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Some Results on Pareto Optimal Choice Sets for Estimating Main Effects and Interactions in 2^n and 3^n Factorial Plans

Jing Xiao and Pallavi Chitturi
Temple University, Philadelphia, USA

Abstract

Choice-based conjoint experiments are used when choice alternatives can be described in terms of attributes. The objective is to infer the value that respondents attach to attribute levels. This method involves the design of profiles on the basis of attributes specified at certain levels. Respondents are presented sets of profiles called choice sets, and asked to select the one they consider best. Information Per Profile (IPP) is used as an optimality criteria to compare designs with different numbers of profiles. The optimality of connected main effects plans based on two consecutive choice sets, S_l and S_{l+1}, has been examined in the literature. However, the optimality of non-consecutive choice sets has not been examined. In this paper we examine the IPP of non-consecutive choice sets and show that IPP can be maximized under certain conditions. Further, we show that non-consecutive choice sets have higher IPP than consecutive choice sets for $n \geq 4$. In addition, we examine the optimality of connected first-order-interaction designs based on three choice sets and show that non-consecutive choice sets have higher IPP than consecutive choice sets under certain conditions. Further, we check the D-, A- and E-optimality of best consecutive and non-consecutive PO choice sets with maximum IPP. Finally, we consider 3^n choice experiments. We look for the optimal PO choice sets and examine their IPP, D-, A- and E-optimality, as well as comparing consecutive and non-consecutive choice sets.

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1 Introduction

Choice-based conjoint experiments are used when choice alternatives can be described in terms of attributes. The objective is to infer the value that respondents attach to attribute levels. This method involves the design of profiles on the basis of attributes specified at certain levels. Respondents are presented sets of profiles called choice sets, and asked to select the one they consider best.

Louviere and Woodworth (1983) discussed the integration of conjoint analysis with econometrics "discrete choice modeling". They provided a theoretical foundation for choice-based conjoint (CBC) analysis. Instead of ranking or rating all profiles as was usually done in earlier approaches of conjoint analysis, a CBC study asks respondents to repeatedly choose one alternative from different sets of profiles. In each profile, respondents are asked to consider several attributes, and usually need to give up benefits in some attributes in order to reduce costs in others. By examining their preferences for profiles in each choice set, researchers may infer the value respondents attach to changes in benefits and costs relative to changes in other attributes. In recent years, CBC analysis has become a popular method in many areas, such as environmental science (Alriksson and Öberg, 2008; Acosta et al., 2014), Geography (Aguilar, 2010), Health (Brown et al., 2008; Makkar et al., 2015) and Marketing (Ding et al., 2005; Chitturi et al., 2010; Halme and Kallio, 2014).

In a valuation study, some profiles are better than others, or dominating, while some profiles are worse, or dominated. Choosing a dominating alternative, or not choosing a dominated one, does not involve an economic choice. If a choice set has a dominating profile, the respondent's choice is trivially made. Similarly, if a choice set has a dominated profile, it will never be selected. Therefore, we want to restrict our attention to choice sets with no dominating or dominated profiles. Such choice sets are known as Pareto Optimal(PO) sets.

Consider the example of a laptop to illustrate the implications of having dominated or dominating alternatives in a set. Suppose a student wants to buy a laptop and has two choices, a laptop with 3 Gigahertz processing speed and 4 hour battery, and another one with 2 Gigahertz processing speed and 6 hour battery. All other attributes of these two laptops are the same. If the student selects the first laptop, it may be inferred that he values processing speed over battery life, assuming other things equal. Conversely, choice of the second laptop would indicate that the increment on battery life was of more value than the increment on processing speed. On the other hand, if he was offered a choice between a laptop with 3 Gigahertz processing speed and 6 hour battery, and a laptop with 2 Gigahertz processing speed and 4 hour battery, assuming a rational choice, he would very likely select the first laptop because it dominates the second one on both attributes (note that dominating profiles can be presented to check that respondents are choosing rationally, see Severin et al., 2013). Nothing is learned about the relative value the respondent attaches to an increment on processing speed versus battery life. Hence, it is desirable that sets of choices presented to respondents do not contain profiles that dominate, or are dominated by, other profiles.

Definition *A subset T of S is said to be a PO subset if for every two distinct profiles $(u_1, ..., u_n)$, $(w_1, ..., w_n) \in T$, there exist subscripts i and j $(i \neq j)$ such that $u_i < w_i$ and $u_j > w_j$.*

The need of PO choice sets in choice experiments was realized by Wiley (1978). Krieger and Green (1991) extended that work and constructed orthogonal and PO subsets (or close to PO subsets). Huber and Hansen (1986) made some comparisons between PO designs and orthogonal designs and reported that PO designs can predict better.

Raghavarao and Wiley (1998) considered a general setting with any number of levels for the attributes and obtained connected main effects plans. Consider a S^n design with n attributes at s levels each. Let x_i represents the level of the ith attribute, $i = 1, 2, ..., n$. They noted that the choice sets $S_l = \{(x_1, x_2, ..., x_n)| \sum x_i = l\}, l = 0, 1, ..., n(s-1)$, are PO. Any subset of a PO choice set is itself a PO subset. They further showed that the design based on $S_{[\frac{n}{2}]}$ and $S_{[\frac{n}{2}]+1}$ is a connected main effects plan,where $[\frac{n}{2}]$ is the integral part of $\frac{n}{2}$.

Zhang (2001) proved that any two-PO subsets S_l and S_k $(0 < l < k < n, l \neq k)$, is a connected main effects plan and three-PO subsets, S_{l-1}, S_l and S_{l+1} $(2 \leq l \leq n-2)$, is a connected first-order-interaction plan. Zhang (2001) also assessed the optimality of these designs using Information Per Profile (IPP) as a criterion. Raghavarao et al. (2010) explored the design of experiment (DOE) issues that occur when constructing profiles and showed how to modify commonly used designs for solving conjoint analysis problems.

There are several optimality criteria used for evaluating experimental designs. Three popular criteria are D-, A- and E-optimality which were introduced by Jack Kiefer in 1958. Carlsson and Martinsson (2003) and Grossßmann et al. (2006) disscussed several different D-optimal designs for conjoint analysis. Street and Burgess (2004, 2005) and Street et al. (2005) found theories to generate mostly D-optimal designs and some A-optimal designs under the assumption that all profiles in the design are equally attractive.

2 Connected Main Effects PO Designs and Information Per Profile (IPP)

For a 2^n experiment, consider two PO subsets S_l and S_k ($l \neq k, 0 < l, k < n$). Let s denote the total number of profiles in S_l and S_k. Assume the model

$$y_i = \mu + \sum_{j=1}^{n} x_{ij}\beta_j + e_i, \tag{2.1}$$

where y_i is the response to the ith profile, $i = 1, ..., s$, the average score of the profile or the logit of the proportion of times the profile is selected. In this model, μ is the general mean, β_j is the main effect of the jth attribute, x_{ij} is the level of the jth attribute in the ith profile (1 or -1), n is the number of attributes and the e_i's are independently and identically distributed with mean zero and variance σ^2. Linear models have been used for designs for CBC experiments, see Zhang (2001) and Chatterjee and Dey (2013).

A design is said to be a connected main effect design if all the main effect contrasts from the model (2.1) can be estimated. Zhang (2001) proved the following important result for PO subsets.

THEOREM 1. *For a 2^n experiment, the design based on two PO subsets S_l and S_k is a connected main-effect plan, where* $(1 \leq l < k \leq n-1)$.

The design matrix of the model (2.1) is denoted by

$$D = [1_s \ \ X],$$

where X is the $s \times n$ matrix whose $(i,j)th$ element is x_{ij} and 1_s is a s-dimensional vector of ones.

The information matrix A is given by

$$A = D'D = \begin{bmatrix} a_0 & a_1 1_n \\ a_1 1_n & (a_0 - a_2)I_n + a_2 J_n \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix, J_n is the $n \times n$ matrix of all ones, $a_0 = s$, $a_1 = \sum_{i=1}^{s} x_{i\alpha}$, and $a_2 = \sum_{i=1}^{s} x_{i\alpha}x_{i\beta}$ for any $\alpha, \beta = 1, ..., n, \alpha \neq \beta$.

By multiplying the first row of A by $-\frac{a_1}{a_0}$ and adding to the other rows, the information matrix of main effects after eliminating μ is

$$C_m = (a_0 - a_2)I_n + (a_2 - \frac{{a_1}^2}{a_0})J_n.$$

This is also seen in Raghavarao et al. (2010) and Raghavarao and Zhang (2002).

The optimality criterion, IPP is used to compare designs with different number of profiles. IPP(θ) is defined as

$$\theta = \frac{n}{a_0 trace(C_m^{-1})}, \tag{2.2}$$

where C_m^{-1} is

$$C_m^{-1} = \frac{1}{a_0 - a_2}[I_n - \frac{a_0a_2 - a_1^2}{a_0(a_0 - a_2) + n(a_0a_2 - a_1^2)} J_n], \tag{2.3}$$

for any connected main effects plan composed of choice sets S_l and S_k. Based on the IPP definition, the optimal design has the largest θ. Zhang (2001) also proved that

$$0 < \theta \leq 1.$$

3 Optimal Non-consecutive PO Sets S_l and S_{n-l}

One drawback of consecutive PO sets $(S_{[\frac{n}{2}]}, S_{[\frac{n}{2}]+1})$ is that they have the most number of profiles in all possible PO sets S_l and S_{l+1}. Therefore, these PO sets should be used only for studies with a small number of attributes. One reasonable approach to reduce the number of profiles is to use PO sets of non-consecutive format, S_l and S_{n-l}. Using PO sets S_l and S_{n-l} can reduce the number of profiles significantly. To see this benefit, consider the following examples. When $n = 7$, the number of profiles of consecutive PO sets (S_3, S_4) is 70, while the number of profiles of non-consecutive PO sets (S_2, S_5) is 42 and the number of profiles of non-consecutive PO sets (S_1, S_6) is 14.

THEOREM 2. *For a* 2^n *experiment, two PO sets* S_l *and* S_k, *(where* $1 \leq l < k \leq n-1$, $k + l = n$*), have maximum* $IPP(\theta) = \frac{4kl(k-l)^2}{n[(n-2)(k-l)^2+n]}$, *when* $l = ||\frac{n-\sqrt{n}}{2}||$ *and* $k = ||\frac{n+\sqrt{n}}{2}||$. *(* $||l||$ *is the integer nearest to l.)*

The proof of Theorem 2 is detailed and a sketch of it is given in the Appendix.

4 Comparison of Optimal PO Sets $(S_l,\ S_{l+1})$ and $(S_l,\ S_{n-l})$

The optimal PO sets $(S_l,\ S_{l+1})$ and $(S_l,\ S_{n-l})$ have different number of profiles for $n \geq 4$. Therefore, D-, A- and E-optimality criteria are not appropriate to compare the optimal PO sets $(S_l,\ S_{l+1})$ and $(S_l,\ S_{n-l})$. IPP is used to measure information of each profile. Therefore, we compare the optimal PO sets $(S_l,\ S_{l+1})$ and $(S_l,\ S_{n-l})$ using IPP(θ). Here, only the optimal PO sets which have maximum IPP(θ) for both cases are considered.

THEOREM 3. *For a 2^n experiment, non-consecutive PO sets (S_l, S_{n-l}) with maximum IPP(θ), (where $1 \leq l \leq n-1$ and $n \geq 4$), have higher IPP(θ) than consecutive PO sets (S_l, S_{l+1}) with maximum IPP(θ).*

PROOF. For optimal PO sets S_l and S_{l+1}, when $l = [\frac{n}{2}]$, IPP(θ) is maximized.

$$\theta_{maxfor(S_l, S_{l+1})} = \frac{2(n-l)(l+1)}{n(n+1)},$$

where $l = [\frac{n}{2}]$.

For optimal PO sets S_l and S_k, where $1 \leq l < k \leq n-1$, $k + l = n$ and $n \geq 4$, following Theorem 2, when $l = ||\frac{n-\sqrt{n}}{2}||$ and $k = ||\frac{n+\sqrt{n}}{2}||$, IPP($\theta$) is maximized.

$$\theta_{maxfor(S_l, S_{n-l})} = \frac{4kl(k-l)^2}{n[(n-2)(k-l)^2 + n]} = \frac{4kl}{n[n-2+\frac{n}{(k-l)^2}]},$$

where $l = ||\frac{n-\sqrt{n}}{2}||$ and $k = ||\frac{n+\sqrt{n}}{2}||$.

It can be shown that $\theta_{maxfor(S_l, S_{n-l})} > \theta_{maxfor(S_l, S_{l+1})}$. The details can be seen in Xiao (2015). Therefore, non-consecutive PO sets (S_l, S_{n-l}), where $1 \leq l \leq n-1$, have greater IPP(θ) than consecutive PO sets (S_l, S_{l+1}). In other words, the design based on $S_{||\frac{n-\sqrt{n}}{2}||}$ and $S_{||\frac{n+\sqrt{n}}{2}||}$ has higher IPP(θ) than the design based on $S_{[\frac{n}{2}]}$ and $S_{[\frac{n}{2}]+1}$.

Table 1 shows the IPP(θ) and the number of profiles for these designs for $3 \leq n \leq 20$. When $n \geq 4$, the optimal PO sets ($S_{||\frac{n-\sqrt{n}}{2}||}$, $S_{||\frac{n+\sqrt{n}}{2}||}$) has higher IPP($\theta$) and fewer profiles than the optimal PO sets ($S_{[\frac{n}{2}]}$, $S_{[\frac{n}{2}]+1}$). Also, when $n = a^2$, a is a positive integer and $a \geq 2$, optimal PO sets (S_l, S_{n-l}) has IPP(θ) of 1. In all other cases (except $n = 3$), IPP(θ) of optimal PO sets (S_l, S_{n-l}) is very close to 1, while IPP(θ) of optimal PO sets (S_l, S_{l+1}) is decreasing from 0.667. Therefore, optimal PO sets (S_l, S_{n-l}) are better than optimal PO sets (S_l, S_{l+1}) using the optimality criterion of IPP(θ).

5 Optimal Non-consecutive First-Order-Interaction PO Designs

Sections 3 and 4 discussed the connected main effects model. Sometimes, interactions may be important and need to be investigated. In this section, we consider the inclusion of two-factor interactions in the model.

The first-order-interaction effects model can be written as

$$y_i = \mu + \sum_{j=1}^{n} x_{ij}\beta_j + \sum_{j=1}^{n-1}\sum_{j'=j+1}^{n} x_{ij}x_{ij'}\gamma_{jj'} + e_i, \tag{5.1}$$

Table 1: IPP comparison of optimal PO sets (S_l, S_{l+1}) and (S_l, S_{n-l})

	Consecutive ($S_{[\frac{n}{2}]}$, $S_{[\frac{n}{2}]+1}$)				Non-consecutive ($S_{\|\|\frac{n-\sqrt{n}}{2}\|\|}$, $S_{\|\|\frac{n+\sqrt{n}}{2}\|\|}$)			
n	$[\frac{n}{2}]$	$[\frac{n}{2}]+1$	a_0	IPP	l	$n-l$	a_0	IPP
3	1	2	6	0.667	1	2	6	0.667
4	2	3	10	0.600	1	3	8	1.000
5	2	3	20	0.600	1	4	10	0.900
6	3	4	35	0.571	2	4	30	0.970
7	3	4	70	0.571	2	5	42	0.989
8	4	5	126	0.556	3	5	112	0.938
9	4	5	252	0.556	3	6	168	1.000
10	5	6	462	0.545	3	7	240	0.974
11	5	6	924	0.545	4	7	660	0.996
12	6	7	1716	0.538	4	8	990	0.992
13	6	7	3432	0.538	5	8	2574	0.989
14	7	8	6435	0.533	5	9	4004	0.999
15	7	8	12870	0.533	6	9	10010	0.982
16	8	9	24310	0.529	6	10	16016	1.000
17	8	9	48620	0.529	6	11	24752	0.990
18	9	10	92378	0.526	7	11	63648	0.999
19	9	10	184756	0.526	7	12	100776	0.996
20	10	11	352716	0.524	8	12	251940	0.997

* a_0 is the number of profiles

where y_i is the response to the ith profile, $i = 1, 2, ..., t$, t is the total number of profiles; μ is the general mean; β_j is the main effect of the jth attribute; $\gamma_{jj'}$ is the interaction effect of the jth and $j'th$ attributes; $x_{ij} = 1$ or -1, depending on the level of the jth attribute in the ith profile is 1 or 0; n is the number of attributes, and e_i's are independently and identically distributed with mean zero and equal variance σ^2.

A design is a connected first-order-interaction design if we can estimate all the main effect and first-order-interaction effect contrasts from the model (5.1). Zhang (2001) proved that no design based on two PO subsets is a connected first-order-interaction design. The design based on three PO subsets is a connected first-order-interaction design. He also showed that the design based on three consecutive choice sets, $S_{\frac{n}{2}-1}$, $S_{[\frac{n}{2}]}$ and $S_{\frac{n}{2}+1}$, is a connected first-order-interaction design. Next, we show that the design based on three non-consecutive choice sets, S_l, $S_{[\frac{n}{2}]}$ and S_k, is a connected first-order-interaction plan.

THEOREM 4. *The design based on* S_l, $S_{[\frac{n}{2}]}$ *and* S_k *is a connected first-order-interaction plan, where* $l+k = n, n \geq 4$ *and* $l = ||\frac{n-\sqrt{n}}{2}||$, $k = ||\frac{n+\sqrt{n}}{2}||$.

PROOF. Without loss of generality, let the attributes A_1 and A_2 have a first-order-interaction. The attributes A_3, A_4, ..., A_n are given by x_3, x_4, ...,

x_n. The profiles $(0,0,x_3,x_4,...,x_n)$, $(1,0,x_3,x_4,...,x_n)$, $(0,1,x_3,x_4,...,x_n)$ and $(1,1,x_3,x_4,...,x_n)$ occur in the design, so clearly both main effects and interaction effect of A_1 and A_2 are estimable. The PO subsets S_l, $S_{[\frac{n}{2}]}$ and S_k, where $l+k=n, n\geq 4$ and $l=||\frac{n-\sqrt{n}}{2}||$, $k=||\frac{n+\sqrt{n}}{2}||$, provide a solution for x_3, x_4, ..., x_n such that the 4 indicated profiles occur in these subsets. Hence the result is proved.

The model (5.1) can be written in matrix form as $Y = D\beta + e$. D is the design matrix whose first column is a t-dimensional vector of all ones, columns 2 to $n+1$ correspond to the main effects of the n attibutes, and the last $n(n-1)/2$ columns correspond to two-factor interactions of all the n attributes.

Let $c_0 = t$, $c_1 = \sum_{i=1}^t x_{ij}$, $c_2 = \sum_{i=1}^t x_{ij}x_{ij'}$, $c_3 = \sum_{i=1}^t x_{ij}x_{ij'}x_{ij''}$ and $c_4 = \sum_{i=1}^t x_{ij}x_{ij'}x_{ij''}x_{ij'''}$ for distinct j, j', j'' and j'''; j, j', j'', $j''' = 1,2,3,...,n$.

The information matrix A under model (5.1) is given by

$$A = D'D = \begin{bmatrix} c_0 & c_1 1'_n & c_2 1'_{\frac{n(n-1)}{2}} \\ c_1 1_n & (c_0-c_2)I_n + c_2 J_n & C_{12} \\ c_2 1_{\frac{n(n-1)}{2}} & C'_{12} & C_{22} \end{bmatrix}, \qquad (5.2)$$

where C_{22} is a $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ matrix defined by

$$C_{22} = c_0 I_{\frac{n(n-1)}{2}} + c_2 B_1 + c_4 B_2,$$

and C_{12} is a $n \times \frac{n(n-1)}{2}$ matrix given by

$$\left[\begin{array}{c|c|c|c|c} c_1 1'_{n-1} & c_3 1'_{n-2} & c_3 1'_{n-3} & \cdots & c_3 \\ \hline & c_1 1'_{n-2} & c_3 1'_{n-3} & \cdots & c_3 \\ \cline{2-5} & & c_1 1'_{n-3} & \cdots & c_3 \\ \cline{3-5} (c_1-c_3)I_{n-1} + c_3 J_{n-1} & & & \cdots & c_3 \\ \cline{4-5} & (c_1-c_3)I_{n-2} + c_3 J_{n-2} & & \vdots & \vdots \\ \cline{4-5} & & (c_1-c_3)I_{n-3} + c_3 J_{n-3} & \cdots & c_3 \\ \cline{5-5} & & & \cdots & c_1 \\ \cline{5-5} & & & \cdots & c_1 \end{array}\right],$$

where B_1 and B_2 are the 1st and 2nd association matrices respectively. The information matrix for estimating main effects under model (5.1) is:

$$\begin{aligned} C_m &= (c_0 - c_2)I_n + c_2 J_n \\ &- \left[\begin{array}{cc} c_1 1_n & C_{12} \end{array}\right] \left[\begin{array}{cc} c_0 & c_2 1'_{\frac{n(n-1)}{2}} \\ c_2 1_{\frac{n(n-1)}{2}} & C_{22} \end{array}\right]^{-1} \left[\begin{array}{cc} c_1 1_n & C_{12} \end{array}\right]'. \end{aligned} \quad (5.3)$$

According to the definition of IPP(θ) and analogous to the equation (2.2), IPP(θ) for estimating main effects under (5.1) can be defined as

$$\theta = \frac{n}{t \times trace(C_m^{-1})}. \quad (5.4)$$

THEOREM 5. *Let θ be the IPP for estimating main effects from the design based on $S_{\frac{n}{2}-1}$, $S_{[\frac{n}{2}]}$ and $S_{\frac{n}{2}+1}$; and θ^* be the IPP for estimating main effects from the design based on S_l, $S_{[\frac{n}{2}]}$ and S_k, where $l + k = n$ and $l = ||\frac{n-\sqrt{n}}{2}||$, $k = ||\frac{n+\sqrt{n}}{2}||$. When n is 4, 6 or 8, $\theta^* = \theta$; however, when n is an even and $10 \le n \le 50$, $\theta^* > \theta$.*

PROOF. Let $n = 2m$. For the design based on S_l, $S_{[\frac{n}{2}]}$ and S_k, where $l + k = n$ and $l = ||\frac{n-\sqrt{n}}{2}||$, $k = ||\frac{n+\sqrt{n}}{2}||$, we have the following,

$$c_0 = 2\binom{n}{l} + \binom{n}{m}, \quad c_1 = c_3 = 0,$$

$$c_2 = 2\binom{n-2}{l-2} - 4\binom{n-2}{l-1} + 2\binom{n-2}{l} + 2\binom{n-2}{m} - 2\binom{n-2}{m-1}.$$

It can be shown that the information matrix A of (5.2) becomes

$$A = \left[\begin{array}{ccc} c_0 & 01'_n & c_2 1'_{\frac{n(n-1)}{2}} \\ 01_n & (c_0 - c_2)I_n + c_2 J_n & 0_{n \times \frac{n(n-1)}{2}} \\ c_2 1_{\frac{n(n-1)}{2}} & 0_{\frac{n(n-1)}{2} \times n} & C_{22} \end{array}\right].$$

The IPP(θ) of information matrix for estimating main effects from the design based on S_l, $S_{[\frac{n}{2}]}$ and S_k, where $l + k = n$ and $l = ||\frac{n-\sqrt{n}}{2}||$, $k = ||\frac{n+\sqrt{n}}{2}||$, becomes

$$\theta^* = \frac{c_0 - c_2}{c_0} \times \frac{c_0 + (n-1)c_2}{c_0 + (n-2)c_2}.$$

Zhang (2001) showed that when n is even, $n = 2m$, the IPP for estimating main effects from the design based on $S_{\frac{n}{2}-1}$, $S_{[\frac{n}{2}]}$ and $S_{\frac{n}{2}+1}$ is

$$\theta = \frac{8(3m-2)(m+1)}{(3m+1)(11m-7)}.$$

Due to the complicated expression of θ^*, it is not easy to compare θ and θ^* directly. Values of θ and θ^* for even values of n ($n \leq 50$) are given in Table 2 and they show that $\theta^* > \theta$ when n is even and $10 \leq n \leq 50$.

6 Other Optimality Criteria for Optimal PO Sets

PO sets (S_l, S_{l+1}) and (S_l, S_{n-l}) are designs with different number of profiles when $l \neq \frac{n-1}{2}$. Optimal PO sets (S_l, S_{l+1}) have a greater number of profiles than optimal PO sets (S_l, S_{n-l}) when $n \geq 4$. It is not appropriate to use D-, A- and E-optimality criteria to compare optimal PO sets (S_l, S_{l+1})

Table 2: Comparison of IPP θ of PO sets ($S_{\frac{n}{2}-1}$, $S_{[\frac{n}{2}]}$ and $S_{\frac{n}{2}+1}$) and θ^* of (S_l, $S_{[\frac{n}{2}]}$ and S_{n-l})

n	θ for $S_{\frac{n}{2}-1}$, $S_{[\frac{n}{2}]}$ and $S_{\frac{n}{2}+1}$	θ^* for S_l, $S_{[\frac{n}{2}]}$ and S_{n-l}
4	0.914	0.914
6	0.862	0.862
8	0.832	0.832
10	0.813	0.993
12	0.799	0.988
14	0.790	0.982
16	0.782	0.977
18	0.776	0.973
20	0.772	0.969
22	0.768	0.965
24	0.765	0.962
26	0.762	0.995
28	0.759	0.994
30	0.757	0.993
32	0.755	0.993
34	0.754	0.991
36	0.752	0.990
38	0.751	0.989
40	0.750	0.988
42	0.749	0.987
44	0.748	0.986
46	0.747	0.986
48	0.746	0.985
50	0.745	0.996

and $(S_l,\ S_{n-l})$, because they usually have a different number of profiles. Instead, IPP(θ) provides a way to compare designs with different number of profiles. However, we can still calculate the D-, A- and E-optimality of best PO sets $(S_l,\ S_{l+1})$ and $(S_l,\ S_{n-l})$ respectively.

THEOREM 6. *PO sets* $(S_{[\frac{n}{2}]}, S_{[\frac{n}{2}]+1})$ *are the D-, A- and E-optimal designs based on* (S_l, S_{l+1}).

PROOF. For PO sets S_l and S_{l+1}, we have the following,

$$a_0 = \left(\begin{array}{c} n+1 \\ l+1 \end{array} \right), \quad a_1 = -\frac{n-2l-1}{n+1}a_0, \quad a_2 = a_0 - 4\left(\begin{array}{c} n-1 \\ l \end{array} \right).$$

The Information matrix can be simplified as

$$C_m = 4\left(\begin{array}{c} n-1 \\ l \end{array} \right) I_n - \frac{4}{n+1}\left(\begin{array}{c} n-1 \\ l \end{array} \right) J_n.$$

So we can have

$$det(C_m) = \frac{1}{n+1}[4\left(\begin{array}{c} n-1 \\ l \end{array} \right)]^n \quad . \tag{6.1}$$

The D-optimality criterion requires us to maximize $det(C_m)$ which is equivalent to maximizing $\left(\begin{array}{c} n-1 \\ l \end{array} \right)$. Thus, considering l as an integer, when $l = ||\frac{n-1}{2}||$, $\left(\begin{array}{c} n-1 \\ l \end{array} \right)$ is maximized. When n is an even positive integer, $n = 2m$, we know $||\frac{n-1}{2}|| = ||m - \frac{1}{2}|| = m$ and $[\frac{n}{2}] = m$. When n is an odd positive integer, $n = 2m+1$, we have $||\frac{n-1}{2}|| = ||m|| = m$ and $[\frac{n}{2}] = m$. Thus, $||\frac{n-1}{2}|| = [\frac{n}{2}]$ for any positive integer n. Therefore, the PO sets $(S_{[\frac{n}{2}]}, S_{[\frac{n}{2}]+1})$ are D-optimal among designs based on (S_l, S_{l+1}).

Zhang (2001) already proved that PO sets (S_l, S_{l+1}) have IPP(θ) which is a function of l, $\theta(l) = \frac{2(l+1)(n-l)}{n(n+1)}$. Thus, we have

$$\begin{aligned} trace(C_m^{-1}) &= \frac{n}{a_0\theta} \\ &= \frac{n}{2} \cdot \frac{1}{\left(\begin{array}{c} n-1 \\ l \end{array} \right)}. \end{aligned} \tag{6.2}$$

A-optimality requires us to minimize $trace(C_m^{-1})$. Equation (6.2) shows that this is equivalent to maximizing $\left(\begin{array}{c} n-1 \\ l \end{array} \right)$. From the above proof, we know that when $l=[\frac{n}{2}]$, $\left(\begin{array}{c} n-1 \\ l \end{array}\right)$ is maximized. Thus, PO sets $(S_{[\frac{n}{2}]}, S_{[\frac{n}{2}]+1})$ are also the A-optimal designs based on (S_l, S_{l+1}).

To have an E-optimal design, the minimum eigenvalue of information matrix C_m needs to be maximized. It can be shown that $\frac{4}{n+1}\left(\begin{array}{c} n-1 \\ l \end{array}\right)$ is the minimum eigenvalue of C_m (the details can be seen in Xiao, 2015). To maximize $\frac{4}{n+1}\left(\begin{array}{c} n-1 \\ l \end{array}\right)$, $l=[\frac{n}{2}]$. From the above proof of D-optimal designs, we know that PO sets $(S_{[\frac{n}{2}]}, S_{[\frac{n}{2}]+1})$ satisfy this requirement and are E-optimal designs based on (S_l, S_{l+1}).

Therefore, optimal PO sets $(S_{[\frac{n}{2}]}, S_{[\frac{n}{2}]+1})$ and $(S_{[\frac{n-1}{2}]}, S_{[\frac{n-1}{2}]+1})$ are the D-, A- and E-optimal designs based on (S_l, S_{l+1}). Table 3 gives $det(C_m)$, $trace(C_m^{-1})$ and Minimum Eigenvalue of PO Sets (S_l, S_{l+1}) with Maximum IPP(θ). All of them are optimal designs with maximum IPP(θ) and D-, A- and E-optimality based on consecutive PO sets (S_l, S_{l+1}).

For PO sets S_l and S_{n-l}, we have the following,

$$a_0 = 2\left(\begin{array}{c} n \\ l \end{array}\right), \quad a_1 = 0, \quad a_2 = \frac{2(n-2l)^2 - 2n}{n(n-1)}\left(\begin{array}{c} n \\ l \end{array}\right). \qquad (6.3)$$

Table 3: $det(C_m)$, $trace(C_m^{-1})$ and minimum eigenvalue of PO sets (S_l, S_{l+1}) with Maximum IPP(θ)

n	$(S_{[\frac{n}{2}]}, S_{[\frac{n}{2}]+1})$	$(S_{[\frac{n-1}{2}]}, S_{[\frac{n-1}{2}]+1})$	a_0	$det(C_m)$	$trace(C_m^{-1})$	$min(ev)$
3	(S_1, S_2)	(S_1, S_2)	6	1.280×10^2	0.7500	2.0
4	(S_2, S_3)	(S_1, S_2)	10	4.147×10^3	0.6667	2.4
5	(S_2, S_3)	(S_2, S_3)	20	1.327×10^6	0.4167	4.0
6	(S_3, S_4)	(S_2, S_3)	35	5.8514×10^8	0.3000	5.7
7	(S_3, S_4)	(S_3, S_4)	70	2.621×10^{12}	0.1750	10.0
8	(S_4, S_5)	(S_3, S_4)	126	1.640×10^{16}	0.1143	15.6
9	(S_4, S_5)	(S_4, S_5)	252	1.058×10^{21}	0.0643	28.0
10	(S_5, S_6)	(S_4, S_5)	462	9.614×10^{25}	0.0397	45.8
11	(S_5, S_6)	(S_5, S_6)	924	9.097×10^{31}	0.0218	84.0
12	(S_6, S_7)	(S_5, S_6)	1716	1.220×10^{38}	0.0130	142.2
13	(S_6, S_7)	(S_6, S_7)	3432	1.715×10^{45}	0.0070	264.0
14	(S_7, S_8)	(S_6, S_7)	6435	3.436×10^{52}	0.0041	457.6
15	(S_7, S_8)	(S_7, S_8)	12870	7.244×10^{60}	0.0022	858.0
16	(S_8, S_9)	(S_7, S_8)	24310	2.184×10^{69}	0.0012	1514.1
17	(S_8, S_9)	(S_8, S_9)	48620	6.960×10^{78}	0.0007	2860.0
18	(S_9, S_{10})	(S_8, S_9)	92378	3.180×10^{88}	0.0004	5118
19	(S_9, S_{10})	(S_9, S_{10})	184756	1.540×10^{99}	0.0002	9724.0
20	(S_{10}, S_{11})	(S_9, S_{10})	352716	1.072×10^{110}	0.0001	17595.8

Note: a_0 is the number of profiles and $min(ev)$ is the minimum eigenvalue

The Information matrix can be simplified as

$$C_m = \frac{8l(n-l)}{n(n-1)} \binom{n}{l} I_n + \frac{2(n-2l)^2 - 2n}{n(n-1)} \binom{n}{l} J_n.$$

Table 4 gives the $det(C_m)$, $trace(C_m^{-1})$ and minimum eigenvalue of PO Sets (S_l, S_{n-l}) with maximum IPP(θ). Table 5 lists the PO Sets (S_l, S_{n-l}) with maximum D-, A- and E-optimality. Table 6 shows the PO designs with maximum IPP(θ) and the D-, A- and E-optimal PO designs based on (S_l, S_{n-l}). We can see that PO sets (S_l, S_{n-l}) with maximum IPP(θ) have maximum $det(C_m)$ or minimum $trace(C_m^{-1})$ in some cases (see * marked in Table 4). It shows that optimal PO sets (S_{l^*}, S_{n-l^*}) with maximum IPP(θ), where $l^* = ||\frac{n-\sqrt{n}}{2}||$, are A-optimal or D-optimal designs for some cases of (S_l, S_{n-l}). Further, in these practical cases of $n \leq 20$, PO sets (S_l, S_{n-l}) with maximum IPP(θ) are always E-optimal designs. Also, in these practical cases of $n \leq 20$, when n is a square number, the number of profiles is equal to the maximized minimum eigenvalue.

Table 4: $det(C_m)$, $trace(C_m^{-1})$ and minimum eigenvalue of PO sets (S_l, S_{n-l}) with Maximum IPP(θ)

n	(S_l, S_{n-l})	a_0	$det(C_m)$	$trace(C_m^{-1})$	$min(ev)$
3	(S_1, S_2)	6	*1.280×10^2	*0.7500	2
4	(S_1, S_3)	8	*4.096×10^3	*0.5000	8
5	(S_1, S_4)	10	7.373×10^4	0.5556	8
6	(S_2, S_4)	30	*6.711×10^8	*0.2063	20
7	(S_2, S_5)	42	2.212×10^{11}	*0.1685	40
8	(S_3, S_5)	112	*2.007×10^{16}	*0.0762	56
9	(S_3, S_6)	168	1.066×10^{20}	*0.0536	168
10	(S_3, S_7)	240	5.452×10^{23}	0.0428	224
11	(S_4, S_7)	660	1.014×10^{31}	*0.0167	540
12	(S_4, S_8)	990	8.425×10^{35}	0.0122	960
13	(S_5, S_8)	2574	2.042×10^{44}	*0.0051	1782
14	(S_5, S_9)	4004	2.695×10^{50}	0.0035	3960
15	(S_6, S_9)	10010	9.035×10^{59}	*0.0015	6006
16	(S_6, S_{10})	16016	1.874×10^{67}	0.0010	16016
17	(S_6, S_{11})	24752	4.482×10^{74}	0.0007	24024
18	(S_7, S_{11})	63648	2.918×10^{86}	0.0003	56576
19	(S_7, S_{12})	100776	1.108×10^{95}	0.0002	99008
20	(S_8, S_{12})	251940	1.036×10^{108}	*0.0001	201552

Note: a_0 is the number of profiles; $min(ev)$ is the minimum eigenvalue; * marks the designs with maximum $det(C_m)$ and / or minimum $trace(C_m^{-1})$

Table 5: PO sets (S_l, S_{n-l}) with maximum D-, A- and E-optimality

	S_l and S_{n-l} with max($det(C_m)$)			S_l and S_{n-l} with min($trace(C_m^{-1})$)			S_l and S_{n-l} with max(min eigenvalue)		
n	(S_l, S_{n-l})	a_0	$det(C_m)$	(S_l, S_{n-l})	a_0	$trace(C_m^{-1})$	(S_l, S_{n-l})	a_0	$max(min(ev))$
3	(S_1, S_2)	6	1.280×10^2	(S_1, S_2)	6	0.7500	(S_1, S_2)	6	2
4	(S_1, S_3)	8	4.096×10^3	(S_1, S_3)	8	0.5000	(S_1, S_3)	8	8
5	(S_2, S_3)	20	1.327×10^6	(S_2, S_3)	20	0.4167	(S_1, S_4)	10	8
6	(S_2, S_4)	30	6.711×10^8	(S_2, S_4)	30	0.2063	(S_2, S_4)	30	20
7	(S_3, S_4)	70	2.621×10^{12}	(S_2, S_5)	42	0.1685	(S_2, S_5)	42	40
8	(S_3, S_5)	112	2.007×10^{16}	(S_3, S_5)	112	0.0762	(S_3, S_5)	112	56
9	(S_4, S_5)	252	1.058×10^{21}	(S_3, S_6)	168	0.0536	(S_3, S_6)	168	168
10	(S_4, S_6)	420	1.221×10^{26}	(S_4, S_6)	420	0.0260	(S_3, S_7)	240	224
11	(S_5, S_6)	924	9.097×10^{31}	(S_4, S_7)	660	0.0167	(S_4, S_7)	660	540
12	(S_5, S_7)	1584	1.589×10^{38}	(S_5, S_7)	1584	0.0084	(S_4, S_8)	990	960
13	(S_6, S_7)	3432	1.715×10^{45}	(S_5, S_8)	2574	0.0051	(S_5, S_8)	2574	1782
14	(S_6, S_8)	6006	4.551×10^{52}	(S_6, S_8)	6006	0.0026	(S_5, S_9)	4004	3960
15	(S_7, S_8)	12870	7.244×10^{60}	(S_6, S_9)	10010	0.0015	(S_6, S_9)	10010	6006
16	(S_7, S_9)	22880	2.931×10^{69}	(S_7, S_9)	22880	0.0008	(S_6, S_{10})	16016	16016
17	(S_8, S_9)	48620	6.960×10^{78}	(S_7, S_{10})	38896	0.0004	(S_6, S_{11})	24752	24024
18	(S_8, S_{10})	87516	4.312×10^{88}	(S_8, S_{10})	87516	0.0002	(S_7, S_{11})	63648	56576
19	(S_9, S_{10})	184756	1.540×10^{99}	(S_8, S_{11})	151164	0.0001	(S_7, S_{12})	100776	99008
20	(S_9, S_{11})	335920	1.466×10^{110}	(S_9, S_{11})	335920	0.0001	(S_8, S_{12})	251940	201552

Note: a_0 is the number of profiles and $max(min(ev))$ is max(minimum eigenvalue)

Table 6: PO design with maximum IPP (θ) and D-, A- and E-optimal PO designs based on (S_l, S_{n-l})

	Maximum IPP (θ)		D-optimal		A-optimal		E-optimal	
n	(S_l, S_{n-l})	a_0	(S_l, S_{n-l})	a_0	(S_l, S_{n-l})	a_0	(S_l, S_{n-l})	a_0
3	(S_1, S_2)	6	(S_1, S_2)	6	(S_1, S_2)	6	(S_1, S_2)	6
4	(S_1, S_3)	8	(S_1, S_3)	8	(S_1, S_3)	8	(S_1, S_3)	8
5	(S_1, S_4)	10	(S_2, S_3)	20	(S_2, S_3)	20	(S_1, S_4)	10
6	(S_2, S_4)	30	(S_2, S_4)	30	(S_2, S_4)	30	(S_2, S_4)	30
7	(S_2, S_5)	42	(S_3, S_4)	70	(S_2, S_5)	42	(S_2, S_5)	42
8	(S_3, S_5)	112	(S_3, S_5)	112	(S_3, S_5)	112	(S_3, S_5)	112
9	(S_3, S_6)	168	(S_4, S_5)	252	(S_3, S_6)	168	(S_3, S_6)	168
10	(S_3, S_7)	240	(S_4, S_6)	420	(S_4, S_6)	420	(S_3, S_7)	240
11	(S_4, S_7)	660	(S_5, S_6)	924	(S_4, S_7)	660	(S_4, S_7)	660
12	(S_4, S_8)	990	(S_5, S_7)	1584	(S_5, S_7)	1584	(S_4, S_8)	990
13	(S_5, S_8)	2574	(S_6, S_7)	3432	(S_5, S_8)	2574	(S_5, S_8)	2574
14	(S_5, S_9)	4004	(S_6, S_8)	6006	(S_6, S_8)	6006	(S_5, S_9)	4004
15	(S_6, S_9)	10010	(S_7, S_8)	12870	(S_6, S_9)	10010	(S_6, S_9)	10010
16	(S_6, S_{10})	16016	(S_7, S_9)	22880	(S_7, S_9)	22880	(S_6, S_{10})	16016
17	(S_6, S_{11})	24752	(S_8, S_9)	48620	(S_7, S_{10})	38896	(S_6, S_{11})	24752
18	(S_7, S_{11})	63648	(S_8, S_{10})	87516	(S_8, S_{10})	87516	(S_7, S_{11})	63648
19	(S_7, S_{12})	100776	(S_9, S_{10})	184756	(S_8, S_{11})	151164	(S_7, S_{12})	100776
20	(S_8, S_{12})	251940	(S_9, S_{11})	335920	(S_9, S_{11})	335920	(S_8, S_{12})	251940

Note: a_0 is the number of profiles

7 Optimal PO Designs For 3^n Experiments

Raghavarao and Wiley (2006) proved that we can estimate all $n(s-1)$ contrasts of main effects by using two consecutive choice sets S_l and S_{l+1}, where $(s-2) \le l \le (n-1)(s-1)$, for a s^n experiment. Non-consecutive choice sets S_l and $S_{n(s-1)-l}$ were rarely assessed in previous research. In this section, we will examine the IPP, D-, A- and E-optimality of non-consecutive choice sets S_l and S_{2n-l} for a 3^n experiment and compare the results with consecutive choice sets S_l and S_{l+1}.

Consider a 3^n experiment with $n(\ge 2)$ attributes A_1, A_2, A_3, $\ldots$, A_n each at 3 levels denoted by 0, 1 and 2. A profile is denoted by $(x_1, x_2, \ldots, x_n)$, where $x_i = 0, 1$ or 2; $i = 1, 2, \ldots, n$. A choice set consists of a set of profiles. Let us consider a PO design with a total of t profiles numbered 1, 2, $\ldots$, t for a 3^n experiment. Model (2.1) can be rewritten in terms of the linear and quadratic effects of the attributes as

$$y_i = \mu + \sum_{j=1}^{n} x_{ij}^1 \beta_j^1 + \sum_{j=1}^{n} x_{ij}^2 \beta_j^2 + e_i, \tag{7.1}$$

where n is the number of attributes; y_i is the response to the ith profile, $i = 1, 2, \ldots, t$; μ is the general mean; β_j^1 and β_j^2 are the linear and quadratic effects of the jth attribute; x_{ij}^1 and x_{ij}^2 are the levels of the jth attribute in the ith profile. Their assigned values are as follows,

$$(x_{ij}^1, x_{ij}^2) = \begin{cases} (1/\sqrt{2}, 1/\sqrt{6}), & \text{if the level of the } jth \text{ attribute in the } ith \text{ profile is 0,} \\ (0, -2/\sqrt{6}), & \text{if the level of the } jth \text{ attribute in the } ith \text{ profile is 1,} \\ (-1/\sqrt{2}, 1/\sqrt{6}), & \text{if the level of the } jth \text{ attribute in the } ith \text{ profile is 2.} \end{cases}$$

We assume that e_i's are independently and identically distributed with mean zero and equal variance σ^2.

The information matrix A_3 is given by

$$A_3 = \begin{bmatrix} d & 1_n' \otimes a_1' \\ 1_n \otimes a_1 & I_n \otimes G_1 + J_n \otimes (G_2 - G_1) \end{bmatrix},$$

where $d = t$ is the total number of profiles; I_n is a $n \times n$ identity matrix; J_n is a $n \times n$ matrix of all ones; $\otimes$ is the Kronecker product symbol;

$$a_1 = \begin{pmatrix} \frac{1}{\sqrt{2}}(d_0 - d_2) \\ \frac{1}{\sqrt{6}}(d_0 - 2d_1 + d_2) \end{pmatrix},$$

$$G_1 = \begin{bmatrix} \frac{1}{2}(d_0 + d_2) & \frac{1}{\sqrt{12}}(d_0 - d_2) \\ \frac{1}{\sqrt{12}}(d_0 - d_2) & \frac{1}{6}(d_0 + 4d_1 + d_2) \end{bmatrix},$$

$$G_2 = \begin{bmatrix} \frac{1}{2}(d_{00} - 2d_{02} + d_{22}) & \frac{1}{\sqrt{12}}(d_{00} - 2d_{01} + 2d_{21} - d_{22}) \\ \frac{1}{\sqrt{12}}(d_{00} - 2d_{01} + 2d_{21} - d_{22}) & \frac{1}{6}(d_{00} - 4d_{01} + 6d_{02} - 4d_{12} + d_{22}) \end{bmatrix},$$

where d_0, d_1 and d_2 are the total number of profiles with levels 0, 1 and 2, respectively for any given attribute A_i; d_{00}, d_{01}, d_{02}, d_{11}, d_{12} and d_{22} are the total number of profiles with level combinations (0, 0), (0, 1), (0, 2), (1, 1), (1, 2) and (2, 2), respectively, for any given attributes (A_i, A_j); $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$; $i \neq j$.

The information matrix of main effects after eliminating μ is

$$C_{3m} = I_n \otimes (G_1 - G_2) + J_n \otimes (G_2 - \frac{1}{d} a_1 a_1'). \tag{7.2}$$

If the design is a connected main-effect plan, the Information Per Profile (IPP) for estimating single degrees of freedom contrast can be defined as

$$\theta_3 = \frac{2n}{d\, trace(C_{3m}^{-1})}. \tag{7.3}$$

For the designs based on PO sets, S_l and S_{2n-l}, to have two distinct sets, without loss of generality, we can have $1 \leq l \leq n-1$.

Because of the perfect symmetry between these two sets, S_l and S_{2n-l} have the same number of profiles. Thus, we can have

$$d = 2a_l^n, \quad d_0 = 2a_l^{n-1}, \quad d_1 = 2a_{l-1}^{n-1}, \quad d_2 = 2a_{l-2}^{n-1},$$

$$d_{00} = 2a_l^{n-2}, \quad d_{01} = 2a_{l-1}^{n-2}, \quad d_{02} = d_{11} = 2a_{l-2}^{n-2},$$

and

$$d_{12} = 2a_{l-3}^{n-2}, \quad d_{22} = 2a_{l-4}^{n-2},$$

where a_l^n is the number of profiles in PO set S_l for a 3^n experiment given by

$$a_l^n = \sum_{i=0}^{||\frac{n}{2}-1||} \binom{n}{i} \binom{n-i}{l-2i}.$$

Table 7: The connected main-effect plans for PO sets (S_l, S_{2n-l})

n	Connected main-effect plans for PO sets (S_l, S_{2n-l})
2	None
3	None
4	None
5	None
6	(S_2, S_{10}), (S_3, S_9), (S_4, S_8)
7	(S_2, S_{12}), (S_3, S_{11}), (S_4, S_{10}), (S_5, S_9)
8	(S_2, S_{14}), (S_3, S_{13}), (S_4, S_{12}), (S_6, S_{10}), (S_7, S_9)
9	(S_2, S_{16}), (S_3, S_{15}), (S_4, S_{14}), (S_5, S_{13}), (S_6, S_{12}), (S_7, S_{11}), (S_8, S_{10})
10	(S_2, S_{18}), (S_3, S_{17}), (S_5, S_{15}), (S_6, S_{14}), (S_7, S_{13}), (S_8, S_{12}), (S_9, S_{11})
11	(S_2, S_{20}), (S_3, S_{19}), (S_4, S_{18}), (S_5, S_{17}), (S_6, S_{16}), (S_7, S_{15}), (S_8, S_{14}), (S_9, S_{13}), (S_{10}, S_{12})
12	(S_2, S_{22}), (S_3, S_{21}), (S_5, S_{19}), (S_6, S_{18}), (S_7, S_{17}), (S_8, S_{16}), (S_9, S_{15}), (S_{10}, S_{14}), (S_{11}, S_{13})
13	(S_2, S_{24}), (S_4, S_{22}), (S_5, S_{21}), (S_6, S_{20}), (S_7, S_{19}), (S_8, S_{18}), (S_9, S_{17}), (S_{12}, S_{14})
14	(S_2, S_{26}), (S_3, S_{25}), (S_4, S_{24}), (S_5, S_{23}), (S_6, S_{22}), (S_7, S_{21}), (S_8, S_{20}), (S_9, S_{19}), (S_{10}, S_{18}), (S_{11}, S_{17})
15	(S_2, S_{28}), (S_3, S_{27}), (S_5, S_{25}), (S_6, S_{24}), (S_7, S_{23}), (S_8, S_{22}), (S_9, S_{21}), (S_{10}, S_{20}), (S_{12}, S_{18}), (S_{13}, S_{17}), (S_{14}, S_{16})
16	(S_2, S_{30}), (S_3, S_{29}), (S_4, S_{28}), (S_5, S_{27}), (S_6, S_{26}), (S_7, S_{25}), (S_8, S_{24}), (S_9, S_{23}), (S_{10}, S_{22}), (S_{11}, S_{21}), (S_{12}, S_{20}), (S_{13}, S_{19}), (S_{15}, S_{17})
17	(S_2, S_{32}), (S_3, S_{31}), (S_4, S_{30}), (S_5, S_{29}), (S_6, S_{28}), (S_7, S_{27}), (S_8, S_{26}), (S_9, S_{25}), (S_{10}, S_{24}), (S_{11}, S_{23}), (S_{12}, S_{22}), (S_{13}, S_{21}), (S_{14}, S_{20})
18	(S_2, S_{34}), (S_3, S_{33}), (S_4, S_{32}), (S_5, S_{31}), (S_6, S_{30}), (S_7, S_{29}), (S_8, S_{28}), (S_9, S_{27}), (S_{10}, S_{26}), (S_{12}, S_{24}), (S_{13}, S_{23}), (S_{14}, S_{22}), (S_{15}, S_{21})
19	(S_2, S_{36}), (S_3, S_{35}), (S_4, S_{34}), (S_5, S_{33}), (S_6, S_{32}), (S_8, S_{30}), (S_{10}, S_{28}), (S_{12}, S_{26}), (S_{13}, S_{25}), (S_{14}, S_{24}), (S_{15}, S_{23}), (S_{18}, S_{20})
20	(S_3, S_{37}), (S_4, S_{36}), (S_5, S_{35}), (S_7, S_{33}), (S_9, S_{31}), (S_{10}, S_{30}), (S_{11}, S_{29}), (S_{12}, S_{28}), (S_{13}, S_{27}), (S_{15}, S_{25}), (S_{17}, S_{23}), (S_{19}, S_{21})

Table 8: $det(C_m)$, $trace(C_m^{-1})$ and minimum eigenvalue of optimal PO sets (S_l, S_{2n-l}) with maximum IPP(θ_3)

n	l	$2n-l$	d	max(θ_3)	$det(C_m)$	$trace(C_m^{-1})$	min(eigenvalue)
6	4	8	180	0.324	1.318×10^5	0.2061	7.105×10^{-14}
7	5	9	532	0.335	9.025×10^{15}	0.0786	2.558×10^{-13}
8	7	9	2032	0.369	2.876×10^{28}	0.0213	*2.273×10^{-13}
9	8	10	5814	0.366	7.908×10^{42}	0.0085	*9.095×10^{-13}
10	9	11	16700	0.363	1.400×10^{59}	0.0033	*2.728×10^{-12}
11	10	12	48136	0.361	8.868×10^{76}	0.0013	*7.276×10^{-12}
12	11	13	139152	0.359	5.944×10^{95}	0.0005	5.093×10^{-11}
13	12	14	403286	0.357	4.827×10^{117}	0.0002	*2.910×10^{-11}
14	11	17	777504	0.333	7.224×10^{135}	0.0001	4.075×10^{-10}
15	14	16	3409020	0.354	1.986×10^{166}	2.486×10^{-5}	*2.328×10^{-10}

Note: d is the number of profiles. * marks the designs which are not E-optimal

Using the values of d through d_{22} and a_l^n above, we can calculate the information matrix in (7.2) for PO sets, S_l and S_{2n-l} ($1 \leq l \leq n-1$). If the information matrix C_{3m} has a positive determinant, the PO design, (S_l, S_{2n-l}), is a connected main-effect plan. The IPP(θ_3) in (7.3) can be calculated.

Table 7 shows the connected main-effect plans for PO sets S_l and S_{2n-l} ($2 \leq n \leq 20$). We can see that there are some, PO sets S_l and S_{2n-l} which are connected main-effect plans. We can calculate IPP(θ_3) for only these PO sets S_l and S_{2n-l} which are connected main-effect plans. Table 8 gives the optimal PO sets (S_l, S_{2n-l}) with their maximum IPP(θ_3), $det(C_m)$, $trace(C_m^{-1})$ and minimum eigenvalue of the information matrix for $6 \leq n \leq 15$ using R. In

Table 9: Comparison of PO sets (S_l, S_{2n-l}) and (S_l, S_{l+1}) with maximum IPP(θ_3)

	S_l and S_{2n-l}				S_l and S_{l+1}			
n	l	$2n-l$	d	θ_3	l	$l+1$	d	θ_3*
5	3	7	60	0.300	5	6	96	0.153
6	3	9	180	0.324	5	6	267	0.152
7	5	9	532	0.335	6	7	750	0.151
8	4	12	2032	0.369	7	8	2123	0.150
9	5	13	5814	0.366	8	9	6046	0.149
10	7	13	16700	0.363	9	10	17303	0.149
11	9	13	48136	0.361	10	11	49721	0.148
12	9	15	139152	0.359	11	12	143365	0.148
13	9	17	403286	0.357	12	13	414584	0.147
14	11	17	777504	0.333	13	14	1201917	0.147
15	13	17	3409020	0.354	14	15	3492117	0.147

these calculation results, the PO sets have maximum IPP(θ_3) and are also D- and A-optimal designs based on PO sets (S_l, S_{2n-l}).

Zhang (2001) proved that S_{n-1} and S_n are optimal in the class of designs (S_l, S_{l+1}) for $n \leq 15$. Table 9 gives the comparison results between PO sets (S_l, S_{2n-l}) and (S_l, S_{l+1}) for $5 \leq n \leq 15$. We can find that the PO sets (S_l, S_{2n-l}) has higher IPP(θ_3) compared to PO sets (S_l, S_{l+1}) for $5 \leq n \leq 15$ and PO sets (S_l, S_{2n-l}) also have fewer number of profiles.

8 Conclusion

In a valuation study, dominating (or dominated) profiles are ones which are better (or worse) than others. Choosing a dominating profile or not choosing a dominated profile does not provide much information about comparing attributes because respondents can choose a dominating profile or avoid a dominated profile trivially. Therefore, choice sets with no dominating or dominated profiles are preferable. Such choice sets are known as Pareto Optimal(PO) choice sets. Raghavarao and Wiley (2006) proposed a strategy for generating PO choice sets and gave general connected S^n designs to estimate main effects, two-way interactions and three-way interactions. Further, Raghavarao et al. (2010) explored the design of experiment issues that occur when constructing profiles and showed how to modify commonly used designs for solving conjoint analysis problems.

In this paper, we examined the IPP(θ) of non-consecutive PO design, (S_l, S_{n-l}). The advantage of a non-consecutive PO design is that it reduces the number of profiles and make the experiment smaller in some conditions. Using Information Per Profile (IPP(θ)) as a criterion, we found the optimal non-consecutive PO sets, (S_{l^*}, S_{n-l^*}), where $l^* = ||\frac{n-\sqrt{n}}{2}||$. Then, we compared non-consecutive and consecutive PO designs and concluded that when $n \geq 4$, the optimal non-consecutive PO design, (S_{l^*}, S_{n-l^*}), where $l^* = ||\frac{n-\sqrt{n}}{2}||$, has greater IPP($\theta$) and fewer profiles than the optimal consecutive PO designs, $(S_{[\frac{n}{2}]}, S_{[\frac{n}{2}]+1})$. Also, when $n = a^2$, a is an positive integer and $a \geq 2$, optimal non-consecutive PO sets, (S_{l^*}, S_{n-l^*}), where $l^* = ||\frac{n-\sqrt{n}}{2}||$, have IPP($\theta$) of 1. In other cases (except $n = 3$), IPP(θ) of optimal non-consecutive PO sets, (S_{l^*}, S_{n-l^*}), where $l^* = ||\frac{n-\sqrt{n}}{2}||$, is very close to 1, while IPP(θ) of optimal consecutive PO sets, $(S_{[\frac{n}{2}]}, S_{[\frac{n}{2}]+1})$, is decreasing from 0.667. Therefore, we can see that optimal non-consecutive PO sets, (S_{l^*}, S_{n-l^*}), where $l^* = ||\frac{n-\sqrt{n}}{2}||$, are the best design based on two PO sets, (S_l, S_k) for practical purposes.

Next, we considered the PO design model including first-order-interaction effects. We proved that the non-consecutive design, S_l, $S_{[\frac{n}{2}]}$ and S_k is a connected first-order-interaction plan, where $l+k=n, n \geq 4$ and $l = ||\frac{n-\sqrt{n}}{2}||$, $k = ||\frac{n+\sqrt{n}}{2}||$. Then we examined the IPP(θ) of the non-consecutive PO design. We showed that when n is 4, 6 or 8, IPP(θ) of non-consecutive PO design is equal to the IPP(θ) of consecutive PO design, $S_{\frac{n}{2}-1}$, $S_{[\frac{n}{2}]}$ and $S_{\frac{n}{2}+1}$. When n is an even and $n \geq 10$, IPP(θ) of non-consecutive PO design is greater than the IPP(θ) of consecutive PO design, $S_{\frac{n}{2}-1}$, $S_{[\frac{n}{2}]}$ and $S_{\frac{n}{2}+1}$. We also found that IPP(θ) of the non-consecutive design, S_l, $S_{[\frac{n}{2}]}$ and S_k, where $l+k=n, n \geq 4$ and $l = ||\frac{n-\sqrt{n}}{2}||$, $k = ||\frac{n+\sqrt{n}}{2}||$, is very close to 1. Thus, we can see that for practical purposes, the non-consecutive PO design is the best PO design.

Further, we examined other optimality criteria, such as D-, A- and E-optimality, of both consecutive and non-consecutive PO sets with maximum IPP(θ). We found that optimal PO sets $(S_{[\frac{n}{2}]}, S_{[\frac{n}{2}]+1})$ are the D-, A- and E-optimal designs based on consecutive PO sets, (S_l, S_{l+1}). For non-consecutive PO sets, (S_{l^*}, S_{n-l^*}), where $l^* = ||\frac{n-\sqrt{n}}{2}||$ are D- or A-optimal designs in some cases and always E-optimal designs in the practical cases of $n \leq 20$. Considering some practical limits in the real world, experimenters can choose designs based on multiple criteria, such as number of profiles, IPP(θ), D-, A- and E-optimality.

Finally, we extended our discussion to 3^n experiments. We looked for the feasible PO non-consecutive choice sets with information matrix with positive determinant. We found that not every non-consecutive choice sets of a 3^n experiment is a connected main-effect plan. For those non-consecutive choice sets which are connected main-effect plans, we examined their IPP, D-, A- and E-optimality and showed that the PO sets with maximum IPP(θ_3) are also D- and A-optimal designs based on PO sets (S_l, S_{2n-l}) for $5 \leq n \leq 15$. We also compared non-consecutive and consecutive choice sets and found that the optimal PO sets (S_l, S_{2n-l}) has IPP(θ_3) higher than that of the PO sets (S_l, S_{l+1}) for $5 \leq n \leq 15$.

Appendix

Proof of Theorem 2

For two PO choice sets S_l and S_k, where $l \neq k, 0 < l, k < n$ and $l+k=n$, we have $\begin{pmatrix} n \\ l \end{pmatrix} = \begin{pmatrix} n \\ k \end{pmatrix}$. Without loss of generality, we assume $l < k$.

We can have the equations (6.3). The IPP (θ) can be simplified to a function of l as follows,

$$\theta(l) = \frac{4kl(k-l)^2}{n[(n-2)(k-l)^2+n]}. \tag{8.1}$$

Let $k - l = d$, where $1 \leq d \leq n-2$, so $l = \frac{n-d}{2}$ and $k = \frac{n+d}{2}$. Equation (8.1) becomes

$$\theta(d) = \frac{(n^2-d^2)d^2}{n[(n-2)d^2+n]}.$$

Let $b = d^2 \geq 1$, then

$$\theta(b) = \frac{(n^2-b)b}{n[(n-2)b+n]}.$$

Differentiating θ we get

$$\theta' = \frac{-(n-2)b^2+n(n^2-2b)}{n[(n-2)b+n]^2}.$$

For $n \geq 3$, so that $n-2>0$. Since $1 \leq l < k \leq n-1$, $n[(n-2)b+n]^2 > 0$. Set $\theta' = 0$, so $-(n-2)b^2 + n(n^2-2b) = 0$. Solving this equation for b, we have $b = n$ (because $b \geq 1$, the negative solution was ignored). Since $d \geq 1$, we have $d = \sqrt{n}$. Therefore when $d = \sqrt{n}$, we can have $\theta_{max} = 1$. Considering k and l are positive integers, we can conclude that when $l = ||\frac{n-\sqrt{n}}{2}||$ and $k = ||\frac{n+\sqrt{n}}{2}||$, IPP($\theta$) reaches its maximum value.

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Jing Xiao
Pallavi Chitturi
Temple University,
Philadelphia, PA, USA
E-mail: tub82918@temple.edu
chitturi@temple.edu

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