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A Flexible Choice of Critical Constants for the Improved Hybrid Hochberg-Hommel Procedure

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Abstract

In Gou et al. (Biometrika **101**(4), 899–911 2014), we proposed an improved hybrid Hochberg-Hommel (HH_0) step-up multiple test procedure. The procedure used a fixed, specified set of critical constants. In this follow-up note, we show that an entire class of critical constants can be derived by relaxing one of the assumptions in that paper. This allows a choice of more powerful critical constants. These critical constants do not control the type I error under positive dependence among the test statistics as shown by simulations. Therefore, they should be used only under independence and negative dependence when not only they control the type I error but also yield higher power. If positive dependence is expected, as is common in practice, then the HH_0 procedure should be used since it is robust with respect to type I error control under dependence.

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1 Introduction

Consider the problemof testing $n \geq 2$ null hypotheses, $H_1, \ldots, H_n$, based on their p-values, denoted by $p_1, \ldots, p_n$. Let $p_{1:n} \leq \cdots \leq p_{n:n}$ denote the ordered p-values and let $H_{1:n}, \ldots, H_{n:n}$ denote the corresponding null hypotheses. We assume that $p_1, \ldots, p_n$ are independent. Any multiple test procedure for testing the n hypotheses must satisfy the following familywise error rate (FWER) strong control requirement (Hochberg and Tamhane, 1987):

$$\mathrm{FWER} = \Pr\left(\text{Reject at least one true } H_i\right) \leq \alpha \tag{1.1}$$

for any given $\alpha \in (0, 1)$ under all possible combinations of the true and false H_i's.

Gou et al. (2014) proposed the following step-up multiple test procedure, which they referred to as a hybrid Hommel-Hochberg (HH) procedure: Fix two sets of critical constants:

$$c_1 \geq \cdots \geq c_n, \quad d_1 \geq \cdots \geq d_n \text{ where } 1 \geq c_i \geq d_i \text{ for all } i \text{ and } c_n = d_n. \tag{1.2}$$

At Step 1, if $p_{n:n} \leq c_1\alpha$ then reject any hypothesis with a p-value $\leq d_1\alpha$; otherwise accept $H_{n:n}$ and proceed to the next step. In general, at Step i if $p_{n-i+1:n} \leq c_i\alpha$ then reject any hypothesis with a p-value $\leq d_i\alpha$; otherwise accept $H_{n-i+1:n}$ and proceed to the next step. At the last step, stop testing and reject $H_{1:n}$ if $p_{1:n} \leq c_n\alpha = d_n\alpha$; otherwise accept $H_{1:n}$.

Gou et al. (2014) studied a special case of this procedure, denoted as the HH_0 procedure, which uses

$$c_i = \frac{i+1}{2i}, d_i = \frac{1}{i}\ (1 \leq i \leq n-1) \quad \text{and} \quad c_n = d_n = \frac{1}{n}. \tag{1.3}$$

This choice was originally suggested by Rom (2013). In Theorem 3 of Gou et al. (2014) we showed that (1.3) is the only possible choice if the c_i's must remain finite when $\alpha \to 0$. However, it is not the c_i's but the $c_i\alpha$'s, which must remain finite and in fact less than 1 as $\alpha \to 0$. We can see from equation (8) in Gou et al. (2014) that the $c_i\alpha$'s can remain finite even when the $d_i \neq 1/i$ and hence the c_i's do not necessarily have to equal $(i+1)/2i \leq 1$. So we can relax the condition imposed on the c_i's in Gou et al. (2014). In this paper we propose a recursive algorithm for computing general (c_i, d_i) by modifying the constraint $c_i \leq 1$ in (1.2) to $c_i < 1/\alpha$ so that $c_i\alpha < 1$.

2 Recursive Relations for Critical Constants

In Theorem 1 of Gou et al. (2014) we showed that the HH procedure is a shortcut to the closed procedure (Marcus et al., 1976) that tests each intersection hypothesis $H(I) = \bigcap_{i\in I} H_i$, where I is a nonempty subset of size $m \leq n$ of the index set $\{1, \ldots, n\}$, by using the following rejection region given by equation (4) of Gou et al. (2014):

$$E_i(m) = \begin{cases} p_{(m)} \leq c_1\alpha & (i=1) \\ p_{(m)} > c_1\alpha, p_{(m-1)} > c_2\alpha, \ldots, p_{(m-i+2)} > c_{i-1}\alpha & \\ p_{(m-i+1)} \leq c_i\alpha, p_{(1)} \leq d_i\alpha & (i=2, \ldots, m). \end{cases} \tag{2.1}$$

Denote the probabilities of this rejection region by $A_i(m) = \Pr(E_i(m))$.

To satisfy (1.1) we must have

$$\sum_{i=1}^{m} A_i(m) \leq \alpha \quad \text{for all} \quad m = 1, \ldots, n. \tag{2.2}$$

In order to maximize power, we must test $H(I)$ at the full level α and so there must be an equality in the above equation. Such full α-level local tests of $H(I)$ used in a closed procedure are called α-exhaustive.

Next define

$$B_i(m) = \begin{cases} \Pr\left(p_{m:m} \le c_1\alpha\right) & i = 1, \\ \Pr\left(p_{m:m} > c_1\alpha, \ldots, p_{m-i+2:m} > c_{i-1}\alpha, p_{m-i+1:m} \le c_i\alpha\right) & i = 2, \ldots, m, \\ \Pr\left(p_{m:m} > c_1\alpha, \ldots, p_{1:m} > c_m\alpha\right) & i = m+1. \end{cases} \tag{2.3}$$

Note that while $A_i(m)$ is a function of both the c_i's and the d_i's, $B_i(m)$ is a function only of the c_i's. By starting with $B_1(1) = c_1\alpha$ and $A_1(1) = d_1\alpha$, the probabilities $A_i(m)$ $(1 \le i \le m, 1 \le m \le n)$ can be computed by using the following recursive relations:

$$B_i\left(m-1\right) = \alpha\left(\frac{m-1}{m-i}\right) c_i B_i\left(m-2\right) \quad (m \ge 3, 1 \le i \le m-1), \tag{2.4}$$

$$B_m\left(m-1\right) = 1 - \sum_{i=1}^{m-1} B_i\left(m-1\right) \quad (m \ge 2), \tag{2.5}$$

$$A_i(m) = \alpha\left(\frac{m}{m-i+1}\right)\left[\frac{c_i^{m-i+1} - (c_i - d_i)^{m-i+1}}{c_i^{m-i}}\right] B_i\left(m-1\right). \tag{2.6}$$

These relations are similar to those given in Cai and Sarkar (2008) and Gou and Tamhane (2014) for the generalized Simes critical constants. Their proofs are given in the Appendix. The proofs utilize the independence assumption.

We will illustrate the calculations for $m = 1, 2, 3$.

m = 1 We have $A_1(1) = d_1\alpha$, which is the type I error of the local test. Hence $d_1 \le 1$ and if the local test must be α-exhaustive then $d_1 = 1$.

m = 2 We have $B_1(1) = c_1\alpha$ and $B_2(1) = 1 - B_1(1) = 1 - c_1\alpha$. Then

$$A_1(2) = \frac{c_1^2 - (c_1 - d_1)^2}{c_1}\alpha B_1(1) = [c_1^2 - (c_1 - d_1)^2]\alpha^2$$

and

$$A_2(2) = 2d_2\alpha B_2(1) = 2d_2\alpha(1 - c_1\alpha).$$

The type I error of the local test is $A_1(2) + A_2(2)$, which depends on $\{c_1\}$, $\{d_1, d_2\}$ and α.

m = 3 We have $B_1(2) = \alpha^2 c_1 B_1(1)$, $B_2(2) = 2\alpha c_2 B_2(1)$ and $B_3(2) = 1 - B_1(2) - B_2(2)$. Next,

$$A_1(3) = \frac{c_1^3 - (c_1 - d_1)^3}{c_1^2}\alpha B_1(2), A_2(3) = \frac{3[c_2^2 - (c_2 - d_2)^2]}{2c_2}\alpha B_2(2)$$

and

$$A_3(3) = 3d_3\alpha B_3(2).$$

The type I error of the local test is $A_1(3) + A_2(3) + A_3(3)$, which depends on $\{c_1, c_2\}$, $\{d_1, d_2, d_3\}$ and α.

In general, to compute the type I error of the local test with m hypotheses, we first compute $B_i(m-2)$ for $i = 1, \cdots, m-1$ and $A_i(m-1)$ for $i = 1, \cdots, m-1$. We then compute $B_i(m-1)$ for $i = 1, \cdots, m-1$ from (2.4). Next we compute $B_m(m-1)$ from (2.5). Finally, we compute $A_i(m)$, for $i = 1, \cdots, m$ from (2.6). When $i = m$, we have $A_m(m) = md_m\alpha B_m(m-1)$, so c_m does not appear in the expression for $A_m(m)$. The type I error of the local test with m hypotheses is $\sum_{i=1}^{m} A_i(m)$, which depends on $\{c_1, \cdots, c_{m-1}\}$, $\{d_1, \cdots, d_m\}$ and α.

The condition (2.2) with an equality for α-exhaustive local tests is equivalent to

$$C_m(\alpha; c_1, \ldots, c_{m-1}; d_1, \ldots, d_m) \equiv 1 - \frac{1}{\alpha}\sum_{i=1}^{m} A_i(m) = 0, \quad m = 1, \cdots, n. \tag{2.7}$$

Note that when going from $m-1$ to m, two new critical constants, c_{m-1} and d_m, must be computed. Thus if we have either (a) pre-specified c_i's, or (b) pre-specified d_i's, or (c) pre-specified relations between c_{i-1}'s and d_i's, then we can directly solve (2.7) for the remaining unknown. For example, the HH_0 procedure uses pre-specified $d_i = 1/i$, so (2.7) can be solved for the single unknown c_i. The resulting c_i's depend on α in general, whereas the $c_i = (i+1)/2i$ used in HH_0 do not depend on α but they are slightly conservative since they do not exhaust α.

The general algorithm works as follows. We begin by fixing $d_1 \in [1/n, 1]$. This choice is dictated by the fact that if $d_1 < 1/n$ then the stepwise procedure will be less powerful than the Bonferroni procedure since it will use a critical constant less than the Bonferroni critical constant α/n. On the other hand, if $d_1 > 1$ then it will not control the FWER at level α since its critical constant will be greater than α. We also constrain all $c_i < 1/\alpha$ by making $c_1 < 1/\alpha$.

Suppose that we have already determined $\{c_1, \ldots, c_{m-2}\}$ and $\{d_1, \ldots, d_{m-1}\}$. Then for c_{m-1} and d_m, we first solve the following inequality for d_m subject to $d_m \le d_{m-1}$ for two limiting values of c_{m-1}:

$$C_m(\alpha; c_1, \ldots, c_{m-2}, c_{m-1}; d_1, \ldots, d_m) \ge 0. \tag{2.8}$$

If we set c_{m-1} equal to its largest possible value, namely, c_{m-2} then we get the smallest possible value d_m^* of d_m. On the other hand, if we set c_{m-1}

equal to its smallest possible value, namely d_{m-1} then we get the largest possible value d_m^{**} of d_m. The choice of d_m in the interval $[d_m^*, d_m^{**}]$ is not unique but we may follow some rule such as the midpoint of the interval. Having selected $d_m \in [d_m^*, d_m^{**}]$, we can then solve for c_{m-1} from

$$C_m(\alpha; c_1, \ldots, c_{m-2}, c_{m-1}; d_1, \ldots, d_m) = 0.$$

If a solution that satisfies $c_m \leq c_{m-1}$ does not exist then we cannot exhaust α for the local test.

The sequence of calculations is as shown in the following flow chart.

$$\begin{array}{ccccccccccc} & & c_1 & & c_2 & & \cdots & & c_{n-2} & & c_{n-1} \\ & & \uparrow & \searrow & \uparrow & \searrow & \cdots & \searrow & \uparrow & \searrow & \uparrow \\ d_1 & \rightarrow & d_2 & & d_3 & & \cdots & & d_{n-1} & & d_n \end{array}$$

As an illustration, consider $n = 2$ and $\alpha = 0.05$. At Step 1, we need to choose $d_1 \in [1/2, 1]$. Suppose we choose $d_1 = 1$. Then $C_1(\alpha; d_1) = 1 - d_1 = 0$, which satisfies the inequality (2.8). At Step 2, by setting $c_1 = d_1$, we must have $C_2(\alpha; d_1; d_1, d_2) = 1 - 2d_2 - \left(d_1^2 - 2d_1 d_2\right)\alpha = (1-\alpha)(1-2d_2) \geq 0$. We make this an equality by choosing $d_2 = 1/2$. Then at Step 3, we find c_1 from $C_2(\alpha; c_1; d_1, d_2) = 1 - 2d_2 - \left(2c_1 d_1 - 2c_1 d_2 - d_1^2\right)\alpha = (1 - c_1)\alpha \geq 0$. We make this an equality by choosing $c_1 = 1$. Following this sequence of calculations gives the critical constants for $w = 0$ in Table 1. On the other hand, if at Step 2, instead of choosing $d_2 = 1/2$ we choose $d_2 = 1/3$. Then at Step 3, we find c_1 from $C_2(\alpha; c_1; d_1, d_2) = 1 - 2d_2 - \left(2c_1 d_1 - 2c_1 d_2 - d_1^2\right)\alpha = 1/3 - (4c_1/3 - 1)\alpha \geq 0$. We make this an equality by choosing $c_1 = 5.75$. Following this sequence of calculations gives the critical constants for $w = 1$ in Table 1. It should be noted that both these choices exhaust α.

Calculations become more complicated if we choose $d_1 < 1$. To illustrate this, suppose we choose $d_1 = 0.8$. Then $C_1(\alpha; d_1) = 1 - d_1 = 0.2 > 0$, which satisfies the inequality (2.8). At Step 2, by setting $c_1 = d_1$, we must have

Table 1: Critical Constants for the HH(w) Procedure ($\alpha = 5\%$)

	Rom	$w = 0$		$w = 0.25$		$w = 0.50$		$w = 0.75$		$w = 1$	
i	$c_i = d_i$	c_i	d_i	c_i	d_i	c_i	d_i	c_i	d_i	c_i	d_i
1	1.000	1.000	1.000	2.462	1.000	3.714	1.000	4.800	1.000	5.750	1.000
2	0.500	0.750	0.500	1.951	0.458	3.218	0.417	4.675	0.375	5.750	0.333
3	0.338	0.670	0.333	1.623	0.313	2.533	0.292	3.309	0.271	4.904	0.250
4	0.254	0.629	0.250	1.429	0.238	2.214	0.225	2.968	0.213	3.160	0.200
5	0.204	0.200	0.200	0.192	0.192	0.183	0.183	0.175	0.175	0.167	0.167

$C_2(\alpha; d_1; d_1, d_2) = 1 - 2d_2 - \left(d_1^2 - 2d_1 d_2\right)\alpha \geq 0$, which is equivalent to the condition:

$$0 \leq d_2 \leq \min\left(d_1, \frac{1 - d_1^2\alpha}{2(1 - d_1\alpha)}\right).$$

When $d_1 = 0.8$, this range is $0 \leq d_2 \leq 0.5042$. Suppose we choose $d_2 = 0.4$. Then at Step 3, we have $C_2(\alpha; c_1; d_1, d_2) = 1 - 2d_2 - \left(2c_1 d_1 - 2c_1 d_2 - d_1^2\right)\alpha$. Therefore, in order to exhaust α, we must have

$$c_1 = \min\left(\frac{1}{\alpha}, \frac{1 - 2d_2 + d_1^2\alpha}{2(d_1 - d_2)\alpha}\right).$$

Substituting $d_1 = 0.8$ and $d_2 = 0.4$, we get $c_1 = 5.8$.

3 Table of Critical Constants

Because of the non-uniqueness of the choices of the (c_i, d_i) and the consequent difficulty of finding an optimal choice among them, we did not use the general algorithm given above to calculate the critical constants (c_i, d_i) given in Table 1. Instead we fixed the d_i's by parameterizing them and then computed the c_i's. The particular parametrization we used is the convex combination of $1/i$ and $1/(i+1)$ according to the formula:

$$d_1 = 1, d_i = \frac{w}{i+1} + \frac{1-w}{i} \quad (i \geq 2).$$

Thus the d_i are constrained to the interval $[1/(i+1), 1/i]$ if $w \in [0, 1]$. The reason for limiting $d_i \leq 1/i$ has already been mentioned, namely that if $d_i > 1/i$ and hence $d_1 > 1$, the type I error will exceed α. There is no similar theoretical reason to limit $d_i \geq 1/(i+1)$. We will see in Table 2

Table 2: Estimates of the % Familywise Error Rate ($\alpha = 5\%$)

		$\rho < 0$			$\rho = 0$			$\rho = 0.5$		
n	n_0	$w = 0$	$w = 0.5$	$w = 1$	$w = 0$	$w = 0.5$	$w = 1$	$w = 0$	$w = 0.5$	$w = 1$
	1	2.210	3.152	3.642	3.288	4.198	4.442	4.667	4.962	4.991
3	2	3.855	3.769	3.403	4.255	4.661	4.757	4.575	5.847	6.463
	3	5.127	4.588	4.037	4.949	4.942	4.885	4.666	6.059	7.018
	1	1.503	2.686	3.223	2.361	3.727	4.198	4.337	4.827	4.893
	2	2.631	3.256	3.306	3.244	4.153	4.501	4.259	5.689	6.439
5	3	3.629	3.852	3.805	3.908	4.414	4.508	4.408	5.987	7.019
	4	4.482	4.433	4.235	4.463	4.645	4.761	4.408	5.925	7.130
	5	5.108	4.862	4.523	4.911	4.905	4.899	4.418	5.866	7.192

that as $d_i \to 1/(i+1)$, i.e., as $w \to 1$, the FWER is not controlled under positive dependence. Anti-conservatism is even more serious if $d_i < 1/(i+1)$ or $w > 1$. Therefore we chose to limit $d_i \geq 1/(i+1)$.

For illustration we chose $w = 0, 0.25, 0.50, 0.75, 1$ and then computed the associated c_i's by solving (2.7). We denote the HH procedure that uses these critical constants by HH(w), e.g., HH($w = 0.5$). We also included the Rom (1990) critical constants for comparison purposes as they provide an exact version of the Hochberg critical constants, $c_i = d_i = 1/i$ $(1 \leq i \leq n)$. Note that the Rom procedure can be shown to be a special case of the HH(w) procedure.

Notice that as w gets larger, the c_i's get larger and the d_i's get smaller. This inverse empirical relationship between the c_i's and the d_i's is intuitively obvious but is difficult to prove analytically. For $w = 0$ (in which case $d_i = 1/i$), we see that only $c_4 = 0.629$ is different (slightly larger) from that used by HH_0, namely $c_4 = 0.625$; the d_i-values are the same for both procedures. Thus the HH($w = 0$) procedure is nearly identical (slightly more powerful) than the HH_0 procedure. For this reason we have not simulated the powers of the HH_0 procedure separately.

4 FWER Simulations

From its construction we know that the flexible HH procedure controls FWER under independence. To investigate whether it also controls FWER under dependence, we carried out simulations under the following scenarios. We generated multivariate normal variates, $z_1, z_2, \ldots, z_n$, where $z_i \sim N(\delta, 1^2)$ with $\delta = 0$ for $i = 1, \ldots, n_0$, $\delta = 2$ for $i = n_0 + 1, \ldots, n$ and $\text{Corr}(z_i, z_j) = \rho$. Thus there were n_0 true null hypotheses and $n_1 = n - n_0$ false null hypotheses. The p-values were generated using the transformation $p_i = 1 - \Phi(z_i)$ for $i = 1, \ldots, n$ where $\Phi(\cdot)$ is the standard normal c.d.f. The following combinations of various parameters were studied: $n = 3, 5, n_0 = 1, \ldots, n$, $\rho = -0.8/(n-1), 0, 0.5$ and $w = 0, 0.5, 1$. (Note that ρ must be $> -1/(n-1)$ for the correlation matrix to be positive definite.) Thus $\rho = -0.4$ for $n = 3$ and $\rho = -0.2$ for $n = 5$ in case of $\rho < 0$. A total of 10^6 replications were performed for each case.

We see that the flexible HH procedure controls FWER conservatively for $\rho \leq 0$, but for $\rho > 0$ the procedure is anti-conservative except for $w = 0$ (which is essentially the HH_0 procedure). We suspect that the anti-conservatism of the flexible HH procedure under positive dependence when $w > 0$ is due to the fact that all $c_i > 1$ except $c_n = d_n = (1-w)/n + w/(n+1) < 1$. Contrast this with the case $w = 0$ where all $c_i \leq 1$. A possible reason for the anti-conservatism is that, as shown by Sarkar (1998) for the

Simes test, if the test statistics are MTP_2 distributed then the type I error is $\leq c_1\alpha$. So if $c_1 > 1$ then the upper bound on the type I error is greater than α and hence it is possible for the type I error to exceed α. Note, however, that Sarkar's result does not apply directly to the flexible HH procedure since it uses two sets of critical constants $\{c_i, d_i\}$ while the Simes test uses only one set of critical constants $\{c_i\}$. Nonetheless, we can expect that a similar result will hold in this case.

5 Power Simulations

We ran simulations for power comparisons for $n = 5$ under different scenarios. The p-values were generated using the same simulation set up given in Section 4. We simulated two types of powers: familywise power (F-power), which is the probability of rejecting at least one false hypothesis, and average power (A-power), which is the expected proportion of rejected false hypotheses for $\delta = 1$ and 2. The results for the two powers were similar, so to save space, here we report only the F-powers for $\delta = 2$ in Table 3.

First we observe that the Rom procedure is least powerful except when $n_1 = 1$ when its power is the highest and is the same as that of the HH($w = 0$) procedure. The Hochberg procedure is of course uniformly less powerful than the Rom procedure. Thus these two procedures are not competitive with the HH($w = 0$) procedure. In fact, in Gou et al. (2014) we had shown that the Hochberg procedure is uniformly less powerful than the HH_0 procedure.

Next we observe that the F-power increases monotonically with w and the highest power is attained at $w = 1$, i.e., $d_i = 1/(i+1)$, in all cases except $n_1 = 1$ (one false null hypothesis, four true null hypotheses). However, as seen from Table 2, the FWER also increases monotonically with w exceeding α when $\rho > 0$. Thus we cannot recommend HH($w = 1$) because of its lack of FWER control.

As noted before, the reason for this anti-conservatism is that the corresponding $c_i > 1$. To take advantage of the higher power of HH(w) for

Table 3: F-Power ($\alpha = 5\%$)

δ	n_1	Rom	$w = 0$	$w = 0.25$	$w = 0.50$	$w = 0.75$	$w = 1$
	1	37.63*	37.63*	37.58	37.51	37.42	37.31
	2	61.51	62.06	62.91	63.27	63.41	63.43*
2	3	75.92	77.48	79.27	80.14	80.56	80.59*
	4	85.09	86.94	89.27	90.44	91.01	91.04*
	5	90.01	92.59	94.92	96.06	96.67	96.93*

* denotes the maximum F-power

Table 4: Critical Constants for the HH$'(w)$ Procedure for $w = 0.05$ ($\alpha = 5\%$)

i	1	2	3	4	5
c_i	1.000	0.988	0.863	0.790	0.198
d_i	1.000	0.500	0.329	0.248	0.198

$w > 0$ while still controlling the FWER, we can modify the HH(w) procedure by putting a constraint $c_1 = d_1 = 1, d_2 = 1/2$ and then compute the remaining (c_i, d_i) using a selected value of w. It can be shown that the range of w must be limited to a rather narrow interval around 0 in order to ensure that the c_i's and d_i's are monotone. We may regard this as an alternative scheme for computing the critical constants and denote the corresponding procedure by HH$'(w)$ (HH$'(w)$ is HH(w) with additional restrictions $c_1 = d_1 = 1, d_2 = 1/2$). For example, for $n = 5$ if we choose $w = 0.05$ then we get the critical constants shown in Table 4.

By comparing these critical constants with those for $w = 0$ from Table 1 we see that the c_i-values in Table 4 are larger, while the d_i-values are slightly smaller. Therefore we can expect some power gain for HH$'(w = 0.05)$. But as Table 5 shows the power gain is quite small, just around 1%. Therefore it may not be worthwhile to use this procedure and simply use the HH$(w = 0)$ procedure or the HH$_0$ procedure.

Finally we note that only the case $w = 0$, which corresponds to the HH$_0$ procedure, provides uniformly more powerful alternative than the Hochberg procedure. In fact, we had shown in Gou et al. (2014) that the HH$_0$ procedure rejects all null hypotheses rejected by Hochberg and possibly more. For all other values of $0 < w \leq 1$, although the proposed procedure has higher overall statistical power, it does not guarantee uniform dominance

Table 5: F-power Comparison of HH$'(w = 0.05)$ with HH$(w = 0)$ for $\delta = 2, \alpha = 5\%$

	$\rho = -0.2$		$\rho = 0$		$\rho = 0.5$	
n_1	HH ($w = 0$)	HH$'$ ($w = 0.05$)	HH ($w = 0$)	HH$'$ ($w = 0.05$)	HH ($w = 0$)	HH$'$ ($w = 0.05$)
1	37.56*	37.54	37.62*	37.61	37.55*	37.54
2	65.05	65.30*	62.02	62.28*	54.34	54.55*
3	83.47	84.03*	77.49	77.99*	64.07	64.57*
4	94.22	94.86*	86.93	87.53*	70.47	71.24*
5	99.03	99.31*	92.59	93.18*	75.00	75.91*

* denotes the maximum F-power

Table 6: p-Values and Critical Constants for the Example ($\alpha = 0.05$)

		HH_0		$\text{HH}(w=1)$	
i	$p_{n-i+1:n}$	$c_i\alpha$	$d_i\alpha$	$c_i\alpha$	$d_i\alpha$
1	0.060	0.0500	0.0500	0.2875	0.0500
2	0.035	0.0375	0.0250	0.2875	0.0167
3	0.022	0.0333	0.0167	0.2452	0.0125
4	0.007	0.0313	0.0125	0.1580	0.0100
5	0.005	0.0100	0.0100	0.0083	0.0083

over the Hochberg procedure. As an example, consider $n = 3, p_1 = 0.30, p_2 = 0.02, p_3 = 0.01$ and $\alpha = 0.05$. In this case the Hochberg procedure rejects H_2 and H_3, but $\text{HH}(w = 1)$ rejects only H_3.

6 Example

We will use two limiting cases of the $\text{HH}(w)$ procedure, namely $\text{HH}(w = 0)$ (or actually HH_0) and $\text{HH}(w = 1)$. Consider a subset of five p-values obtained by taking every alternate p-value from Table 8 in Gou et al. (2014). These p-values along with the critical constants $c_i\alpha$ and $d_i\alpha$ for $n = 5, \alpha = 0.05$ for the HH_0 procedure and for the $\text{HH}(w = 1)$ procedure are tabulated in Table 6.

HH_0 stops at Step 2 since $p_{4:5} = 0.035 < 0.0375$ and hence rejects any H_i with $p_i < 0.025$, i.e., hypotheses $H_{3:5}$, $H_{2:5}$ and $H_{1:5}$. On the other hand, $\text{HH}(w = 1)$ stops at Step 1, since $p_{5:5} = 0.060 < 0.2875$ and hence rejects any H_i with $p_i < 0.05$, i.e., hypotheses $H_{4:5}$, $H_{3:5}$, $H_{2:5}$ and $H_{1:5}$. Thus it rejects one more hypothesis.

7 Concluding Remarks

In this paper we have given a method to calculate a flexible set of critical constants (c_i, d_i) for the improved HH procedure proposed in Gou et al. (2014). These critical constants can be calculated by first fixing the d_i's parameterized by taking a linear combination of $1/i$ and $1/(i + 1)$, and then calculating the c_i's. The advantage of the flexible HH procedure is that more powerful critical constants can be determined than the fixed constants given by (1.3) used in the HH_0 procedure. The proposed procedure for $w > 0$, i.e., $d_i < 1/i$, does not control the FWER for positively correlated test statistics, unless c_1 is constrained to be 1. Since in practice, test statistics are commonly positively correlated, we recommend using the $\text{HH}(w = 0)$ procedure or more simply the HH_0 procedure. If independence or negative dependence is expected then the HH(w) procedure with $w > 0$ may be used for its higher power.

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Appendix

Proofs of Eqs. (2.4), (2.5) and (2.6)

The probabilities $A_i(m)$ and $B_i(m)$ are defined as follows:

$$A_i(m) = \begin{cases} \Pr\left(p_{m:m} \le c_1\alpha\right) & \text{if } i = 1 \\ \Pr\left(p_{m:m} > c_1\alpha, \ldots, p_{m-i+2:m} > c_{i-1}\alpha, p_{m-i+1:m} \le c_i\alpha,\right. & \\ \left. p_{1:m} \le d_i\alpha\right) & \text{if } i = 2, \ldots, m. \end{cases} \tag{A.1}$$

$$B_i(m) = \begin{cases} \Pr\left(p_{m:m} \le c_1\alpha\right) & \text{if } i = 1 \\ \Pr\left(p_{m:m} > c_1\alpha, \ldots, p_{m-i+2:m} > c_{i-1}\alpha, p_{m-i+1:m} \le c_i\alpha\right) & \text{if } i = 2, \ldots, m \\ \Pr\left(p_{m:m} > c_1\alpha, \ldots, p_{1:m} > c_m\alpha\right) & \text{if } i = m+1. \end{cases} \tag{A.2}$$

We first prove the recursive relationship

$$A_i(m) = \alpha\left(\frac{m}{m-i+1}\right)\left[\frac{c_i^{m-i+1} - (c_i - d_i)^{m-i+1}}{c_i^{m-i} - (c_i - d_i)^{m-i}}\right] A_i(m-1) \tag{A.3}$$

where $i = 1, \ldots, m-1$. The independence assumption is used all of the following derivations.

Note that

$$\begin{aligned} A_i(m) &= \Pr\left(p_{m:m} > c_1\alpha, \ldots, p_{m-i+2:m} > c_{i-1}\alpha, p_{m-i+1:m} \le c_i\alpha, p_{1:m} \le d_i\alpha\right) \\ &= \binom{m}{m-i+1} \Pr\left(p_{i-1:i-1} > c_1\alpha, \ldots, p_{1:i-1} > c_{i-1}\alpha\right) \\ &\quad \times \Pr\left(p_{m-i+1:m-i+1} \le c_i\alpha, p_{1:m-i+1} \le d_i\alpha\right) \\ &= \binom{m}{m-i+1} \Pr\left(p_{i-1:i-1} > c_1\alpha, \ldots, p_{1:i-1} > c_{i-1}\alpha\right) \\ &\quad \times \left[\Pr\left(p_{m-i+1:m-i+1} \le c_i\alpha\right) \Pr\left(p_{m-i+1:n-i+1} \le c_i\alpha, p_{1:m-i+1} > d_i\alpha\right)\right] \\ &= \binom{n}{m-i+1} \Pr\left(p_{i-1:i-1} > c_1\alpha, \ldots, p_{1:i-1} > c_{i-1}\alpha\right) \\ &\quad \times \left[(c_i\alpha)^{m-i+1} - (c_i\alpha - d_i\alpha)^{m-i+1}\right] \\ &= \binom{m}{m-i+1} \alpha^{m-i+1}\left[c_i^{m-i+1} - (c_i - d_i)^{m-i+1}\right] \\ &\quad \times P\{p_{i-1:i-1} > c_1\alpha, \ldots, p_{1:i-1} > c_{i-1}\alpha\}. \end{aligned} \tag{A.4}$$

Similarly,

$$\begin{aligned} A_i(m-1) &= \Pr\left(p_{m-1:m-1} > c_1\alpha, \ldots, p_{m-i+1:m-1} > c_{i-1}\alpha, p_{m-i:m-1}\right. \\ &\qquad \left. \leq c_i\alpha, p_{1:m-1} \leq d_i\alpha\right) \\ &= \binom{m-1}{m-i}\alpha^{m-i}[c_i^{m-i} - (c_i - d_i)^{m-i}] \\ &\quad \times \Pr\left(p_{i-1:i-1} > c_1\alpha, \ldots, p_{1:i-1} > c_{i-1}\alpha\right). \qquad \text{(A.5)} \end{aligned}$$

Substituting for $\Pr(p_{i-1:i-1} > c_1\alpha, \ldots, p_{1:i-1} > c_{i-1}\alpha)$ from (A.5) in (A.4) gives the recursive relation between $A_i(m)$ and $A_i(m-1)$ in (A.3).

Next, note that when $i = 1, \ldots, m$, $B_i(m)$ is a special case of $A_i(m)$ for $c_i = d_i$. Let $c_i = d_i$ in the recursive relationship between $A_i(m)$ and $A_i(m-1)$ in (A.3), it follows the recursive relationship between $B_i(m)$ and $B_i(m-1)$:

$$B_i(m) = \alpha c_i \left(\frac{m}{m-i+1}\right) B_i(m-1), \qquad \text{(A.6)}$$

where $i = 1, \ldots, m-1$.

The recursive relation between between $A_i(m)$, $A_i(m-1)$ and $B_i(m-1)$ in (A.7) follows similarly.

$$A_i(m) = \alpha \left(\frac{m}{m-i+1}\right) \left[\frac{c_i^{m-i+1} - (c_i - d_i)^{m-i+1}}{c_i^{m-i}}\right] B_i(m-1), \qquad \text{(A.7)}$$

where $i = 1, \ldots, m$.

Note that

$$\begin{aligned} B_i(m-1) &= \Pr\left(p_{m-1:m-1} > c_1\alpha, \ldots, p_{m-i+1:m-1} > c_{i-1}\alpha, p_{m-i:m-1} \leq c_i\alpha\right) \\ &= \binom{m-1}{m-i}\alpha^{m-i}c_i^{m-i}\Pr\left(p_{i-1:i-1} > c_1\alpha, \ldots, p_{1:i-1} > c_{i-1}\alpha\right). \qquad \text{(A.8)} \end{aligned}$$

Similarly, substituting for $\Pr(p_{i-1:i-1} > c_1\alpha, \ldots, p_{1:i-1} > c_{i-1}\alpha)$ from (A.8) in (A.4) gives the recursive relation between $A_i(m)$ and $B_i(m-1)$ in (A.7).

Finally, note that

$$\sum_{i=1}^{m+1} B_i(m) = 1,$$

it follows that

$$B_{m+1}(m) = 1 - \sum_{i=1}^{m} B_i(m). \qquad \text{(A.9)}$$

References

cai, g. and sarkar, s.k. (2008). Modified Simes' critical values under independence. *Statistics and Probability Letters* **78**, 12, 1362–1368.

gou, j. and tamhane, a.c. (2014). On generalized Simes critical constants. *Biometrical Journal* **56**, 6, 1035–1054.

gou, j., tamhane, a. c., xi, d. and rom, d. (2014). A class of improved hybrid Hochberg-Hommel type step-up multiple test procedures. *Biometrika* **101**, 4, 899–911.

hochberg, y. (1988). A sharper Bonferroni procedure for multiple tests of significance. *Biometrika* **75**, 4, 800–802.

hochberg, y. and tamhane, a.c. (1987). *Multiple Comparison Procedures*. Wiley, New York.

hommel, g. (1988). A stagewise rejective multiple test procedure based on a modified Bonferroni test. *Biometrika* **75**, 2, 383–386.

marcus, r., peritz, e. and gabriel, k.r. (1976). On closed testing procedures with special reference to ordered analysis of variance. *Biometrika* **63**, 655–660.

rom, d.m. (1990). A sequentially rejective test procedure based on a modified Bonferroni inequality. *Biometrika* **77**, 3, 663–665.

rom, d.m. (2013). An improved Hochberg procedure for multiple tests of significance. *British Journal of Mathematical and Statistical Psychology* **66**, 1, 189–196.

sarkar, s.k. (1998). Some probability inequalities for ordered MTP2 random variables: a proof of the Simes conjecture. *The Annals of Statistics* **26**, 2, 494–504.

simes, r.j. (1986). An improved Bonferroni procedure for multiple tests of significance. *Biometrika* **73**, 751–754.

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