712	0.851	0.855	0.848
	EB	0.856	0.728	0.865	0.867	0.856
	TB	0.863	0.759	0.870	0.878	0.864
	AEL	0.860	0.729	0.882	0.885	0.864
n=100	EL	0.863	0.779	0.869	0.873	0.850
	EB	0.870	0.797	0.877	0.884	0.862
	TB	0.873	0.822	0.878	0.887	0.869
	AEL	0.877	0.796	0.882	0.886	0.864

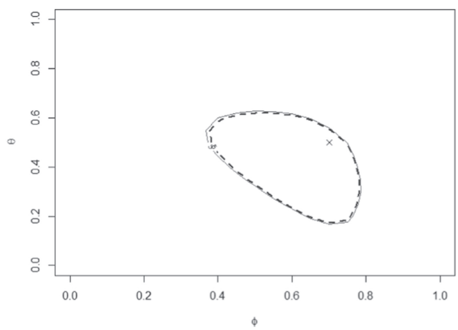
EL= empirical likelihood, *TB*=*EL* with theoretical Bartlett correction,
EB=*EL* with estimated Bartlett correction, *AEL* adjusted EL



(a) n=40



(b) n=70



(c) n=100

Figure 1: 90% adjusted empirical likelihood confidence region for ARMA(1,1) model with $a_t \sim N(0,1)$, '+' is the true parameter value

Table 4: Coverage probabilities for the parameters of ARMA(1,1) models

n	Method	$\phi = 0.1$ $\theta = 0.4$	$\phi = 0.7$ $\theta = 0.1$	$\phi = -0.7$ $\theta = 0.5$	$\phi = -0.7$ $\theta = -0.2$	$\phi = -0.2$ $\theta = -0.7$	$\phi = 0.5$ $\theta = -0.7$
Model: $a_t \sim N(0,1)$							
n=20	EL	0.779	0.725	0.691	0.729	0.729	0.729
	EB	0.801	0.747	0.709	0.748	0.749	0.745
	TB	0.817	0.787	0.723	0.776	0.772	0.759
	AEL	0.853	0.798	0.763	0.788	0.830	0.808
n=30	EL	0.833	0.756	0.733	0.779	0.738	0.759
	EB	0.845	0.766	0.740	0.798	0.760	0.770
	TB	0.853	0.801	0.762	0.815	0.774	0.778
	AEL	0.874	0.800	0.767	0.813	0.813	0.800
n=40	EL	0.818	0.804	0.755	0.776	0.765	0.782
	EB	0.833	0.819	0.767	0.793	0.782	0.791
	TB	0.848	0.834	0.776	0.811	0.796	0.796
	AEL	0.861	0.829	0.788	0.805	0.810	0.809
n=70	EL	0.836	0.829	0.810	0.827	0.822	0.826
	EB	0.846	0.840	0.812	0.833	0.360	0.834
	TB	0.849	0.851	0.818	0.842	0.843	0.834
	AEL	0.858	0.842	0.823	0.840	0.853	0.851
n=100	EL	0.865	0.854	0.858	0.860	0.837	0.826
	EB	0.873	0.860	0.869	0.877	0.841	0.833
	TB	0.878	0.865	0.873	0.882	0.846	0.835
	AEL	0.880	0.862	0.873	0.877	0.850	0.846
Model: $a_t \sim \chi^2_5 - 5$							
n=20	EL	0.771	0.723	0.701	0.725	0.717	0.707
	EB	0.792	0.744	0.716	0.743	0.740	0.721
	TB	0.811	0.784	0.741	0.785	0.765	0.735
	AEL	0.858	0.778	0.772	0.788	0.817	0.790
n=30	EL	0.815	0.723	0.728	0.751	0.752	0.752
	EB	0.826	0.746	0.746	0.767	0.770	0.757
	TB	0.840	0.790	0.758	0.806	0.781	0.766
	AEL	0.867	0.780	0.761	0.798	0.813	0.790
n=40	EL	0.816	0.766	0.775	0.783	0.722	0.772
	EB	0.827	0.781	0.792	0.798	0.787	0.783
	TB	0.841	0.804	0.806	0.822	0.797	0.788
	AEL	0.851	0.796	0.805	0.813	0.814	0.814

Table 4: (continued)

n	Method	$\phi=0.1$ $\theta=0.4$	$\phi=0.7$ $\theta=0.1$	$\phi=-0.7$ $\theta=0.5$	$\phi=-0.7$ $\theta=-0.2$	$\phi=-0.2$ $\theta=-0.7$	$\phi=0.5$ $\theta=-0.7$
n=70	EL	0.844	0.818	0.830	0.838	0.811	0.813
	EB	0.854	0.834	0.841	0.852	0.826	0.823
	TB	0.859	0.851	0.847	0.860	0.830	0.823
	AEL	0.861	0.838	0.848	0.857	0.839	0.837
n=100	EL	0.869	0.857	0.822	0.843	0.844	0.841
	EB	0.877	0.868	0.837	0.855	0.849	0.846
	TB	0.881	0.876	0.843	0.864	0.854	0.847
	AEL	0.883	0.868	0.840	0.856	0.858	0.855

EL empirical likelihood, *TB=EL* with theoretical Bartlett correction, *EB=EL* with estimated Bartlett correction, *AEL* adjusted EL

where B is the backward shift operator ($BZ_t = Z_{t-1}$), with $\Theta(B) = 1+\theta_1 B+\theta_2 B^2+...+\theta_q B^q$, $\Phi(B) = 1-\phi_1 B-\phi_2 B^2-...-\phi_p B^p$. and white noise process $a_t \sim N(0,\sigma^2)$. The absolute values of the roots of these two polynomials are all greater than 1 to guarantee the stationarity and invertibility of the model. We also assume that $\Theta(B)$ and $\Phi(B)$ have no common factors to avoid the redundancy of the parameters. The spectral density of ARMA(p,q) model is given by

$$g(\omega,\beta) = \frac{\sigma^2}{2\pi}\frac{|\Theta(e^{-i\omega})|^2}{|\Phi(e^{-i\omega})|^2}, \qquad \omega \in [0,\pi],$$

where $\beta = (\phi_1,...,\phi_p,\theta_1,...,\theta_q,\sigma^2)'$. The variance, σ^2 of the noise is usually considered as a nuisance parameter as it has no effect on the main characteristics of the process other than modifying the scale of the process. Hence, we consider the profile spectral likelihood function. Let $\beta = (\beta_{(1)},\sigma^2)$, where $\beta_{(1)}$ is the parameter of interest. Monti (1997) showed that by maximizing (3.2) with respect to σ^2, the spectral log-likelihood function becomes

$$\ln\{\hat{L}(\beta_{(1)})\} = -n\ln\left\{n^{-1}\sum_{j=1}^{n}\frac{I(\omega_j)}{g_j^1(\beta_{(1)})}\right\} - \sum_{j=1}^{n}\ln\{g_j^1(\beta_{(1)})\} - n,$$

where

$$g_j^1(\beta_{(1)}) = \frac{1}{2\pi}\frac{|\Theta(e^{-i\omega})|^2}{|\Phi(e^{-i\omega})|^2}.$$

Consequently, the profile empirical likelihood ratio is given by

$$\hat{W}^*(\beta_{(1)}) = 4\sum_{j=1}^{n+1}\ln\left[1+\xi(\beta_{(1)})'\tilde{\psi}_j(I(\omega_j),\beta_{(1)})\right], \qquad (5.1)$$

Table 5: Comparison of Coverage probabilities for the parameter of MA(1) models

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.7$	$\theta = 0.75$	$\theta = 0.8$	$\theta = 0.85$	$\theta = 0.9$
Model:	$a_t \sim N(0,1)$							
n=20	EL	0.834	0.823	0.789	0.767	0.725	0.637	0.489
	AEL	0.869	0.869	0.860	0.836	0.828	0.831	0.881
	EL^M	0.794	0.732	0.639	0.576	0.522	0.389	0.100
	AEL^M	0.838	0.764	0.678	0.613	0.557	0.415	0.110
n=40	EL	0.893	0.876	0.834	0.813	0.779	0.723	0.595
	AEL	0.911	0.892	0.862	0.845	0.815	0.762	0.666
	EL^M	0.868	0.839	0.779	0.767	0.746	0.697	0.638
	AEL^M	0.889	0.852	0.814	0.784	0.767	0.732	0.648
n=70	EL	0.890	0.877	0.840	0.828	0.775	0.741	0.651
	AEL	0.907	0.891	0.855	0.842	0.798	0.758	0.675
	EL^M	0.880	0.850	0.831	0.811	0.777	0.739	0.694
	AEL^M	0.894	0.863	0.863	0.843	0.790	0.754	0.715
Model:	$a_t \sim \chi^2_5 - 5$							
n=20	EL	0.839	0.808	0.770	0.752	0.722	0.653	0.510
	AEL	0.874	0.855	0.825	0.821	0.823	0.821	0.871
	EL^M	0.805	0.757	0.683	0.625	0.561	0.415	0.113
	AEL^M	0.849	0.801	0.718	0.670	0.589	0.450	0.125
n=40	EL	0.856	0.851	0.815	0.791	0.751	0.685	0.584
	AEL	0.880	0.878	0.835	0.814	0.782	0.723	0.660
	EL^M	0.853	0.816	0.768	0.748	0.719	0.676	0.603
	AEL^M	0.875	0.829	0.787	0.766	0.745	0.708	0.628
n=70	EL	0.893	0.881	0.847	0.827	0.787	0.744	0.637
	AEL	0.902	0.895	0.854	0.841	0.798	0.758	0.663
	EL^M	0.864	0.851	0.826	0.801	0.766	0.719	0.656
	AEL^M	0.879	0.864	0.844	0.844	0.779	0.736	0.683

EL empirical likelihood under our method, *AEL* adjusted EL under our method, EL^M empirical likelihood under Monti's method, AEL^M adjusted EL under Monti's method

where the estimator of $\beta_{(1)}$ is the M-estimator corresponding to the ψ-function

$$\tilde{\psi}\{I(\omega_j), \beta_{(1)}\} = \frac{I(\omega_j)}{g_j^1(\beta_{(1)})} \frac{\partial \ln\{g_j^1(\beta_{(1)})\}}{\partial \beta_{(1)}},$$

and $\xi(\beta_{(1)})$ satisfies,

$$\sum_{j=1}^{n+1} [1 + \xi(\beta_{(1)})_j^{\prime\tilde{\psi}} \{I(\omega_j), \beta_{(1)}]^{-1} \tilde{\psi}_j \{I(\omega_j), \beta\} = 0$$

and $\tilde{\psi}_{n+1} = -\frac{a_n}{n} \sum_{j=1}^{n} \tilde{\psi}_j$ and $a_n = \max(1, \log(n)/2)$. Using arguments similar to Nordman and Lahiri (2006), we show that $\hat{W}^*(\beta_{(1)})$ is still asymptotically distributed as χ^2_{p+q}. The proof is provided in the Appendix.

As follows, we apply the proposed AEL method to three real data to construct confidence contours.

5.1. Clinical Process Concentration. We consider the example as Box et al. (2008). The data set describes a series of 197 chemical process concentration readings. They propose an ARMA (1,1) model,

$$Z_t = \phi Z_{t-1} + a_t - \theta a_{t-1}.$$

Figure 2 shows the 90% adjusted empirical likelihood confidence region (solid line) for a parameter $\beta_{(1)} = (\phi, \theta)'$. It is compared with an empirical likelihood confidence region (dashed line). We observe that the confidence contour based on the proposed AEL contains the one obtained based on the unadjusted EL while retaining the data-driven shape.

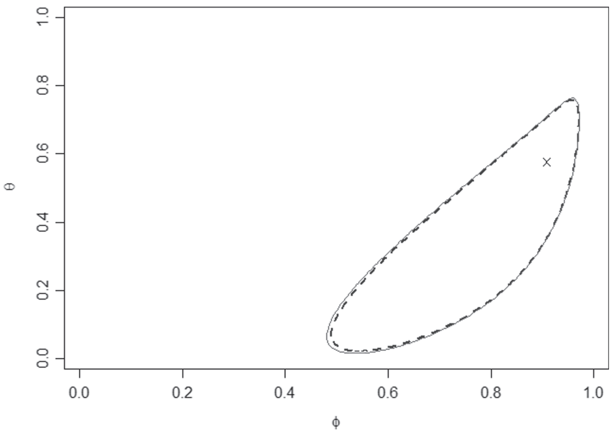


Figure 2: 90% adjusted empirical likelihood (*solid line*) and empirical likelihood (*dashed line*) confidence regions for the parameters of an ARMA(1,1) model fitted to the series of chemical process concentration readings (Box et al. 2008)

5.2. Abundance of Canadian Hare from 1905 to 1935. The connection between the cyclic changes in abundance of the canadian snowshoe hare (varying hare) and the canadian lynx was studied by Stenseth et al. (1997). These are generally assumed to be linked to each other because of the predator and the prey relations. The snow hare data derive from the main drainage of Hudson Bay, Canada. Cryer and Chan (2008) analyzed this data through a time series model. We take the same data for our proposed method. The Box-Cox power transformation suggested the square root transformation is appropriate for the original data. After the transformation, the ACF

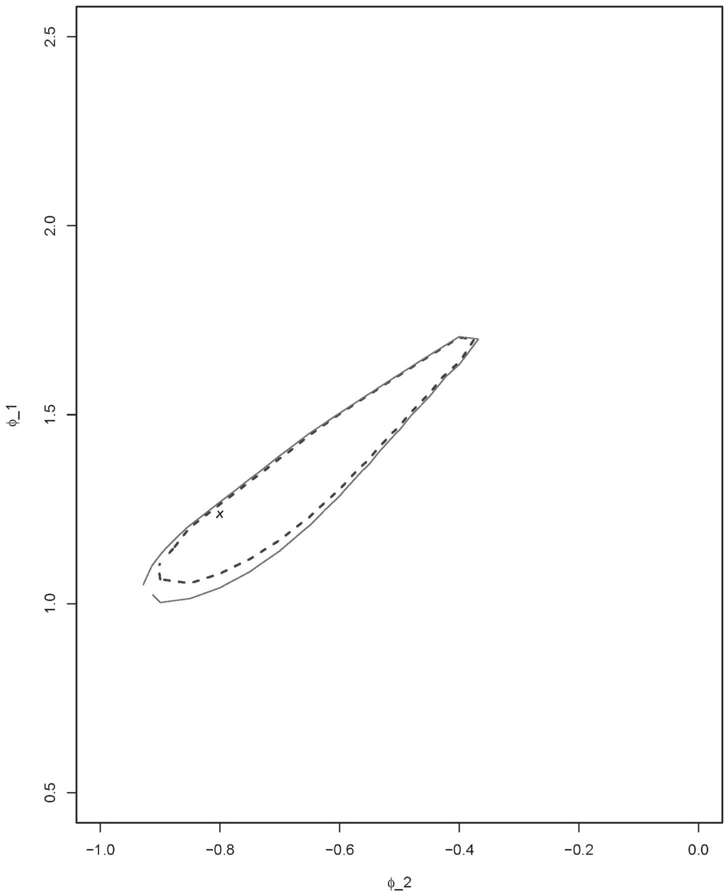


Figure 3: 90% adjusted empirical likelihood (*solid line*) and empirical likelihood (*dashed line*) confidence regions for the parameters of an AR(2) model fitted to the series of Hare Abundance data (Cryer and Chan 2008)

and PACF recommend that the transformed data can be fitted by AR(2) model,

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} - a_t.$$

The data is considered to be small sample size since only 31 observations are obtained. Figure 3 shows the 90% adjusted empirical likelihood confidence region (solid line) for a parameter $\beta_{(1)} = (\phi_1, \phi_2)'$. It is compared with the empirical likelihood confidence region (dashed line). We observe that the confidence contour based on the proposed AEL contains the one obtained based on the unadjusted EL while retaining the data-driven shape, which is also confirmed from the simulations in Section 4.

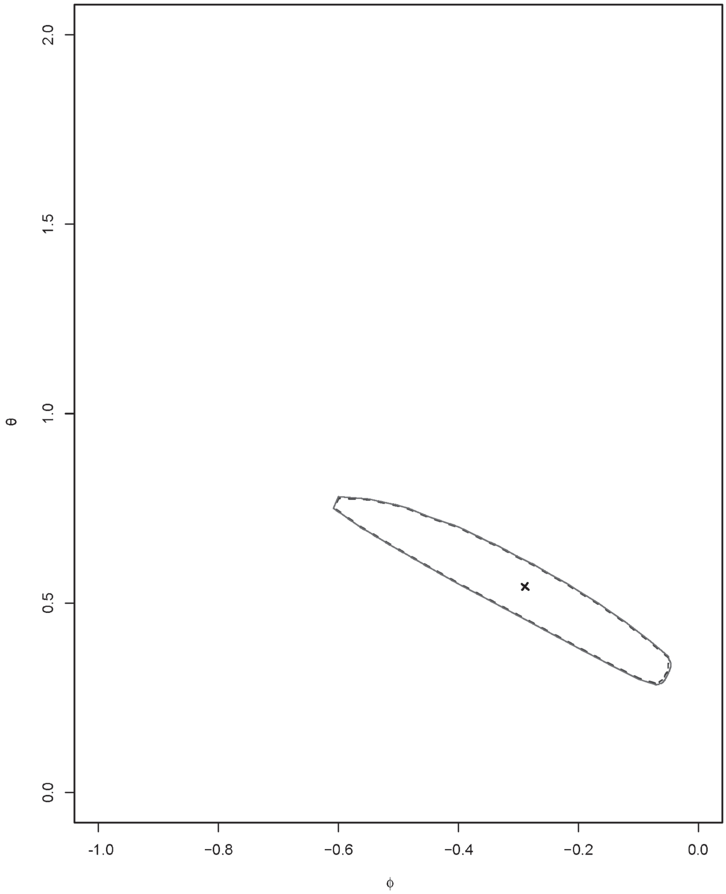


Figure 4: 90% adjusted empirical likelihood (*solid line*) and empirical likelihood (*dashed line*) confidence regions for the parameters of an ARMA(1,1) model fitted to the series of Hare Abundance (Cryer and Chan 2008)

5.3. Monthly Price of Crude Oil. The data consists of 241 monthly prices of a barrel of crude oil from January, 1986 through January, 2006. We obtain the data from Cryer and Chan (2008). Log transformation followed by differencing of the data has been suggested and ARMA(1,1) model is recommended. Figure 4 shows the 90% adjusted empirical likelihood confidence region (solid line) for a parameter $\beta_{(1)} = (\phi, \theta)'$. It is compared with the empirical likelihood confidence region (dashed line). Similar to the first two examples, the confidence contour based on the proposed AEL contains the one obtained based on the unadjusted EL while retaining the data-driven shape. We notice that both contours are almost overlapped due to the large sample size, which can be observed from the simulation sin Section 4.

6 Discussion

In this paper, we propose an adjusted empirical likelihood (AEL) to extend Nordman and Lahiri's (2006) method for dependent observations based on the adjusted empirical likelihood (AEL) by Chen et al. (2008), which was originally developed fro independent observations. We establish the asymptotic null distribution of the AEL statistic for stationary time series as the standard chi-square distribution which is same as the one obtained by Nordman and Lahiri (2006). Simulations for AR(1), MA(1), MAR(2), and ARMA(1,1) models with different sample sizes have been conducted to illustrate the performance of the proposed AEL method. Coverage probabilities of the AEL method have also been compared to the unadjusted EL method and the Bartlett correction ones. The unadjusted EL method used for comparisons is the one proposed by Nordman and Lahiri (2006) for any Gaussian or non-Gaussian error distribution. Comparisons indicate that the proposed AEL method compares favorably with the other methods, especially for the small sample sizes. Such a method is applied to three real data to construct the confidence regions for the parameters of ARMA models. We observe that the confidence regions obtained by the AEL contains the one obtained by the unadjusted EL while retaining the data-driven shape.

Emerson and Owen (2009) proposed adding two points and changing the location of the extra points as an extension to AEL approach for the hypothesis testing of multivariate population mean. Such a method addresses the under-coverage issue of original EL method by Owen (1988) and the limitation of Chen et al. (2008). They showed that it results in better calibration. As pointed out by Liu and Chen (2010), the optimal level of adjustment in their method still remains unknown. Liu and Chen (2010) proposed an alternative approach also by additing two artifical points to specify the optimal level of adjustment to achieve the high-order precision. Meanwhile they showed that the modified AEL has the high-order precision as the Bartlett EL (BEL), as well as constructing less biased estimator of Bartlett correction factor. In our ongoing work, we plan to study the performance of both modified AEL methods on different stationary and non-stationary time series models as well as with the AEL used in this paper. Furthermore, we also plan to construct a non-parametric procedure based on the EL and the AEL for the change point problem in time series.

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The authors declare that they have no conflict of interest.

References

BOX, G. E. P., JENKINS, G. M. and REINSEL, G. C. (2008). *Time Series Analysis: Forecasting & Control.* John Wiley & Sons, Inc., New Jersey.

CHAN, N. H. and LING, S. (2006). Empirical Likelihood for GARCH models. *Econ. Theory* **22**, 402–428.

CHAN, N. H. and LIU, L. (2010). Bartlett correctability of empirical likelihood in time series. *J. Jpn. Stat. Soc.* **40**, 1–5.

CHEN, J., VARIYATH, A. M. and ABRAHAM, B (2008). Adjusted empirical likelihood and its properties. *J. Comput. Graph. Stat.* **17**, 426–443.

CRYER, J. D. and CHAN, K. (2008). *Time Series Analysis With Applications in R.* Springer Science & Business Media, New York.

DICICCIO, T. J., HALL, P. and ROMANO, J. P (1991). Empirical Likelihood is Bartlett-correctable. *Ann. Stat.* **19**, 1053–61.

DZHAPARIDZE, K. (1986). *Parameter Estimation and Hypothesis Testing in Spectral Analysis of Stationary Time Series.* Springer, New York.

EMERSON, S. C. and OWEN, A. B (2009). Calibration of the empirical likelihood method for a vector mean. *Electron. J. Stat.* **3**, 1161–1192.

KITAMURA, Y. (1997). Empirical likelihood methods with weakly dependent processes. *Ann. Stat.* **25**, 2084–2102.

LIU, Y. and CHEN, J (2010). Adjusted empirical likelihood with high-order precision. *Ann. Stat.* **38**, 1341–1362.

MONTI, A. C. (1997). Empirical likelihood confidence regions in time series models. *Biometrika* **84**, 395–405.

MYKLAND, P. A. (1995). Dual likelihood. *Ann. Stat.* **23**, 396–421.

NORDMAN, D. J. and LAHIRI, S. N (2006). A frequency domain empirical likelihood for short- and long-range dependence. *Ann. Statist.* **34**, 3019–50.

NORDMAN, D. J. and LAHIRI, S. N (2014). A review of empirical likelihood methods for time series. *J. Stati. Plan. Inference* **155**, 1–18.

OGATA, H. (2005). Empirical likelihood for non Gaussian stationary processes. *Scientiae Mathematicae Japonicae Online* **e-2005**, 465–473.

OWEN, A. B. (1988). Empirical likelihood ratio confidence intervals for a single functional. *Biometrika* **75**, 237–49.

OWEN, A. B. (1990). Empirical likelihood ratio confidence regions. *Ann. Stat.* **18**, 90–120.

OWEN, A. B. (1991). Empirical likelihood for linear models. *Ann. Stat.* **19**, 1725–47.

OWEN, A. B. (2001). *Empirical Likelihood.* Chapman & Hall/CRC, New York.

PIYADI GAMAGE, R. D., NING, W. and GUPTA, A. K. (2017). Adjusted empirical likelihood for long-memory time series models. *J. Stat. Theory Pract.* **11**, 1, 220–223.

STENSETH, N. C., FALCK, W. and BJØRNSTAD, O. N (1997). Population regulation in snowshoe hare and Canadian lynx: Asymmetric food web configurations between hare and lynx. *Proc. Nat. Acad. Sci. USA* **94**, 10, 5147–5152.

WHITTLE, P. (1953). Estimation and information in time series. *Arkiv Matematik* **2**, 423–34.

YAU, C. Y. (2012). Empirical likelihood in long-memory time series models. *J. Time Series Anal.* **33**, 269–75.

Appendix

In this section, we give the brief proof of Theorem 1. Detailed proof is available upon request. PROOF. First we prove that $\xi = O_p(n^{-\frac{1}{2}})$. The adjusted empirical likelihood ratio function is,

$$W^*(\beta_{(1)}) = -4 \sup \left\{ \sum_{j=1}^{n+1} log[(n+1)p_j] | p_j \geq 0; \sum_{j=1}^{n+1} p_j = 1; \sum_{j=1}^{n+1} \psi_j(I(\omega_j), \beta_{(1)}) = 0 \right\}$$

where $\psi_{n+1} = -\frac{a_n}{n} \sum_{j=1}^{n} \psi_j = -a_n \bar{\psi}_n$ and $a_n = \max(1, log(n)/2) = o_p(n)$. We will show that $W^*(\beta_{(1)}) \sim {\chi_k}^2$. First, we need to show that $\xi = O_p(n^{-\frac{1}{2}})$. Denote $\psi_j \equiv \psi_j(I(\omega_j), \beta)$. Assume $Var\{\psi(I(\omega), \beta_{(1)})\}$ is finite and has rank $q < m (= dim(\psi))$. Let the eigenvalues of $Var\{\psi(I(\omega), \beta_{(1)})\}$ be ${\sigma_1}^2 \leq {\sigma_2}^2 \leq ... \leq {\sigma_m}^2$. WLOG, assume ${\sigma_1}^2 = 1$. Let ξ be the solution of

$$\sum_{j=1}^{n+1} \frac{\psi_j}{1 + \xi' \psi_j} = 0. \tag{A.1}$$

Let $\psi^* = \max_{1 \leq j \leq n} \parallel \psi_j \parallel$. Since $|\psi_j| \geq 0, j = 1, ..., n$ are independent, by Lemma 3 of Owen (1990), we have $\psi^* = \max_{1 \leq j \leq n} \parallel \psi_j \parallel = o_p(n^{\frac{1}{2}})$ if $E(|\psi_j|^2) < \infty$. By CLT, we have $\bar{\psi}_n = \frac{1}{n} \sum_{j=1}^{n} \psi_j = O_p(n^{-\frac{1}{2}})$.

Let $\xi = \rho\theta$ where $\rho \geq 0$ and $\parallel \theta \parallel = 1$. As long as $a_n = o_p(n)$ and using a similar argument as in Chen's (2008) we get $\rho = \parallel \xi \parallel = O_p(n^{-\frac{1}{2}})$ which implies $\xi = O_p(n^{-\frac{1}{2}})$.

Now we need to prove that $W^*(\beta_{(1)}) \sim {\chi_k}^2$. Under suitable regularity conditions (Dzhaparidze 1986), $n^{-\frac{1}{2}}(\beta_{(1),n} - \beta_{(1)}) \sim N(0, V)$ asymptotically, where V is the covariance matrix of $\beta_{(1)}$ and $\beta_{(1),n}$ is the Whittle's estimator of $\beta_{(1)}$. Therefore, $\beta_{(1)} - \beta_{(1),n} = O_p(n^{-\frac{1}{2}})$ which implies $\beta_{(1)} = \beta_{(1),n} + n^{-\frac{1}{2}}u$ with $|u| < +\infty$. From Eq. A.1, we have

$$\begin{aligned} 0 &= \sum_{j=1}^{n+1} \frac{\psi_j}{1+\xi'\psi_j} \\ &= \frac{1}{n}\sum_{j=1}^{n+1} \psi_j[1-\xi'\psi_j] \\ &= \bar{\psi}_n + \frac{1}{n}\psi_{n+1} - \frac{1}{n}\sum_{j=1}^{n+1} \psi_j\xi'\psi_j{}' \ \left(\tfrac{1}{n}\psi_{n+1} = o_p(n^{-\frac{1}{2}})\right) \\ &= \bar{\psi}_n - \frac{1}{n}\sum_{j=1}^{n} \psi_j\xi'\psi_j{}' + o_p(n^{-\frac{1}{2}}). \qquad (A.3) \end{aligned}$$

Using Taylor expansion at $\beta_{(1)} = \beta_{(1),n}$, we have,

$$\begin{aligned} \bar{\psi}_n &= \frac{1}{n}\sum_{j=1}^{n}\psi_j = \frac{1}{n}\sum_{j=1}^{n} \left.\frac{\partial\psi_j(I(\omega_j),t)}{\partial t'}\right|_{\beta_{(1),n}} (\beta_{(1)} - \beta_{(1),n}) + O_p(n^{-1}) \\ &= \hat{A}(\beta_{(1),n})(\beta_{(1)} - \beta_{(1),n}) + o_p(n^{-\frac{1}{2}}), \qquad (A.4) \end{aligned}$$

where

$$\hat{A}(\beta_{(1),n}) = \frac{1}{n}\sum_{j=1}^{n} \left.\frac{\partial\psi_j(I(\omega_j),t)}{\partial t'}\right|_{\beta_{(1),n}}.$$

Since $\psi_j(I(\omega_j), \beta_{(1),n}) \to \psi_j(I(\omega_j), \beta_{(1)})$ in probability, we have,

$$\begin{aligned} \frac{2}{n}\sum_{j=1}^{n}\psi_j\psi_j{}' &= \frac{2}{n}\sum_{j=1}^{n}\psi_j(I(\omega_j),\beta_{(1),n})\psi_j(I(\omega_j),\beta_{(1),n})' + O_p(n^{-\frac{1}{2}}) \\ &= \hat{\Sigma}(\beta_{(1),n}) + O_p(n^{-\frac{1}{2}}), \qquad (A.5) \end{aligned}$$

where

$$\hat{\Sigma}(\beta_{(1),n}) = \frac{2}{n}\sum_{j=1}^{n}\psi_j(I(\omega_j),\beta_{(1),n})\psi_j(I(\omega_j),\beta_{(1),n})'.$$

With Eqs. A.3, A.4 and A.5 we have,

$$\xi = \hat{\Sigma}(\beta_{(1),n})^{-1}\hat{A}(\beta_{(1),n})(\beta_{(1)} - \beta_{(1),n}) + o_p(n^{-\frac{1}{2}}). \qquad (A.6)$$

By Taylor expansion, we obtain,

$$\begin{aligned}
W^*(\beta_{(1)}) &= 4\sum_{j=1}^{n+1} \log[1+\xi'\psi_j(I(\omega_j),\beta_{(1)})] \\
&= 2\big[2\sum_{j=1}^{n+1} \xi'\psi_j(I(\omega_j),\beta_{(1)}) - \sum_{j=1}^{n+1} (\xi'\psi_j(I(\omega_j),\beta_{(1)}))^2\big] + o_p(1).
\end{aligned}$$

and using Eqs. A.4, A.5 and A.6 gives,

$$W^*(\beta_{(1)}) = n(\beta_{(1)} - \beta_{(1),n})'\hat{V}^{-1}(\beta_{(1)} - \beta_{(1),n}) + o_p(1),$$

where

$$\hat{V} = \hat{A}(\beta_{(1),n})^{-1}\hat{\Sigma}(\beta_{(1),n})\{\hat{A}(\beta_{(1),n})'\}^{-1}.$$

Hence, by Lemma 6 of Nordman and Lahiri (2006) $W^*(\beta_{(1)})$ converges to a standard chi-square distribution with k degrees of freedom as $n \to \infty$.

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